This is a brief note to spell out a basic but useful fact about relative Borel-Moore homology. The fact itself is just a special case of a corollary in Bredon's *Sheaf Theory* [Bre97], but it takes a little bit of work to unwind his notation in this special case, and it seems useful to have the special case written down explicitly.

Proposition *Let X be a Hausdorff, locally compact, paracompact space and let $U \subseteq X$ an open subspace. Let $\mathcal{L}$ be a local system on X, defined over a principal ideal domain R, whose fibres are finitely generated R-modules. We then have:*

$$H_*^{lf}(X, X \smallsetminus U; \mathcal{L}) \cong H_*^{lf}(U; \mathcal{L}). \tag{1}$$

Here, H_*^{lf} denotes the homology of infinite, locally finite chains, often referred to as *Borel-Moore homology*. More generally, the Borel-Moore homology of a space (or pair of spaces) is defined in [Bre97, Chap. V] with coefficients in any sheaf $\mathcal{M}$ and with respect to any "family of supports" Φ. Taking Φ to be the family of all *compact* subspaces recovers ordinary singular homology (so, in this more general sense, ordinary homology is a special case of Borel-Moore homology) whereas taking Φ to be the family of all *closed* subspaces corresponds to H_*^{lf}, which is usually what "*Borel-Moore homology*" refers to if Φ is not specified.

Proof of the Proposition. We will check that the isomorphism (1) is a special case of

$$H_*^{\Phi}(X, X \smallsetminus U; \mathcal{M}) \cong H_*^{\Phi \cap U}(U; \mathcal{M}), \tag{2}$$

which is proven in [Bre97, Corollary V.5.10, page 312] under the assumptions that X is Hausdorff and locally compact, $U \subseteq X$ is an open subspace (whose complement $X \smallsetminus U$ we denote by F), $\mathcal{M}$ is a sheaf on X defined over a principal ideal domain R and Φ is a family of supports on X such that $(\mathcal{M}, \Phi|F)$ is "elementary".

Precisely, let us set $\mathcal{M} = \mathcal{L}$ and take Φ to be the family of all closed subspaces of X. It then follows from the definition of the subspace topology that $\Phi \cap U = \{K \cap U \mid K \in \Phi\}$ is the family of all closed subspaces of U. Thus (2) in this special case is precisely (1) and it just remains to check that $(\mathcal{L}, \Phi|F)$ is elementary, where $\Phi|F$ is the family of all closed subspaces of $F = X \smallsetminus U$.

Recall from [Bre97, Definition V.3.6] that a sheaf $\mathcal{M}$ is *elementary* if it is locally constant and its stalks are finitely generated over R. In other words, it is elementary precisely if it is a local system whose fibres are finitely generated R-modules, which is exactly what we have assumed for $\mathcal{L}$ in the statement of the Proposition. Thus $\mathcal{M} = \mathcal{L}$ is elementary.

It is remarked immediately after [Bre97, Definition V.3.6] (using [Bre97, Lemma V.3.7]) that if $\mathcal{M}$ is elementary and every element of Ψ is paracompact then $(\mathcal{M}, \Psi)$ is elementary. We therefore just have to show that every element of $\Psi = \Phi|F$ is paracompact. This follows from our assumption that X is paracompact, together with the fact that any closed subspace of a paracompact space is again paracompact. ☐

Remark When $\mathcal{L}$ is a constant local system, the isomorphism (1) could be proven by interpreting $H_*^{lf}(-)$ as the reduced homology of the (relative) one-point compactification. This strategy works more generally for local systems $\mathcal{L}$ on X that extend to the relative one-point compactification of the pair $(X, X \smallsetminus U)$. However, there are cases where one is interested in (1) for local systems $\mathcal{L}$ that do not so extend.

References

[Bre97] Glen E. Bredon. *Sheaf theory*. Second. Vol. 170. Graduate Texts in Mathematics. Springer-Verlag, New York, 1997, pp. xii+502.