Heisenberg homology on surface configurations

Christian Blanchet[*], Martin Palmer[†], Awais Shaukat[‡]

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Abstract

Motivated by the Lawrence-Krammer-Bigelow representations of the classical braid groups, we study the homology of unordered configurations in an orientable genus-g surface with one boundary component, over non-commutative local systems defined from representations of the discrete Heisenberg group. For a general representation we obtain a twisted action of the mapping class group. In the case of the Schrödinger representation or its finite dimensional analogues, by composing with a Stone-von Neumann isomorphism we are able to untwist and obtain a representation to the projective unitary group which lifts to a unitary representation of the *stably universal* central extension of the mapping class group.

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Introduction

The braid group B_m was defined by Artin in terms of geometric braids in $\mathbb{R}^3$; equivalently, it is the fundamental group of the configuration space $\mathcal{C}_m(\mathbb{R}^2)$ of m unordered points in the plane. Another equivalent description is as the mapping class group $\mathfrak{M}(\mathbb{D}_m) = \mathrm{Diff}(\mathbb{D}_m, S^1)/\mathrm{Diff}_0(\mathbb{D}_m, S^1)$ of the closed 2-disc with m interior points removed. (The *mapping class group* of a surface is the group of isotopy classes of self-diffeomorphisms fixing the boundary pointwise.)

There is also a natural action of $\mathrm{Diff}(\mathbb{D}_m, S^1)$ on configuration spaces $\mathcal{C}_n(\mathbb{D}_m)$; considering the induced action on the homology of these configuration spaces, Lawrence [30] defined a representation of B_m for each $n \geqslant 1$. The $n = 2$ version is known as the *Lawrence-Krammer-Bigelow representation*, and a celebrated result of Bigelow [12] and Krammer [29] states that this representation of B_m is *faithful*, i.e. injective.

[*]Université de Paris and Sorbonne Université, CNRS, IMJ-PRG, F-75006 Paris, France
[†]Simion Stoilow Mathematical Institute of the Romanian Academy, Bucharest, Romania
[‡]Abdus Salam School of Mathematical Sciences, Lahore, Pakistan

On the other hand, for almost all other surfaces Σ, the question of whether the mapping class group $\mathfrak{M}(\Sigma)$ admits a faithful, finite-dimensional representation over a field (whether it is *linear*) is open. The mapping class group of the torus is $SL_2(\mathbb{Z})$, which is evidently linear, and the mapping class group of the closed orientable surface of genus 2 was shown to be linear by Bigelow and Budney [13], as a corollary of the linearity of B_5. However, nothing is known in genus $g \geqslant 3$.

Our programme is to study the action of the positive-genus and connected-boundary mapping class groups $\mathfrak{M}(\Sigma_{g,1})$ on the homology of the configuration spaces $\mathcal{C}_n(\Sigma_{g,1})$, equipped with local systems that are similar to the Lawrence-Krammer-Bigelow construction. We first argue that *abelian* local systems would not be good enough. In general, for any surface Σ and $n \geqslant 2$, the abelianisation of $\pi_1(\mathcal{C}_n(\Sigma))$ is canonically isomorphic to $H_1(\Sigma) \times C$, where C is a cyclic group of order ∞ if Σ is planar, of order $2n-2$ if $\Sigma = \mathbb{S}^2$ and of order 2 in all other cases (see for example [16, Proposition 6.11]). In the case $\Sigma = \mathbb{D}_m$, the abelianisation is $\mathbb{Z}^m \times \mathbb{Z}$, and the Lawrence representations are defined using the local system given by the quotient $\pi_1(\mathcal{C}_n(\mathbb{D}_m)) \twoheadrightarrow \mathbb{Z}^m \times \mathbb{Z} \twoheadrightarrow \mathbb{Z} \times \mathbb{Z}$, where the second map is addition of the first m factors. However, in the non-planar case (in particular if $\Sigma = \Sigma_{g,1}$), we *lose information* by passing to the abelianisation, since the cyclic factor C – which counts the self-winding or "writhe" of a loop of configurations – has order 2 rather than order ∞.

To obtain a better analogue of the Lawrence representations in the setting $\Sigma = \Sigma_{g,1}$ for $g > 0$, we consider instead a larger, non-abelian quotient of $\pi_1(\mathcal{C}_n(\Sigma))$, the discrete Heisenberg group $\mathcal{H} = \mathcal{H}(\Sigma)$, which may be defined as the central extension of the first homology $H = H_1(\Sigma, \mathbb{Z})$ associated to the intersection 2-cocycle. This is a 2-nilpotent group that arises very naturally as a quotient of the surface braid group $\pi_1(\mathcal{C}_n(\Sigma))$ by forcing a single element to be central. In the case $n \geqslant 3$ it is known by [4] to be the 2-*nilpotentisation* of the surface braid group (in fact it is the maximal nilpotent quotient of the surface braid group), but for $n = 2$ it differs from the 2-nilpotentisation. A key property of this nilpotent quotient is that it still detects the self-winding (or "writhe") of a loop of configurations *without reducing modulo two*. Any representation V of the discrete Heisenberg group $\mathcal{H}(\Sigma)$ defines a local system on the configuration space $\mathcal{C}_n(\Sigma)$.

An and Ko studied in [1] extensions of the Lawrence-Krammer-Bigelow representations to homological representations of surface braid groups; see also [7]. Their purpose was to extend the homological representation of the classical braid group to some homology of configurations in an n-punctured surface and produce representations of the surface braid groups. In our case the surface has no puncture, and the goal is to represent the full mapping class group. Our Heisenberg local systems have a similar flavour but are significantly simpler; moreover we obtain a strong improvement by specialising to the Schrödinger representations.

Notation 1. Henceforth we will use the abbreviation $\Sigma = \Sigma_{g,1}$ for an integer $g \geqslant 1$.

General representation. Our first main result is a calculation of a Borel-Moore relative homology group with coefficients twisted by any representation of the Heisenberg group, together with a twisted action of the mapping class group. In the following, H_*^{BM} will denote Borel-Moore homology and $\mathcal{C}_n(\Sigma, \partial^-(\Sigma))$ is the properly embedded subspace of $\mathcal{C}_n(\Sigma)$ consisting of all configurations intersecting a given arc $\partial^-\Sigma \subset \partial\Sigma$. The twisted action will be formulated as a representation of an *action groupoid*. The key point is that the mapping class group acts on the Heisenberg group which implies an action on our local systems. We denote by $f_{\mathcal{H}} \in \mathrm{Aut}(\mathcal{H})$ the automorphism induced by $f \in \mathfrak{M}(\Sigma)$. For a representation $\rho\colon \mathcal{H} \to GL(V)$ and $\tau \in \mathrm{Aut}(\mathcal{H})$, the τ-twisted representation $\rho \circ \tau$ is denoted by ${}_\tau V$.

Theorem A. *Let $n \geqslant 2$, $g \geqslant 1$ and let V be a representation of the discrete Heisenberg group $\mathcal{H} = \mathcal{H}(\Sigma = \Sigma_{g,1})$ over a ring R.*
(a) *The Borel-Moore homology module $H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)$ is isomorphic to the direct sum of $\begin{pmatrix} 2g+n-1 \\ n \end{pmatrix}$ copies of V. Furthermore, it is the only non-vanishing module in the graded module $H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)$.*
(b) *There is a natural twisted representation of the mapping class group $\mathfrak{M}(\Sigma)$ on the collection of R-modules*

$$H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_\tau V\big)\ , \quad \tau \in \mathrm{Aut}(\mathcal{H})\ ,$$

where the action of $f \in \mathfrak{M}(\Sigma)$ is

$$\mathcal{C}_n(f)_*\colon H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_{\tau\circ f_{\mathcal{H}}} V\big) \longrightarrow H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_\tau V\big) \qquad (1)$$

Remark 2. The case of the trivial representation is already something interesting; indeed, connecting with Moriyama's work [36], we will show that the Johnson filtration is recovered.

Remark 3. The Heisenberg group $\mathcal{H}(\Sigma)$ can be realised as a group of $(g+2) \times (g+2)$ matrices, which gives a *tautological* $(g+2)$-dimensional representation. We then obtain, for each $n \geqslant 2$, a family of twisted representations with polynomially growing dimension equal to $(g+2)\begin{pmatrix} 2g+n-1 \\ n \end{pmatrix}$.

The Schrödinger representation. The famous Stone-von Neumann Theorem states that the Schrödinger representation $W \cong L^2(\mathbb{R}^g)$ is, up to isomorphism, the unique unitary representation of the real Heisenberg group $\mathcal{H}_{\mathbb{R}}(\Sigma)$ with given non-trivial action of its centre determined by a non-zero real number $\hbar$ (the Planck constant). We also denote by W this representation restricted to the discrete group $\mathcal{H}$. For $\tau \in \mathrm{Aut}(\mathcal{H})$ the twisted representation ${}_\tau W$ is isomorphic to W as a unitary representation and the isomorphism is defined up to a unit complex number. Using such isomorphisms we may identify the twisted local system with the original one and obtain an untwisted representation of the mapping class group to the projective unitary group of the homology with local coefficients W. Our second main result is a linear lift of this projective action to the stably universal central extension $\widetilde{\mathfrak{M}}(\Sigma)$.

Theorem B. *For each $n \geqslant 2$ and $g \geqslant 1$ there is a complex unitary representation of $\widetilde{\mathfrak{M}}(\Sigma = \Sigma_{g,1})$ on the complex Hilbert space*

$$\mathcal{V}_n = H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); W\big) \tag{2}$$

that lifts the natural projective action $\mathfrak{M}(\Sigma) \to PU(\mathcal{V}_n)$.

The group $\widetilde{\mathfrak{M}}(\Sigma)$ on which we construct our linear representation is a central extension of the mapping class group $\mathfrak{M}(\Sigma)$ of the form:

$$1 \to \mathbb{Z} \longrightarrow \widetilde{\mathfrak{M}}(\Sigma) \longrightarrow \mathfrak{M}(\Sigma) \to 1, \tag{3}$$

and is the *stably universal central extension* of $\mathfrak{M}(\Sigma)$ in the following sense. A group G has a *universal central extension* (an initial object in the category of central extensions of G) if and only if $H_1(G;\mathbb{Z}) = 0$. It is of the form $0 \to H_2(G;\mathbb{Z}) \to \widetilde{G} \to G \to 1$ when it exists. For genus $g \geqslant 4$, we have $H_1(\mathfrak{M}(\Sigma_{g,1});\mathbb{Z}) = 0$ and $H_2(\mathfrak{M}(\Sigma_{g,1});\mathbb{Z}) \cong \mathbb{Z}$. Moreover, there are natural inclusion maps

$$\mathfrak{M}(\Sigma_{1,1}) \longrightarrow \mathfrak{M}(\Sigma_{2,1}) \longrightarrow \cdots \longrightarrow \mathfrak{M}(\Sigma_{g,1}) \longrightarrow \mathfrak{M}(\Sigma_{g+1,1}) \longrightarrow \cdots, \tag{4}$$

which induce isomorphisms on $H_1(-;\mathbb{Z})$ and $H_2(-;\mathbb{Z})$ for $g \geqslant 4$ (by homological stability for mapping class groups of surfaces, due originally to Harer [23]; see [44, Theorem 1.1] for the optimal stability range). This implies that, for $g \geqslant 4$, the pullback along (4) of the universal central extension of $\mathfrak{M}(\Sigma_{g+1,1})$ to $\mathfrak{M}(\Sigma_{g,1})$ is the universal central extension of $\mathfrak{M}(\Sigma_{g,1})$. Hence we may define, for all $g \geqslant 1$, the *stably universal central extension* of $\mathfrak{M}(\Sigma_{g,1})$ to be the pullback along (4) of the universal central extension of $\mathfrak{M}(\Sigma_{h,1})$ for any $h \geqslant \max(g, 4)$.

When the Planck constant is 2π times a rational number, the discrete Heisenberg group has finite-dimensional Schrödinger representations, which may be realised either by theta functions, by induction or by an abelian TQFT. We will follow [20, 21, 22] which connect nicely the different approaches when $\hbar = \frac{2\pi}{N}$ for a positive even integer N. We denote by $W_N = L^2((\mathbb{Z}/N)^g)$ the N^g-dimensional representation that is the unique irreducible representation of the finite quotient $\mathcal{H}_N = \mathcal{H}/I_N$ by the normal subgroup $I_N = \{(2Nk, Nx) \mid k \in \mathbb{Z}, x \in H\}$ where each central element $(k, 0)$ acts by $e^{\frac{i\pi k}{N}}$. The analogue of the Stone-von Neumann Theorem in this context [21, Theorem 2.4] allows us to construct an untwisted representation of the mapping class group to a projective unitary group which also supports a linear lift to the stably universal central extension.

Theorem C. *For each $g \geqslant 1$, $n \geqslant 2$ and $N \geqslant 2$ with N even, there is a complex unitary representation of $\widetilde{\mathfrak{M}}(\Sigma = \Sigma_{g,1})$ on the $\binom{2g+n-1}{n} N^g$-dimensional complex Hilbert space*

$$\mathcal{V}_{N,n} = H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); W_N\big) \tag{5}$$

that lifts the natural projective action $\mathfrak{M}(\Sigma) \to PU(\mathcal{V}_{N,n})$.

Remark 4. For any complex vector space V, the adjoint action of $GL(V)$ on $\mathrm{End}_{\mathbb{C}}(V)$ induces a canonical embedding $PGL(V) \hookrightarrow GL(\mathrm{End}_{\mathbb{C}}(V))$. Applying this to the natural projective action $\mathfrak{M}(\Sigma) \to PU(\mathcal{V}_{N,n})$, we obtain an *untwisted complex representation*

$$\mathfrak{M}(\Sigma) \longrightarrow GL(\mathrm{End}_{\mathbb{C}}(\mathcal{V}_{N,n})) \tag{6}$$

of dimension $\binom{2g+n-1}{n}^2 N^{2g}$. We observe that:

Observation 5. Injectivity of (6) is equivalent to the statement that

$$\text{the pre-image of } \mathbb{C}^* \text{ along (5)}: \widetilde{\mathfrak{M}}(\Sigma) \to U(\mathcal{V}_{N,n}) \text{ is equal to } \mathbb{Z}, \tag{7}$$

where $\mathbb{C}^*$ means the scalar operators in $U(\mathcal{V}_{N,n})$ and $\mathbb{Z}$ refers to the kernel of (3). Thus a proof of (7) for any pair (N, n) with $N, n \geqslant 2$ would imply that the mapping class group $\mathfrak{M}(\Sigma)$ is linear.

Subgroups of the mapping class group. We will identify the kernel of the mapping class group action on the Heisenberg group as the *Chillingworth subgroup* $\mathrm{Chill}(\Sigma)$. This subgroup, included in the Torelli group $\mathfrak{T}(\Sigma)$, may be defined as the subgroup of mapping classes acting trivially on homotopy classes of non-singular vector fields. For a general representation of the Heisenberg group $\mathcal{H}(\Sigma)$, the twisted action on homologies induces a representation of $\mathrm{Chill}(\Sigma)$. We also show that a mapping class f belongs to the Torelli subgroup if and only if the automorphism $f_{\mathcal{H}}$ is inner, in which case the conjugating element is defined up to centre. It follows that, for $f \in \mathfrak{T}(\Sigma)$ and any representation V of $\mathcal{H}$, we may identify the $f_{\mathcal{H}}$-twisted local system with the original one and obtain an untwisted projective representation of the Torelli group on the homology with local coefficients V. Our third main result normalises this action on the Torelli group itself, meaning that a cocycle associated with the projective action is null-homologous.

Theorem D. *For each $n \geqslant 2$, $g \geqslant 1$ and each representation V of the discrete Heisenberg group $\mathcal{H} = \mathcal{H}(\Sigma)$ over a ring R, there exists a lift of the projective action on homology which defines a representation of the Torelli group $\mathfrak{T}(\Sigma = \Sigma_{g,1})$ on the R-module*

$$H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\big). \tag{8}$$

The stably universal central extension $\widetilde{\mathfrak{M}}(\Sigma)$ of the mapping class group trivialises on the Torelli group (Lemma 47), so Theorems B and C above already provide representations of the Torelli group without passing to a central extension. The advantage of Theorem D is that it applies to *any* representation V of the discrete Heisenberg group, not just the Schrödinger representations W and W_N.

Returning now to the Schrödinger representation, we identify another large subgroup of the mapping class group $\mathfrak{M}(\Sigma)$ on which we may construct linear representations without passing to a central extension. In [33], Morita introduced a crossed homomorphism $\mathfrak{M}(\Sigma) \to H^1(\Sigma)$, $f \mapsto \mathfrak{d}_f$ representing a generator for $H^1(\mathfrak{M}(\Sigma), H^1(\Sigma)) \cong \mathbb{Z}$. We will recover this crossed homomorphism from the action $f \mapsto f_{\mathcal{H}}$ on the Heisenberg group.

We define the *Morita subgroup* $\mathrm{Mor}(\Sigma)$ of the mapping class group $\mathfrak{M}(\Sigma)$ to be the kernel of Morita's crossed homomorphism. Note that this subgroup depends on the parametrisation of Σ.

The unitary group $U(\mathcal{V}_n)$ of the Hilbert space $\mathcal{V}_n =$ (2) is a central extension of the projective unitary group $PU(\mathcal{V}_n)$ by the circle group. Hence the natural projective representation $\mathfrak{M}(\Sigma) \to PU(\mathcal{V}_n)$ lifts to a linear representation of some central extension $\overline{\mathfrak{M}}(\Sigma)$ of the mapping class group by the circle. Another natural central extension of the mapping class group is the *metaplectic extension* $\widehat{\mathfrak{M}}(\Sigma)$, which is a central extension by $\mathbb{Z}/2$. We show that the restriction $\overline{\mathrm{Mor}}(\Sigma)$ of the central extension $\overline{\mathfrak{M}}(\Sigma)$ to the Morita subgroup contains the restriction $\widehat{\mathrm{Mor}}(\Sigma)$ of the metaplectic extension $\widehat{\mathfrak{M}}(\Sigma)$, and also that the extension $\widehat{\mathrm{Mor}}(\Sigma) \to \mathrm{Mor}(\Sigma)$ is trivial. This implies:

Theorem E. *For each $n \geqslant 2$ and $g \geqslant 1$ the projective action on Borel-Moore homology lifts to a unitary representation of the subgroup $\mathrm{Mor}(\Sigma) \subseteq \mathfrak{M}(\Sigma = \Sigma_{g,1})$ on the complex Hilbert space*

$$\mathcal{V}_n = H_n^{BM}\bigl(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); W\bigr). \tag{9}$$

Remark 6. The representations of Theorem E and of Theorem D (taking V equal to the Schrödinger representation in the latter) agree on the intersection $\mathrm{Mor}(\Sigma) \cap \mathfrak{T}(\Sigma)$, which is the *Chillingworth subgroup* $\mathrm{Chill}(\Sigma)$ of the mapping class group.

See Figure 1 for a visual summary of the three subgroups of the mapping class group that we consider. The homomorphism $\mathfrak{s}$ is the action of the mapping class group on its first homology $H = H_1(\Sigma; \mathbb{Z})$ and $\mathfrak{d}$ is Morita's crossed homomorphism (see §3). The pair $(\mathfrak{s}, \mathfrak{d})$ thus defines a homomorphism to the semi-direct product $Sp(H) \ltimes H$ lifting $\mathfrak{s}$.

$$\begin{tikzcd}
& \mathfrak{T}(\Sigma) \arrow[dr] & & H \\
\mathrm{Chill}(\Sigma) \arrow[ur] \arrow[rr] \arrow[dr] & & \mathfrak{M}(\Sigma) \arrow[ur, "\mathfrak{d}"] \arrow[r, "(\mathfrak{s},\mathfrak{d})"] \arrow[dr, "\mathfrak{s}"'] & Sp(H) \ltimes H \arrow[u] \arrow[d] \\
& \mathrm{Mor}(\Sigma) \arrow[ur] & & Sp(H)
\end{tikzcd}$$

Figure 1: The *Torelli group* $\mathfrak{T}(\Sigma) = \ker(\mathfrak{s})$, the *Morita subgroup* $\mathrm{Mor}(\Sigma) = \ker(\mathfrak{d})$ and their intersection, the *Chillingworth subgroup* $\mathrm{Chill}(\Sigma) = \ker((\mathfrak{s}, \mathfrak{d}))$.

Kernels. To describe an upper bound on the kernels of our representations, we first recall the *Johnson filtration* of the mapping class group.

The mapping class group $\mathfrak{M}(\Sigma)$ acts naturally on the fundamental group $\pi_1(\Sigma)$ of the surface. Each term of the lower central series of a group is fully invariant, so there is a well-defined induced action of $\mathfrak{M}(\Sigma)$ on the quotient $\pi_1(\Sigma)/\Gamma_{i+1}$, which is the largest $(i+1)$-step nilpotent quotient of $\pi_1(\Sigma)$. The *Johnson filtration* $\mathfrak{J}(*)$ is then defined by setting $\mathfrak{J}(i)$ to be the kernel of this induced action. Thus $\mathfrak{J}(0)$ is the whole mapping class group and $\mathfrak{J}(1)$ is the Torelli group. The intersection of all terms in the filtration is trivial, i.e., it is an *exhaustive* filtration of the mapping class group [27]. The Chillingworth subgroup sits between $\mathfrak{J}(1)$ and $\mathfrak{J}(2)$:

$$\mathfrak{M}(\Sigma) = \mathfrak{J}(0) \;\supseteq\; \mathfrak{T}(\Sigma) = \mathfrak{J}(1) \;\supseteq\; \mathrm{Chill}(\Sigma) \;\supseteq\; \mathfrak{J}(2) \;\supseteq\; \mathfrak{J}(3) \;\supseteq\; \cdots$$

One may also consider the induced action of the mapping class group $\mathfrak{M}(\Sigma)$ on the universal metabelian quotient $\pi_1(\Sigma)/\pi_1(\Sigma)^{(2)}$ of the fundamental group of the surface (the quotient by its second derived subgroup); its kernel is the *Magnus kernel* of $\mathfrak{M}(\Sigma)$, which we denote by $\mathrm{Mag}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$. In §7 (Proposition 72) we prove:

Proposition F. *For each $n \geqslant 2$, $g \geqslant 1$ and representation V of the discrete Heisenberg group $\mathcal{H} = \mathcal{H}(\Sigma)$, the kernels of the representations constructed in Theorems A–E are contained in $\mathfrak{J}(n) \cap \mathrm{Mag}(\Sigma)$.*

Computability. We emphasise that our representations are explicit and computable. First, the underlying R-modules in Theorems A–E are direct sums of finitely many copies of the R-module V that underlies the chosen representation of the discrete Heisenberg group $\mathcal{H}(\Sigma)$. This is Theorem A(a); an explicit basis is described in Theorem 11.

Moreover, the actions of elements of the mapping class group on the canonical basis provided by Theorem 11 may be explicitly computed. To demonstrate this, we calculate in §8 explicit matrices for our representations in the case when $n = 2$ and $V = \mathbb{Z}[\mathcal{H}]$ is the regular representation of $\mathcal{H} = \mathcal{H}(\Sigma)$. For example, when $g = 1$, the Dehn twist around the boundary of $\Sigma_{1,1}$ acts by the 3×3 matrix over $\mathbb{Z}[\mathcal{H}] = \mathbb{Z}[u^{\pm 1}]\langle a^{\pm 1}, b^{\pm 1}\rangle/(ab = u^2 ba)$ depicted in Figure 9 (page 57).

Outline. In §1 we define and study the quotient $\mathcal{H}$ of the surface braid group. In §2 we study the Borel-Moore homology (with local coefficients) of configuration spaces on Σ, showing that, with coefficients in $V = \mathbb{Z}[\mathcal{H}]$, it is a free module with an explicit free generating set. Next, in §3, we show that the action of the mapping class group on the surface braid group descends to $\mathcal{H}$. Then we study this induced action on $\mathcal{H}$ in detail, and in particular determine its kernel, as well as the subgroup of the mapping class group that acts by *inner* automorphisms under this action. The latter group turns out to be the Torelli group, whereas the kernel turns out to be the Chillingworth subgroup. In §4 we then describe a general trick for untwisting twisted representations of groups by passing to a central extension.

Section 5 puts all of this together and constructs the twisted representations of Theorem A and the untwisted representations of Theorem D. In particular, §5.1 explains the notion of a *twisted representation* of a group and §5.2 constructs twisted representations of the full mapping class group. In §6 we then prove Theorems B, C and E: untwisted representations of the stably universal central extension of the mapping class group and untwisted representations of the Morita subgroup (without passing to a central extension), with coefficients in the Schrödinger representation of $\mathcal{H}$ or its finite-dimensional analogues. The untwisting in these cases uses the *Segal-Shale-Weil projective representation* of the symplectic group.

In §7 we discuss relations with the Moriyama and Magnus representations of mapping class groups, and deduce that the kernels of our representations of $\mathfrak{M}(\Sigma)$ are contained in $\mathfrak{J}(n) \cap \mathrm{Mag}(\Sigma)$, where $\mathfrak{J}(*)$ is the Johnson filtration and $\mathrm{Mag}(\Sigma)$ is the Magnus kernel. In §8 we explain how to compute explicit matrices for our representations with respect to the free basis coming from §2. We carry out this computation in the case of configurations of $n = 2$ points and where $V = \mathbb{Z}[\mathcal{H}]$ is the regular representation of $\mathcal{H}$; this special case of our construction is the most direct analogue of the Lawrence-Krammer-Bigelow representations of the braid groups.

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Contents

1 A non-commutative local system on configuration spaces of surfaces

Let $\Sigma = \Sigma_{g,1}$ be a compact, connected, orientable surface of genus $g \geqslant 1$ with one boundary component. For $n \geqslant 2$, the n-point unordered configuration space of Σ is

$$\mathcal{C}_n(\Sigma) = \{\{c_1, c_2, ..., c_n\} \subset \Sigma \mid c_i \neq c_j \text{ for } i \neq j\},$$

topologised as a quotient of a subspace of Σ^n. The surface braid group $\mathbb{B}_n(\Sigma)$ is then defined as $\mathbb{B}_n(\Sigma) = \pi_1(\mathcal{C}_n(\Sigma))$. We will use the presentation of this group given by

Bellingeri and Godelle in [5], which in turn follows from Bellingeri's presentation [3]. It has generators $\sigma_1, \ldots, \sigma_{n-1}$, $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ and relations:

$$\begin{cases} (\mathbf{BR1})\ [\sigma_i, \sigma_j] = 1 & \text{for } |i-j| \geqslant 2, \\ (\mathbf{BR2})\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1, \\ (\mathbf{CR1})\ [\alpha_r, \sigma_i] = [\beta_r, \sigma_i] = 1 & \text{for } i > 1 \text{ and all } r, \\ (\mathbf{CR2})\ [\alpha_r, \sigma_1\alpha_r\sigma_1] = [\beta_r, \sigma_1\beta_r\sigma_1] = 1 & \text{for all } r, \\ (\mathbf{CR3})\ [\alpha_r, \sigma_1^{-1}\alpha_s\sigma_1] = [\alpha_r, \sigma_1^{-1}\beta_s\sigma_1] = & \\ \qquad\qquad = [\beta_r, \sigma_1^{-1}\alpha_s\sigma_1] = [\beta_r, \sigma_1^{-1}\beta_s\sigma_1] = 1 & \text{for all } r < s, \\ (\mathbf{SCR})\ \sigma_1\beta_r\sigma_1\alpha_r\sigma_1 = \alpha_r\sigma_1\beta_r & \text{for all } r. \end{cases}$$

We note that composition of loops is written from right to left. Our relation (**CR3**) is a slight modification of the relation (**CR3**) of [5], but it is equivalent to it via the relation (**CR2**).

The first homology group $H_1(\Sigma) = H_1(\Sigma;\mathbb{Z})$ is equipped with a symplectic intersection form

$$\omega\colon H_1(\Sigma) \times H_1(\Sigma) \longrightarrow \mathbb{Z}$$

and the *Heisenberg group* $\mathcal{H} = \mathcal{H}(\Sigma)$ is defined to be the central extension of $H_1(\Sigma)$ determined by the 2-cocycle ω. Concretely, it is the set-theoretic product $\mathbb{Z} \times H_1(\Sigma)$ with the operation

$$(k,x)(l,y) = (k + l + \omega(x,y), x + y). \tag{10}$$

Denote by $\psi\colon \mathcal{H} \twoheadrightarrow H_1(\Sigma)$ the projection onto the second factor and by $i\colon \mathbb{Z} \hookrightarrow \mathcal{H}$ the inclusion of the first factor; the central extension may then be written as:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ i\ } \mathcal{H} \xrightarrow{\ \psi\ } H_1(\Sigma) \longrightarrow 0$$

There is a general recipe for computing a presentation of an extension of two groups, given presentations of these two groups (see for example [16, §A.3]). In particular, for a *central* extension $1 \to H \to G \to K \to 1$ with $H = \langle X|R\rangle$ and $K = \langle Y|S\rangle$, we have $G = \langle X \sqcup Y | R \sqcup \widetilde{S} \sqcup T\rangle$, where $\widetilde{S}$ is any collection of relations that are true in G and that project to the relations S in K and where T is a collection of relations saying that the generators X are central in G.

Applying this to our setting, we obtain the following presentation of $\mathcal{H}$, where we write $u = (1,0)$ and where $a_1, \ldots, a_g$, $b_1, \ldots, b_g$ is a symplectic basis of $H_1(\Sigma)$.

Proposition 7. *The Heisenberg group $\mathcal{H} = \mathcal{H}(\Sigma)$ admits a presentation with generators u, $\tilde{a}_i = (0, a_i)$, $\tilde{b}_i = (0, b_i)$ for $1 \leqslant i \leqslant g$ and relations:*

$$\begin{cases} \textit{all pairs of generators commute, except:} \\ \tilde{a}_i\tilde{b}_i = u^2\tilde{b}_i\tilde{a}_i \qquad \textit{for each } i. \end{cases} \tag{11}$$

Proof. We apply the above procedure to the presentations $\mathbb{Z} = \langle X|R\rangle$ and $H_1(\Sigma) = \langle Y|S\rangle$ where $X = \{u\}$, $Y = \{\tilde{a}_1, \ldots, \tilde{a}_g, \tilde{b}_1, \ldots, \tilde{b}_g\}$, the relations R are empty and the relations S say that all pairs of elements of Y commute. The relations T say that u commutes with each of $\{\tilde{a}_1, \ldots, \tilde{a}_g, \tilde{b}_1, \ldots, \tilde{b}_g\}$, so to show that (11) is a correct presentation of $\mathcal{H}$ it will suffice to show that the relations $\tilde{a}_i\tilde{b}_i = u^2\tilde{b}_i\tilde{a}_i$ and $\tilde{a}_i\tilde{b}_j = \tilde{b}_j\tilde{a}_i$ for $i \neq j$ are true in $\mathcal{H}$, because we may then take $\widetilde{S}$ to be this collection of relations, since it projects to S. To verify these, we compute that

$$\tilde{a}_i\tilde{b}_j = (0, a_i + b_j) = (0, b_j + a_i) = \tilde{b}_i\tilde{a}_j$$

since $\omega(a_i, b_j) = 0$ when $i \neq j$, and

$$\tilde{a}_i\tilde{b}_i = (1, a_i + b_i) = (1, b_i + a_i) = (2, 0)(-1, b_i + a_i) = u^2\tilde{b}_i\tilde{a}_i,$$

since $\omega(a_i, b_i) = 1$ and $\omega(b_i, a_i) = -1$. ■

It follows immediately from this presentation that:

Corollary 8. *For each $g \geqslant 1$ and $n \geqslant 2$, there is a natural surjective homomorphism*

$$\phi\colon \mathbb{B}_n(\Sigma) \longrightarrow\!\!\!\!\rightarrow \mathcal{H}(\Sigma)$$

sending each σ_i to u and sending $\alpha_i \mapsto \tilde{a}_i$, $\beta_i \mapsto \tilde{b}_i$.

In the case $n \geqslant 3$, this quotient of the surface braid group has previously been considered in [4, 6, 7], which also consider the more general setting where Σ is closed or has several boundary components. The alternative approach in these articles allows one to identify the kernel of ϕ as a characteristic subgroup. We include below a description of the kernel valid for all $n \geqslant 2$.

Proposition 9. (a) *For $n \geqslant 2$, the kernel of ϕ is the normal subgroup generated by the commutators $[\sigma_1, x]$ for $x \in \mathbb{B}_n(\Sigma)$.*
(b) *For $n \geqslant 3$, the kernel of ϕ is the subgroup of 3-commutators $\Gamma_3(\mathbb{B}_n(\Sigma))$.*

For a proof of statement (b), we refer to [4, Theorem 2]. More precisely, statement (10) on page 1416 of [4] is the analogous fact for the closed surface Σ_g: that there is a surjective homomorphism $\mathbb{B}_n(\Sigma_g) \twoheadrightarrow \mathcal{H}_g/\langle u^{2(n+g-1)}\rangle$ whose kernel is exactly $\Gamma_3(\mathbb{B}_n(\Sigma_g))$. The proof given there works also in our case where the surface has one boundary component and we do not quotient by $\langle u^{2(n+g-1)}\rangle$. In this paper we will use statement (a) and focus on the case $n = 2$ in our explicit computations.

Proof. Let $K_n \subseteq \mathbb{B}_n(\Sigma)$ be the normal subgroup generated by the commutators $[\sigma_1, x]$ for $x \in \mathbb{B}_n(\Sigma)$. The image $\phi(\sigma_1)$ being central, we have $K_n \subseteq \ker(\phi)$, hence we see that ϕ may be factored through a surjective homomorphism $\overline{\phi}\colon \mathbb{B}_n(\Sigma)/K_n \to \mathcal{H}$. If we add centrality of σ_1 to the defining relations for $\mathbb{B}_n(\Sigma)$, we may:

- replace (**BR2**) by $\sigma_i = \sigma_1$ for all i,
- remove (**BR1**), (**CR1**) and (**CR2**),

- replace (**CR3**) by commutators of all pairs of generators except for (α_r, β_r),
- replace (**SCR**) with $\alpha_r\beta_r = \sigma_1^2\beta_r\alpha_r$.

Finally the presentations of $\mathbb{B}_n(\Sigma)/K_n$ and $\mathcal{H}$ coincide and $\overline{\phi}$ is an isomorphism, which proves (a). ∎

In contrast to the case of $n \geqslant 3$, the kernel $\ker(\phi)$ when $n = 2$ lies strictly between the terms Γ_2 and Γ_3 of the lower central series of $\mathbb{B}_2(\Sigma)$.

Proposition 10. *There are proper inclusions*

$$\Gamma_3(\mathbb{B}_2(\Sigma)) \hookrightarrow \ker(\phi) \hookrightarrow \Gamma_2(\mathbb{B}_2(\Sigma)).$$

Proof. By the above proposition, $\ker(\phi)$ is normally generated by commutators, so it must lie inside $\Gamma_2(\mathbb{B}_2(\Sigma))$. On the other hand, the Heisenberg group $\mathcal{H} = \mathcal{H}_g$ is a central extension of an abelian group, hence 2-nilpotent. The kernel of any homomorphism $G \to H$ with target a 2-nilpotent group contains $\Gamma_3(G)$, so $\ker(\phi)$ contains $\Gamma_3(\mathbb{B}_2(\Sigma))$. To see that $\ker(\phi)$ is not equal to Γ_2, it suffices to note that the Heisenberg group is not abelian. To see that $\ker(\phi)$ is not equal to Γ_3, we will construct a quotient

$$\psi\colon \mathbb{B}_2(\Sigma) \longrightarrow Q$$

where Q is 2-nilpotent and $[\sigma_1, a_1] \notin \ker(\psi)$. Given this for the moment, suppose for a contradiction that $\ker(\phi) = \Gamma_3$. Then we have $[\sigma_1, a_1] \in \ker(\phi) = \Gamma_3 \subseteq \ker(\psi)$, due to the fact that Q is 2-nilpotent, which is a contradiction.

It therefore remains to show that there exists a quotient Q with the claimed properties. In fact we will take $Q = D_4$, the dihedral group with 8 elements presented by $D_4 = \langle g, \tau \mid g^2 = \tau^2 = (g\tau)^4 = 1\rangle$. Let us set $\psi(a_i) = \psi(b_i) = g$ and $\psi(\sigma_1) = \tau$. It is easy to verify from the presentations that this is a well-defined surjective homomorphism. The dihedral group D_4 is 2-nilpotent (its centre is generated by $(g\tau)^2$ and the quotient by this element is isomorphic to the abelian group $(\mathbb{Z}/2)^2$), and we compute that $\psi([\sigma_1, a_1]) = (\tau g)^2 \neq 1$, which completes the proof. ∎

2 Heisenberg homology

Using the homomorphism ϕ, any representation V of the Heisenberg group $\mathcal{H}$ becomes a module over $R = \mathbb{Z}[\mathbb{B}_n(\Sigma)]$. Following for example [24, Ch. 3.H] or [17, Ch. 5] we then have homology groups with local coefficients $H_*(\mathcal{C}_n(\Sigma); V)$. When V is the regular representation $\mathbb{Z}[\mathcal{H}]$, we simply write $H_*(\mathcal{C}_n(\Sigma); \mathcal{H})$. Let $\widetilde{\mathcal{C}}_n(\Sigma)$ be the regular covering of $\mathcal{C}_n(\Sigma)$ associated with the kernel of ϕ. Then $H_*(\mathcal{C}_n(\Sigma); \mathcal{H})$ is the homology of the singular chain complex $\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))$ considered as a right $\mathbb{Z}[\mathcal{H}]$-module by deck transformations. Given a left representation V of $\mathcal{H}$, then $H_*(\mathcal{C}_n(\Sigma); V)$ is the homology of the complex $\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \otimes_{\mathbb{Z}[\mathcal{H}]} V$.

Relative homology with local coefficients is defined in the usual way. We also use the *Borel-Moore* homology, defined by

$$H_n^{BM}(\mathcal{C}_n(\Sigma); V) = \varprojlim_T H_n(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma) \setminus T; V),$$

where the inverse limit is taken over all compact subsets $T \subset \mathcal{C}_n(\Sigma)$. In general, writing $\mathcal{K}(X)$ for the poset of compact subsets of a space X, the Borel-Moore homology module $H_n^{BM}(X, A; V)$ is the limit of the functor $H_n(X, A \cup (X \setminus -); V)\colon \mathcal{K}(X)^{\mathrm{op}} \to \mathrm{Mod}_R$ for any local system V on X and any subspace $A \subseteq X$. Under mild conditions, which are satisfied in our setting, the Borel-Moore homology is isomorphic to the homology of the chain complex of locally finite singular chains.

Borel-Moore homology is functorial with respect to proper maps. If $f\colon X \to Y$ is a proper map taking $A \subseteq X$ into $B \subseteq Y$, then there is an induced functor $f^{-1}\colon \mathcal{K}(Y) \to \mathcal{K}(X)$ by taking pre-images, and a natural transformation $H_n(X, A \cup (X \setminus -); f^*(V)) \circ f^{-1} \Rightarrow H_n(Y, B \cup (Y \setminus -); V)$ arising from the naturality of singular homology. Taking limits, we obtain

$$\begin{aligned}H_n^{BM}(X, A; f^*(V)) &= \lim H_n(X, A \cup (X \setminus -); f^*(V)) \\ &\longrightarrow \lim \left(H_n(X, A \cup (X \setminus -); f^*(V)) \circ f^{-1}\right) \\ &\longrightarrow \lim H_n(Y, B \cup (Y \setminus -); V) = H_n^{BM}(Y, B; V).\end{aligned}$$

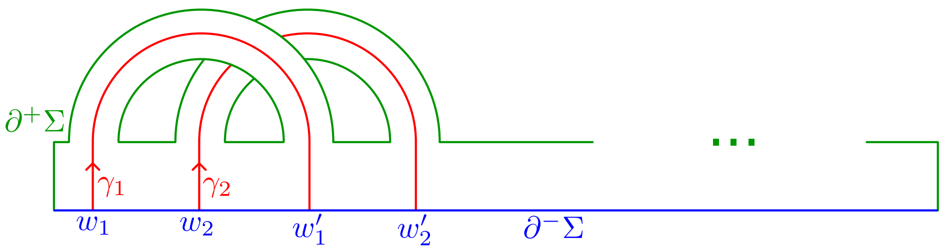
In particular, homeomorphisms are proper maps, so self-homeomorphisms of a space act on its Borel-Moore homology.

We will adapt a method used by Bigelow in the genus-0 case [11] (see also [1, 32, 2]) for computing the relative Borel-Moore homology

$$H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V) = \varprojlim_T (\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)) \cup (\mathcal{C}_n(\Sigma) \setminus T); V),$$

where $\mathcal{C}_n(\Sigma, \partial^-(\Sigma))$ is the closed subspace of configurations containing at least one point in an interval $\partial^-(\Sigma) \subset \partial\Sigma$. In general for a pair (X, Y) the notation $\mathcal{C}_n(X, Y)$ will be used for configurations of n points in X containing at least one point in Y.

The surface Σ can be represented as a thickened interval $[0,1] \times I$ with $2g$ handles, attached as depicted below along $\{1\} \times W$, where W contains in this order the points $w_1, w_2, w_1', w_2', \ldots, w_{2g-1}, w_{2g}, w_{2g-1}', w_{2g}'$. We view Σ as a relative cobordism from $\partial^-(\Sigma) = \{0\} \times I$ (in blue below) to $\partial^+(\Sigma)$ (in green below), where $\partial^+(\Sigma)$ is the closure of the complement of $\partial^-(\Sigma)$ in $\partial(\Sigma)$. For $1 \leqslant i \leqslant 2g$, γ_i denotes the union of the core of the i-th handle with $[0,1] \times \{w_i, w_i'\}$, oriented from w_i to w_i', and $\Gamma = \amalg_i \gamma_i$ (in red below).



Let $\mathcal{K}$ be the set of sequences $k = (k_1, k_2, \ldots, k_{2g})$ such that k_i is a non-negative integer and $\sum_i k_i = n$. We will associate to each $k \in \mathcal{K}$ an element of the Borel-Moore relative homology $H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})$, as follows.

For $k \in \mathcal{K}$ we consider the submanifold $E_k \subset \mathcal{C}_n(\Sigma)$ consisting of all configurations having k_i points on γ_i. This manifold inherits an *orientation* from the orientations of the arcs γ_i together with the ordering (up to even permutations) of the points on Γ by declaring that $x < y$ for $x \in \gamma_i$, $y \in \gamma_j$ if either $i < j$ or $i = j$ and x comes before y according to the orientation of γ_i. Moreover, it is a properly embedded Euclidean half-space $\mathbb{R}^n_+$ in $\mathcal{C}_n(\Sigma)$ with boundary in $\mathcal{C}_n(\Sigma, \partial^-(\Sigma))$. After choosing a path connecting it to the base point in $\mathcal{C}_n(\Sigma)$, E_k represents a homology class in $H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})$ which we also denote by E_k.

Theorem 11 (Theorem A(a))**.** *Let V be any representation of the discrete Heisenberg group $\mathcal{H}$. Then, for $n \geqslant 2$, there is an isomorphism of modules*

$$H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V) \;\cong\; \bigoplus_{k \in \mathcal{K}} V.$$

Furthermore, this is the only non-vanishing module in $H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)$. In particular, when $V = \mathbb{Z}[\mathcal{H}]$, the graded $\mathbb{Z}[\mathcal{H}]$-module $H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})$ is concentrated in degree n and free of dimension $\binom{2g+n-1}{n}$ with basis $\{E_k\}_{k \in \mathcal{K}}$.

Remark 12. Theorem 11 is true (with the same proof) more generally for Borel-Moore homology with coefficients in any representation V of the surface braid group $\mathbb{B}_n(\Sigma) = \pi_1(\mathcal{C}_n(\Sigma))$, not necessarily factoring through the quotient $\mathbb{B}_n(\Sigma) \twoheadrightarrow \mathcal{H}$. However, we will only need Theorem 11 for representations of the Heisenberg group.

Recall that a deformation retraction $h\colon [0,1] \times \Sigma \to \Sigma$ from Σ to $Y \subset \Sigma$ is a continuous map $(t, x) \mapsto h(t, x) = h_t(x)$ such that $h_0 = \mathrm{Id}_\Sigma$, $h_1(\Sigma) \subset Y$, and $(h_t)_{|_Y} = \mathrm{Id}_Y$. We will prove the following lemma in Appendix A.

Lemma 13. *There exists a metric d on Σ inducing the standard topology and a deformation retraction h from Σ to $\Gamma \cup \partial^-(\Sigma)$, such that for all $0 \leqslant t < 1$, the map h_t is a 1-Lipschitz embedding.*

Proof of Theorem 11. We use a metric d and a deformation retraction h from Lemma 13. For $\epsilon > 0$ and $Y \subset \Sigma$ we denote by $\mathcal{C}_n^\epsilon(Y)$ the subspace of configurations $x = \{x_1, x_2, ..., x_n\} \subset Y$ such that $d(x_i, x_j) < \epsilon$ for some $i \neq j$. If Y is closed, then $\mathcal{C}_n^\epsilon(Y)$ is a cofinal family of co-compact subsets of $\mathcal{C}_n(Y)$, which implies that for a pair (Y, Z) of closed subspaces of Σ, we have

$$H_*^{BM}(\mathcal{C}_n(Y), \mathcal{C}_n(Y, Z); V) \;\cong\; \lim_{0 \leftarrow \epsilon} H_*(\mathcal{C}_n(Y), \mathcal{C}_n(Y, Z) \cup \mathcal{C}_n^\epsilon(Y); V) \tag{12}$$

For $0 \leqslant t \leqslant 1$, let $\Sigma_t = h_t(\Sigma)$. For $t < 1$ we have an inclusion

$$(\mathcal{C}_n(\Sigma_t), \mathcal{C}_n(\Sigma_t, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma_t) \;\subset\; (\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma))$$

which is a homotopy equivalence with homotopy inverse $\mathcal{C}_n(h_t)$, which is a map of pairs because h_t is 1-Lipschitz. So we have an inclusion isomorphism

$$H_*(\mathcal{C}_n(\Sigma_t), \mathcal{C}_n(\Sigma_t, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma_t); V) \;\cong\; H_*(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma); V) \tag{13}$$

The compactness of Σ ensures that h_1 is the uniform limit of h_t as $t \to 1$, which implies that for $\epsilon > 0$ we may choose $t = t_\epsilon < 1$ such that for all $p \in \Sigma$ we have $d(h_t(p), h_1(p)) < \frac{\epsilon}{2}$. For such t, let $A_t \subset \mathcal{C}_n(\Sigma_t)$ be the subset of configurations $x = \{x_1, \ldots, x_n\} \subset \Sigma_t$ such that $h_1(h_t^{-1}(x_i)) = h_1(h_t^{-1}(x_j))$ for some $i \neq j$. We have that A_t is closed and (by our definition of $t = t_\epsilon$) contained in the open set $\mathcal{C}_n^\epsilon(\Sigma_t)$. We therefore get an excision isomorphism

$$H_*(\mathcal{C}_n(\Sigma_t)\backslash A_t, (\mathcal{C}_n(\Sigma_t, \partial^-(\Sigma))\cup\mathcal{C}_n^\epsilon(\Sigma_t))\backslash A_t; V) \;\cong\; H_*(\mathcal{C}_n(\Sigma_t), \mathcal{C}_n(\Sigma_t, \partial^-(\Sigma))\cup\mathcal{C}_n^\epsilon(\Sigma_t); V) \tag{14}$$

By applying $h_1 \circ (h_t)^{-1}$ on configurations, we obtain a well-defined map of pairs

$$(\mathcal{C}_n(\Sigma_t) \setminus A_t, (\mathcal{C}_n(\Sigma_t, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma_t)) \setminus A_t) \longrightarrow (\mathcal{C}_n(\Sigma_1), \mathcal{C}_n(\Sigma_1, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma_1)) \;,$$

which is a homotopy inverse to the inclusion. Here $\Sigma_1 = h_1(\Sigma)$ is equal to $\Gamma \cup \partial^-(\Sigma)$. By composing inclusions and excision maps, we obtain the inclusion isomorphism:

$$H_*(\mathcal{C}_n(\Sigma_1), \mathcal{C}_n(\Sigma_1, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma_1); V) \;\cong\; H_*(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma); V). \tag{15}$$

Let $W^- = \{0\} \times W \subset \partial^-(\Sigma)$ and $U_\epsilon \subset \partial^-(\Sigma)$ be defined by the condition $x \in U_\epsilon \Leftrightarrow d(x, W^-) < \frac{\epsilon}{2}$, and $\Gamma_\epsilon = \Gamma \cup U_\epsilon$. In the left-hand side group above, we may apply excision with the closed subset $\mathcal{C}_n(\Sigma_1, \partial^-(\Sigma) \setminus U_\epsilon)$, which gives

$$H_*(\mathcal{C}_n(\Gamma_\epsilon), \mathcal{C}_n(\Gamma_\epsilon, U_\epsilon) \cup \mathcal{C}_n^\epsilon(\Gamma_\epsilon); V) \;\cong\; H_*(\mathcal{C}_n(\Sigma_1), \mathcal{C}_n(\Sigma_1, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma_1); V). \tag{16}$$

We finish with one more excision removing configurations which contain 2 points in the same component of U_ϵ followed by a deformation retraction to configurations in Γ and finally obtain:

$$H_*(\mathcal{C}_n(\Gamma), \mathcal{C}_n(\Gamma, W^-)) \cup \mathcal{C}_n^\epsilon(\Gamma); V) \;\cong\; H_*(\mathcal{C}_n(\Gamma_\epsilon), \mathcal{C}_n(\Gamma_\epsilon, U_\epsilon) \cup \mathcal{C}_n^\epsilon(\Gamma_\epsilon); V). \tag{17}$$

Taking the limit $0 \leftarrow \epsilon$ in the composition of the isomorphisms from equations eqs. (15) to (17), we obtain:

$$H_*^{BM}(\mathcal{C}_n(\Gamma), \mathcal{C}_n(\Gamma, W^-)); V) \;\cong\; H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V). \tag{18}$$

Now we observe that the pair $(\mathcal{C}_n(\Gamma), \mathcal{C}_n(\Gamma, W^-)$ is the disjoint union of the relative cells $(E_k, \partial(E_k))$ for $k \in \mathcal{K}$. It follows that the Borel-Moore homology (18) is trivial when $* \neq n$ and that each Borel-Moore homology class E_k generates a direct summand isomorphic to the coefficients V in degree $* = n$. In particular, when $V = \mathbb{Z}[\mathcal{H}]$, these classes form a basis over $\mathbb{Z}[\mathcal{H}]$ for the degree-n Borel-Moore homology. ∎

3 Action of mapping classes

The *mapping class group* of Σ, denoted by $\mathfrak{M}(\Sigma)$, is the group of orientation preserving diffeomorphisms of Σ fixing the boundary pointwise, modulo isotopies relative to the boundary. The isotopy class of a diffeomorphism f is denoted by $[f]$. An oriented self-diffeomorphism fixing the boundary pointwise $f\colon \Sigma \to \Sigma$ gives us a homeomorphism $\mathcal{C}_n(f)\colon \mathcal{C}_n(\Sigma) \to \mathcal{C}_n(\Sigma)$, defined by $\{x_1, x_2, \ldots, x_n\} \mapsto \{f(x_1), f(x_2), \ldots, f(x_n)\}$. If we ensure that the basepoint configuration of $\mathcal{C}_n(\Sigma)$ is contained in $\partial\Sigma$, then it is fixed by $\mathcal{C}_n(f)$ and this in turn induces a homomorphism $f_{\mathbb{B}_n(\Sigma)} = \pi_1(\mathcal{C}_n(f))\colon \mathbb{B}_n(\Sigma) \to \mathbb{B}_n(\Sigma)$, which depends only on the isotopy class $[f]$ of f.

3.1 Action on the Heisenberg group

We first study the induced action on the Heisenberg group quotient.

Proposition 14. *There exists a unique homomorphism $f_{\mathcal{H}}\colon \mathcal{H} \to \mathcal{H}$ such that the following square commutes:*

$$\begin{tikzcd}
\mathbb{B}_n(\Sigma) \arrow[r, "f_{\mathbb{B}_n(\Sigma)}"] \arrow[d, "\phi"'] & \mathbb{B}_n(\Sigma) \arrow[d, "\phi"] \\
\mathcal{H} \arrow[r, "f_{\mathcal{H}}"] & \mathcal{H}
\end{tikzcd} \tag{19}$$

Thus, there is an action of $\mathfrak{M}(\Sigma)$ on the Heisenberg group $\mathcal{H}$ given by

$$\Psi\colon f \mapsto f_{\mathcal{H}}\colon \mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}(\mathcal{H}). \tag{20}$$

Proof. Since ϕ is surjective, the homomorphism $f_{\mathcal{H}}$ will be uniquely determined by the formula $f_{\mathcal{H}}(\phi(\gamma)) = \phi(f_{\mathbb{B}_n(\Sigma)}(\gamma))$ if it exists. To show that it exists, we need to show that the composition $\phi \circ f_{\mathbb{B}_n(\Sigma)}$ factorises through ϕ, which is equivalent to saying that $f_{\mathbb{B}_n(\Sigma)}$ sends $\ker(\phi)$ into itself.

The braid σ_1 is supported in a sub-disc $D \subset \Sigma$ containing the base configuration. Let $T \subset \Sigma$ be a tubular neighbourhood of $\partial\Sigma$ containing D. Since f fixes $\partial\Sigma$ pointwise, we may isotope f so that it is the identity on T, in particular on D, which implies that $f_{\mathbb{B}_n(\Sigma)}$ fixes σ_1. We then deduce from part (a) of Proposition 9 that $f_{\mathbb{B}_n(\Sigma)}$ sends $\ker(\phi)$ to itself, which completes the proof. ■

3.2 Structure of automorphisms of the Heisenberg group.

Recall that the centre of the Heisenberg group $\mathcal{H}$ is infinite cyclic, generated by the element u. Any automorphism of $\mathcal{H}$ must therefore send u to $u^{\pm 1}$.

Definition 15. We denote the index-2 subgroup of those automorphisms of $\mathcal{H}$ that fix u by $\mathrm{Aut}^+(\mathcal{H})$, and call these *orientation-preserving.*

From the proof of Proposition 14, we observe that, for any $f \in \mathfrak{M}(\Sigma)$, the automorphism $f_{\mathcal{H}}$ is orientation-preserving in the sense of Definition 15. We may therefore refine the action Ψ as follows:

$$\Psi\colon f \mapsto f_{\mathcal{H}}\colon \mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}^+(\mathcal{H}). \tag{21}$$

The quotient of $\mathcal{H}$ by its centre may be canonically identified with $H = H_1(\Sigma)$, so every automorphism of $\mathcal{H}$ induces an automorphism of H. Moreover, if it is orientation-preserving, the induced automorphism of H preserves the symplectic structure, so we have a homomorphism $\mathrm{Aut}^+(\mathcal{H}) \to Sp(H)$ denoted $\varphi \mapsto \overline{\varphi}$. In addition, there is a function $\mathrm{Aut}^+(\mathcal{H}) \to H^* = \mathrm{Hom}(H, \mathbb{Z})$ defined by sending φ to $\varphi^\diamond = \mathrm{pr}_1(\varphi(0, -))$, where we are using the description $\mathcal{H} = \mathbb{Z} \times H$. The fact that $\varphi^\diamond$ is really a *homomorphism* $H \to \mathbb{Z}$ uses the fact that the automorphism of H induced by φ preserves the symplectic structure.

Lemma 16. *The homomorphism* $\mathrm{Aut}^+(\mathcal{H}) \to Sp(H)$ *and the function* $\mathrm{Aut}^+(\mathcal{H}) \to H^*$ *induce an isomorphism*

$$\mathrm{Aut}^+(\mathcal{H}) \cong Sp(H) \ltimes H^*, \quad \varphi \mapsto (\overline{\varphi}, \varphi^\diamond), \tag{22}$$

where the semi-direct product structure on the right-hand side is induced by the natural action of $Sp(H)$ *on* H^**.*

Proof. This is proven in Appendix B. ■

Remark 17. Fixing a symplectic basis of H, the right-hand side of (22) is a subgroup of $GL_{2g}(\mathbb{Z}) \ltimes \mathbb{Z}^{2g}$, which may be embedded into $GL_{2g+1}(\mathbb{Z})$. In this way, any orientation-preserving action of a group G on $\mathcal{H}$ may be viewed as a linear representation of G over $\mathbb{Z}$ of rank $2g + 1$.

Lemma 16 asserts that the general form of an oriented automorphism φ is

$$\varphi(k, x) = (k + \varphi^\diamond(x), \overline{\varphi}(x)) \ ,$$

where $\varphi^\diamond \in H^*$ and $\overline{\varphi} \in Sp(H)$ is the induced symplectic automorphism. From the proof of Proposition 14 we observe that, for any $f \in \mathfrak{M}(\Sigma)$, the automorphism $f_{\mathcal{H}}$ is orientation-preserving in the sense of Definition 15. Hence for a mapping class $f \in \mathfrak{M}(\Sigma)$, the map $f_{\mathcal{H}}$ is represented as follows:

$$f_{\mathcal{H}} \colon (k, x) \mapsto (k + \delta_f(x), f_*(x)), \tag{23}$$

where $\delta_f = f_{\mathcal{H}}^\diamond \in H^* = H^1(\Sigma)$.

3.3 Recovering Morita's crossed homomorphism.

We recall briefly the notion of a *crossed homomorphism*. Let G be a group acting on an abelian group K.

Definition 18. A *crossed homomorphism* $\theta \colon G \to K$ is a function with the property that $\theta(g_2 g_1) = \theta(g_1) + g_1\theta(g_2)$ for all $g_1, g_2 \in G$. A *principal crossed homomorphism* is one of the form $g \mapsto gh - h$ for a fixed element $h \in K$. Notice that every principal crossed homomorphism restricts to zero on the kernel of the action of G on K.

Remark 19. Crossed homomorphisms $G \to K$ are in one-to-one correspondence with lifts

$$\begin{tikzcd} G \arrow[r, dashed] \arrow[dr] & \mathrm{Aut}(K) \ltimes K \arrow[d, two heads] \\ & \mathrm{Aut}(K), \end{tikzcd}$$

where the diagonal arrow is the given action of G on K. Often, we will have $K = H^*$ for a free abelian group H, with the G-action on K induced from one on H. In this case there is a natural (anti-)isomorphism $\mathrm{Aut}(H) \cong \mathrm{Aut}(K)$, and under this identification crossed homomorphisms $G \to H^*$ are in one-to-one correspondence with homomorphisms $G \to \mathrm{Aut}(H) \ltimes H^*$ lifting the action $G \to \mathrm{Aut}(H)$ of G on H.

Notation 20. The crossed homomorphism $G \to H^*$ corresponding to a homomorphism $\Theta\colon G \to \mathrm{Aut}(H) \ltimes H^*$ will be denoted by $\Theta^\diamond$.

Remark 21. Crossed homomorphisms form an abelian group under pointwise addition, and principal crossed homomorphisms form a subgroup. The quotient may be identified with the first cohomology group $H^1(G;K)$.

We will need the following lemma later on.

Lemma 22. *Let G be a group acting on an abelian group K, and denote by $N \subseteq G$ the kernel of this action. Let $S \subseteq N$ be a subset such that*

$$T = \{gsg^{-1} \mid s \in S, g \in G\} \subseteq N$$

generates N. If two crossed homomorphisms $\theta_1, \theta_2\colon G \to K$ agree on S, then they agree on N.

Note that we do *not* assume that S normally generates N; we assume only that N is generated by S together with all of its conjugates by elements of the larger group G.

Proof. Since T generates N, it will suffice to show that θ_1 and θ_2 agree on T. Let $s \in S$ and $g \in G$. We know by hypothesis that $\theta_1(s) = \theta_2(s)$, and we need to show that $\theta_1(g^{-1}sg) = \theta_2(g^{-1}sg)$. First, observe that, for $i = 1, 2$, we have

$$\theta_i(g) + g.\theta_i(g^{-1}) = \theta_i(g^{-1}g) = \theta_i(1) = 0.$$

Using this, and the fact that $s \in N$, so it acts trivially on K, we deduce that

$$\begin{aligned} \theta_i(g^{-1}sg) &= \theta_i(g) + g.\theta_i(s) + gs.\theta_i(g^{-1}) \\ &= \theta_i(g) + g.\theta_i(s) + g.\theta_i(g^{-1}) \\ &= g.\theta_i(s). \end{aligned}$$

Thus $\theta_1(g^{-1}sg) = g.\theta_1(s) = g.\theta_2(s) = \theta_2(g^{-1}sg)$, as required. ■

In [33], Morita introduced a crossed homomorphism $\mathfrak{M}(\Sigma) \to H^1(\Sigma)$, $f \mapsto \mathfrak{d}_f$ representing a generator for $H^1(\mathfrak{M}(\Sigma), H^1(\Sigma)) \cong \mathbb{Z}$. (cf. Proposition 30). We will recover this crossed homomorphism from the action $f \mapsto f_{\mathcal{H}}$ on the Heisenberg group.

Proposition 23. *The map $\delta\colon \mathfrak{M}(\Sigma) \to H^1(\Sigma)$, $f \mapsto \delta_f$, is a crossed homomorphism equal to Morita's $\mathfrak{d}$.*

Proof. We first show that δ is a crossed homomorphism. Let f, g be mapping classes; then we have, for $(k, x) \in \mathcal{H}$,

$$(g \circ f)_{\mathcal{H}}(k, x) = g_{\mathcal{H}}(k + \delta_f(x), f_*(x)) = (k + \delta_f(x) + \delta_g(f_*(x)), (g \circ f)_*(x)) \ ,$$

and so we do get $\delta_{g\circ f}(x) = \delta_f(x) + f^*(\delta_g)(x)$.

We will use as generators the loops given by the first strand in the generators α_i, β_i of the braid group $\mathbb{B}_n(\Sigma)$, and keep the same notation. For $\gamma \in \pi_1(\Sigma)$, let us denote by γ_i the element in the free group generated by α_i, β_i that is the image of γ under the homomorphism that maps the other generators to 1. Then we have a decomposition

$$\gamma_i = \alpha_i^{\nu_1}\beta_i^{\mu_1} \dots \alpha_i^{\nu_m}\beta_i^{\mu_m} \ ,$$

where ν_j and μ_j are 0, -1 or 1. The integer $d_i(\gamma)$ is then defined[1] by

$$\begin{aligned} d_i(\gamma) &= \sum_{j=1}^{m} \nu_j \sum_{k=j}^{m} \mu_k - \sum_{j=1}^{m} \mu_j \sum_{k=j+1}^{m} \nu_k \\ &= \sum_{j=1}^{m}\sum_{k=1}^{m} \iota_{jk}\nu_j\mu_k, \end{aligned}$$

where $\iota_{jk} = +1$ when $j \leqslant k$ and $\iota_{jk} = -1$ when $j > k$. The definition for the Morita crossed homomorphism is as follows:

$$\mathfrak{d}_f([\gamma]) = \sum_{i=1}^{g} d_i(f_\sharp(\gamma)) - d_i(\gamma) \ . \tag{24}$$

If $\gamma \in \pi_1(\Sigma)$ is the first strand of a pure braid also denoted γ, then the above decomposition of γ used for the definition of d_i is also a decomposition in the generators of the braid group, and from the definition of the product in $\mathcal{H}$ we have that

$$\phi(\gamma) = \left(\sum_{i=1}^{g} d_i(\gamma), [\gamma]\right) \in \mathcal{H} \ .$$

This can be checked by recursion on the length of γ. It can also be deduced from [33, Lemma 6.1]. The equality $\mathfrak{d}_f = \delta_f$ follows. ■

Consider the inclusion of surfaces $\Sigma_{g,1} \hookrightarrow \Sigma_{h,1}$ given by boundary connected sum with $\Sigma_{h-g,1}$. This induces an inclusion of mapping class groups

$$\mathfrak{M}(\Sigma_{g,1}) \longrightarrow \mathfrak{M}(\Sigma_{h,1}) \tag{25}$$

by extending diffeomorphisms by the identity on $\Sigma_{h-g,1}$. Recall from the introduction that we define the *Morita subgroup* $\mathrm{Mor}(\Sigma_{g,1})$ to be the kernel of $\mathfrak{d}\colon \mathfrak{M}(\Sigma_{g,1}) \to H^1(\Sigma_{g,1})$. The following lemma and corollary will be used in §5 and §6.

[1] There is a small misprint in [33].

Lemma 24. *The diagram*

$$\begin{tikzcd} \mathfrak{M}(\Sigma_{g,1}) \arrow[r, hook] \arrow[d, "\mathfrak{d}"'] & \mathfrak{M}(\Sigma_{h,1}) \arrow[d, "\mathfrak{d}"] \\ H^1(\Sigma_{g,1}) \arrow[r] & H^1(\Sigma_{h,1}) \end{tikzcd} \tag{26}$$

commutes, where the bottom arrow is the map induced by the inclusion $\Sigma_{g,1} \hookrightarrow \Sigma_{h,1}$ *on* $H_1(-)$*, conjugated by Poincaré duality.*

Corollary 25. *The homomorphism* (25) *restricts to* $\mathrm{Mor}(\Sigma_{g,1}) \hookrightarrow \mathrm{Mor}(\Sigma_{h,1})$*.*

Proof of Lemma 24. Just as in the definition of the Morita crossed homomorphism above, we identify $H^1(\Sigma_{g,1})$ with $\mathrm{Hom}(\pi_1(\Sigma_{g,1}), \mathbb{Z})$. Under this identification, the bottom arrow in (26) is pre-composition with $\mathrm{pr}\colon \pi_1(\Sigma_{h,1}) = \pi_1(\Sigma_{g,1}) * \pi_1(\Sigma_{h-g,1}) \twoheadrightarrow \pi_1(\Sigma_{g,1})$.

Let $f \in \mathfrak{M}(\Sigma_{g,1})$ and write $\hat{f} \in \mathfrak{M}(\Sigma_{h,1})$ for its image under (25). Let $\gamma \in \pi_1(\Sigma_{h,1})$ and write $\gamma = \gamma_1 * \gamma_2$ under the decomposition $\pi_1(\Sigma_{h,1}) = \pi_1(\Sigma_{g,1}) * \pi_1(\Sigma_{h-g,1})$. By construction, we have

$$d_i(\gamma) = d_i(\gamma_1) \qquad \text{and} \qquad d_i(\hat{f}_\sharp(\gamma)) = d_i(f_\sharp(\gamma_1))$$

for $1 \leqslant i \leqslant g$. Moreover, since $\hat{f}$ acts by the identity on $\Sigma_{h-g,1}$, we also have

$$d_i(\hat{f}_\sharp(\gamma)) = d_i(\gamma)$$

for $g+1 \leqslant i \leqslant h$. From the defining formula (24) we deduce that

$$\mathfrak{d}_{\hat{f}}([\gamma]) = \sum_{i=1}^{h} d_i(\hat{f}_\sharp(\gamma)) - d_i(\gamma) = \sum_{i=1}^{g} d_i(f_\sharp(\gamma_1)) - d_i(\gamma_1) = \mathfrak{d}_f([\gamma_1]) = (\mathfrak{d}_f \circ \mathrm{pr})([\gamma]),$$

and so (26) commutes. ■

3.4 Action of the Torelli subgroup.

Recall that the *Torelli subgroup* $\mathfrak{T}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$ consists of those elements of the mapping class group whose natural action on $H_1(\Sigma)$ is trivial. The restriction of the crossed homomorphism $\delta\colon f \mapsto \delta_f$ to the Torelli group is a homomorphism. We will first describe this homomorphism in relation with the action of the Torelli group on homotopy classes of vector fields. Recall that the set $\Xi(\Sigma)$ of homotopy classes of non-vanishing vector fields supports a natural simply transitive action of $H^1(\Sigma)$ (affine structure), and the action of $\mathfrak{M}(\Sigma)$ is compatible with this action. It follows that the Torelli group acts by translation on $\Xi(\Sigma)$, which defines a homomorphism $e\colon \mathfrak{T}(\Sigma) \to H^1(\Sigma)$. A formula for $e(f)([\gamma])$, where γ is a regular curve, is given by the variation of the winding number. For convenience we recall some details about the winding number below.

Fix a Riemannian metric. A non-vanishing vector field X gives a trivialisation of the unit tangent bundle $T_1(\Sigma) \cong \Sigma \times S^1$. The winding number $\omega_X(\gamma)$ of a regular oriented

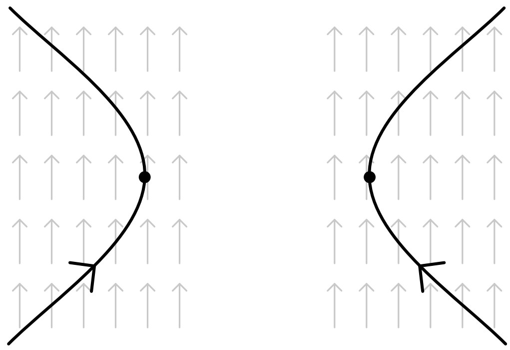


Figure 2: The sign of a point on γ that is tangent to X.

curve γ is the degree of the second component of the unit tangent vector. It can be computed as follows. Assuming that γ is transverse to X except at a finite set $\gamma \pitchfork X$ of points, where it looks locally as in Figure 2, then

$$\omega_X(\gamma) = \sum_{p \in \gamma \pitchfork X} \mathrm{sgn}(p),$$

where $\mathrm{sgn}(p)$ is defined in Figure 2.

The Chillingwoth homomorphism e, studied in [15, 26], is defined by

$$e_X(f)([\gamma]) = \omega_X(f \circ \gamma) - \omega_X(\gamma) \ .$$

Its kernel is the *Chillingworth subgroup*. Note that e does not depend on X, but extends to a crossed homomorphism $e_X : \mathfrak{M}(\Sigma) \to H^1(\Sigma)$ which does.

The following lemma is Proposition 3.7 of [14]. The proof there uses [35]. We give an independent proof below.

Lemma 26. *The homomorphisms δ and e coincide on the Torelli group and have image $\delta(\mathfrak{T}(\Sigma)) = 2H^1(\Sigma)$.*

From the formula (23) we get that the kernel of the action Ψ is included in the Torelli group so that we get this kernel as a corollary of Lemma 26.

Proposition 27. *For any genus $g \geqslant 1$, we have* $\ker(\Psi) = \mathrm{Chill}(\Sigma)$.

Proof. As $\ker(\Psi) \subseteq \mathfrak{T}(\Sigma)$, we may restrict to the Torelli group, at which point we see from formula (23) and Lemma 26 that $\ker(\Psi) = \ker(\delta) = \ker(e) = \mathrm{Chill}(\Sigma)$. ∎

Denote by $\mathrm{Inn}(\mathcal{H})$ the group of inner automorphisms of the Heisenberg group $\mathcal{H}$. From Lemma 26, we also deduce the following.

Proposition 28. *For any genus $g \geqslant 1$, we have* $\Psi^{-1}(\mathrm{Inn}(\mathcal{H})) = \mathfrak{T}(\Sigma)$.

Proof. Conjugation in the Heisenberg group $\mathcal{H}$ is given by the formula

$$(l,x)(k,y)(-l,-x) = (l,y)(k,x)(-l,-x) = (k+2\omega(x,y),y) \tag{27}$$

First, if $f_{\mathcal{H}}$ acts by inner automorphisms, then its induced action on H must be trivial. This means that f lies in the Torelli group. Conversely, if $f \in \mathfrak{T}(\Sigma)$, we have from Lemma 26 that δ_f is in $2H^1(\Sigma)$. Using Poincaré duality, we obtain $x \in H$ such that $\delta(y) = 2\omega(x,y)$ for every y. With the formula (27) we get that $f_{\mathcal{H}}$ is inner. ∎

Proof of Lemma 26. The Torelli group is generated by genus one bounding pairs [25, Theorem 2], and this generating set is a single conjugacy class in the full mapping class group. By Lemma 22 and the fact that both δ and e are crossed homomorphisms defined on the full mapping class group, it will suffice to show that they agree on one particular genus one bounding pair, and take values in $2H^1(\Sigma)$ on this element. Specifically, we will take this element to be

$$f = BP(\gamma,\delta) = T_\gamma . T_\delta^{-1},$$

the genus one bounding pair diffeomorphism depicted in Figure 3, and we will show that both elements $e(f)$ and δ_f of $H^1(\Sigma) \cong \mathrm{Hom}(H_1(\Sigma),\mathbb{Z})$ are equal to the homomorphism $H_1(\Sigma) \to \mathbb{Z}$ given by

$$a_1 \mapsto 2 \quad , \quad a_i \mapsto 0 \text{ for } i \geqslant 2 \quad \text{and} \quad b_i \mapsto 0 \text{ for } i \geqslant 1. \tag{28}$$

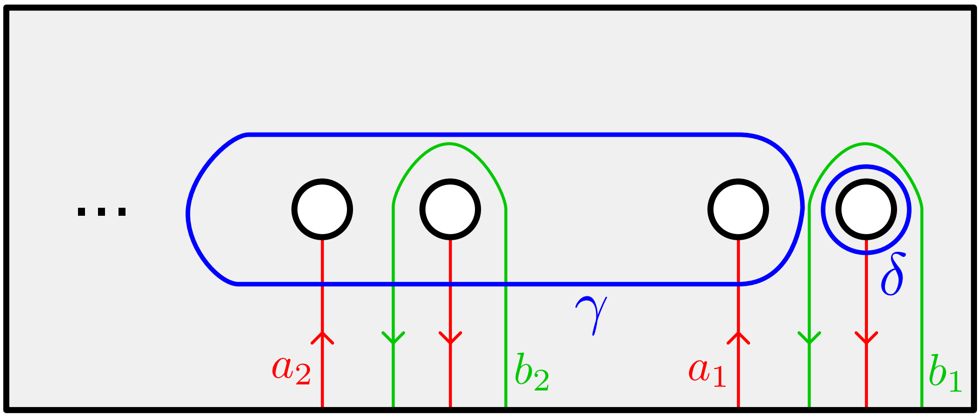


Figure 3: The surface Σ is obtained by identifying the $2g$ interior boundary components (4 depicted above) in g pairs by reflections. The bounding pair map from the proof of Lemma 26 is $BP(\gamma,\delta) = T_\gamma . T_\delta^{-1}$, for the blue curves γ and δ. The red and green arcs form a symplectic basis for the first homology of Σ relative to the bottom edge $\partial^-\Sigma$.

We first calculate δ_f from the automorphism $f_{\mathcal{H}}$. We may directly read off from Figure 3 the effect of f on the elements a_i and b_i of $\mathcal{H}$. It clearly acts trivially except possibly on the three elements $\tilde{a}_2 = (0, a_2)$, $\tilde{b}_2 = (0, b_2)$ and $\tilde{a}_1 = (0, a_1)$, since the others may be realised disjointly from $\gamma \cup \delta$, and:

$$\begin{aligned}
\tilde{a}_1 &\mapsto [\tilde{a}_2, \tilde{b}_2].\tilde{a}_1 = u^2 \tilde{a}_1 = (2, a_1) \\
\tilde{a}_2 &\mapsto [\tilde{a}_2, \tilde{b}_2].\tilde{a}_1 \tilde{b}_1 \tilde{a}_1^{-1}.\tilde{a}_2.\tilde{a}_1 \tilde{b}_1^{-1} \tilde{a}_1^{-1}.[\tilde{a}_2, \tilde{b}_2]^{-1} = \tilde{a}_2 \\
\tilde{b}_2 &\mapsto [\tilde{a}_2, \tilde{b}_2].\tilde{a}_1 \tilde{b}_1 \tilde{a}_1^{-1}.\tilde{b}_2.\tilde{a}_1 \tilde{b}_1^{-1} \tilde{a}_1^{-1}.[\tilde{a}_2, \tilde{b}_2]^{-1} = \tilde{b}_2.
\end{aligned}$$

This gives (28) for δ_f.

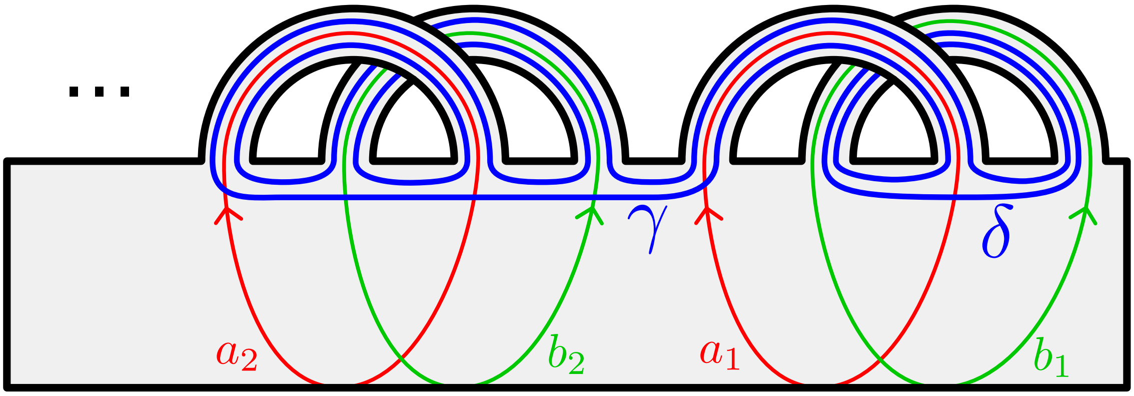


Figure 4: An alternative model for the surface Σ, the bounding pair (γ, δ) and the symplectic basis for the first homology of Σ relative to the bottom edge $\partial^- \Sigma$.

To calculate $e(f)$, we use the alternative model for the surface Σ, the bounding pair (γ, δ) and the symplectic basis a_i, b_i for H depicted in Figure 4. This model for Σ has the advantage of having an obvious unit vector field X, which simply points *upwards* according to the standard framing of the page.

Using this vector field X and comparing to Figure 2, we observe that the winding numbers of the symplectic generators a_i and b_i (more precisely, their smooth, closed representatives pictured in Figure 4) are given by

$$\omega_X(a_i) = -1 \qquad \text{and} \qquad \omega_X(b_i) = +1.$$

We recall that, by definition, $e_X(f)(c) = \omega_X(f \circ \bar{c}) - \omega_X(\bar{c}) \in \mathbb{Z}$ for any $c = [\bar{c}] \in H$. We clearly have $f \circ \bar{c} = \bar{c}$ for $\bar{c} = a_i$ or b_i with $i \geqslant 3$ or for $\bar{c} = b_1$, since these curves may be represented disjointly from $\gamma \cup \delta$. Hence $e_x(f)([\bar{c}]) = 0$ for these $\bar{c}$. The curve $f \circ a_1$ is depicted in Figure 5.

There are precisely three points on this curve where its tangent vector is equal to the vector field X, i.e., where its tangent vector is pointing vertically upwards: two are positive and one is negative (compare the local models in Figure 2), hence

$$\begin{aligned}
e_X(f)(a_1) &= \omega_X(f \circ a_1) - \omega_X(a_1) \\
&= (2 - 1) - (-1) \\
&= 2.
\end{aligned}$$

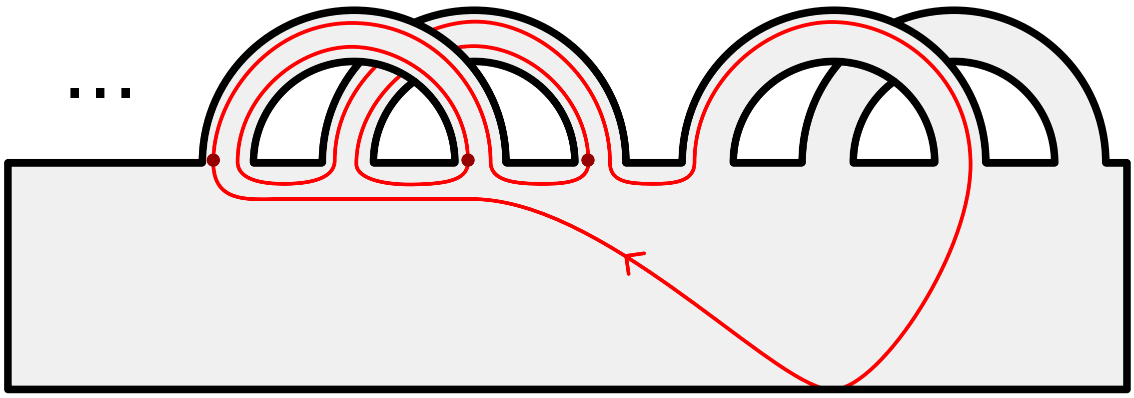


Figure 5: The curve $f \circ a_1$ for $f = T_\gamma . T_\delta^{-1}$. The three points where its tangent vector points vertically upwards are marked with dark red points: the left-most one is negative according to Figure 2, and the other two are positive.
[At first sight it may look like there are two more, but these are not allowed since they do not fit either of the local models of Figure 2. We therefore perturb the curve slightly to get rid of these two tangencies with the vector field X. Alternatively, we may perturb it differently, to turn each of these disallowed tangencies into a pair of two allowed tangencies with opposite signs, which will therefore cancel in the expression for $\omega_X(f \circ a_1)$.]

Now let $\bar{c}$ be either a_2 or b_2. In this case the effect of f is simply to conjugate $\bar{c}$ by γ, so we have that

$$\begin{aligned} \omega_X(f \circ \bar{c}) &= \omega_X(\gamma) + \omega_X(\bar{c}) - \omega_X(\gamma) \\ &= \omega_X(\bar{c}), \end{aligned}$$

since positive/negative tangencies with X for γ are negative/positive tangencies with X for γ^{-1} respectively, and so $e_X(f)([\bar{c}]) = \omega_X(f \circ \bar{c}) - \omega_X(\bar{c}) = 0$. Thus we have shown that the homomorphism $e(f) \colon H \to \mathbb{Z}$ is given by (28). ■

3.5 The Trapp representation.

We next recall the *Trapp representation* [43], and show that our representation of $\mathfrak{M}(\Sigma)$ on $\mathcal{H}$ may be identified with it (up to "coboundaries") when the genus of Σ is at least 2. This recovers Proposition 27, since the kernel of the Trapp representation is precisely the Chillingworth subgroup $\mathrm{Chill}(\Sigma)$ under this condition [43, Corollary 2.7].

Definition 29. The representation of Trapp [43] is defined as a homomorphism

$$\Phi_X \colon \mathfrak{M}(\Sigma) \longrightarrow Sp(H) \ltimes H^* \subset GL_{2g+1}(\mathbb{Z}) \tag{29}$$

(*cf.* Remark 17), lifting the symplectic action $\mathfrak{M}(\Sigma) \to Sp(H)$. Viewed as a homomorphism into $\mathrm{Aut}(H) \ltimes H^*$, it therefore corresponds by Remark 19 to a crossed homomorphism $\mathfrak{M}(\Sigma) \to H^*$. This crossed homomorphism is the variation of the winding number with respect to a fixed non singular vector field X on Σ.

We now wish to compare the two homomorphisms

$$\Phi_X = (29) \quad \text{and} \quad \Psi = (22) \circ (21) \quad : \quad \mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}(H) \ltimes H^*$$

corresponding, respectively, to the crossed homomorphisms

$$\Phi_X^\diamond = e_X \quad \text{and} \quad \Psi^\diamond = \delta \quad : \quad \mathfrak{M}(\Sigma) \longrightarrow H^*.$$

Proposition 30. *For $g \geqslant 2$, the crossed homomorphisms e_X and δ represent the same cohomology class in $H^1(\mathfrak{M}(\Sigma); H^*) \cong \mathbb{Z}$. In other words, they are equal modulo principal crossed homomorphisms.*

Proof. We will use the homomorphism

$$H^1(\mathfrak{M}(\Sigma); H^*) \longrightarrow \mathrm{Hom}(\mathfrak{T}(\Sigma), H^*) \tag{30}$$

given by restricting a crossed homomorphism $\mathfrak{M}(\Sigma) \to H^*$ to the Torelli group. This is well-defined since, as mentioned before, principal crossed homomorphisms are trivial on the Torelli group. The right-hand side of (30) is rather large: in fact, by a theorem of Johnson [28], the abelianisation of $\mathfrak{T}(\Sigma)$ is isomorphic to $\wedge^3 H \oplus (\text{torsion})$, so $\mathrm{Hom}(\mathfrak{T}(\Sigma), H^*) \cong \mathrm{Hom}(\wedge^3 H, H^*)$, which is free abelian of rank $2g\binom{2g}{3}$. However, it has the advantage that it is easier to detect when elements are equal, since it is just a group of homomorphisms (rather than crossed homomorphisms modulo principal ones). On the other hand, the left-hand side of (30) is much smaller. Indeed, Morita proved in [33, Proposition 6.4] that the group $H^1(\mathfrak{M}(\Sigma); H^*)$ is infinite cyclic. (In fact, it is generated by $[\mathfrak{d}]$, which we know by Proposition 23 is equal to $[\delta]$, but we will not need this.) In Lemma 26 we have proven that δ and e_X coincide (and are non-trivial) on the Torelli subgroup. Since $\mathrm{Hom}(\mathfrak{T}(\Sigma), H^*)$ is torsion-free, the homomorphism (30) is injective and the result follows. ■

Remark 31. In summary, we have considered three crossed homomorphisms

$$\delta, \mathfrak{d}, e_X \colon \mathfrak{M}(\Sigma) \longrightarrow H^1(\Sigma),$$

where δ is the crossed homomorphism corresponding to the natural action Ψ of the mapping class group on the Heisenberg group, $\mathfrak{d}$ is Morita's combinatorially-defined crossed homomorphism and e_X is the Chillingworth crossed homomorphism (depending on a choice of non-vanishing vector field X on Σ). We have shown (Proposition 23) that $\delta = \mathfrak{d}$ on $\mathfrak{M}(\Sigma)$ and (Lemma 26) that $\delta = e_X$ when restricted to $\mathfrak{T}(\Sigma)$. In Proposition 30, we used the latter fact to deduce the stronger statement that $\delta = e_X$ on $\mathfrak{M}(\Sigma)$ modulo principal crossed homomorphisms. However, we note that only the weaker statement of Lemma 26 was needed to deduce (Proposition 27) that $\ker(\Psi) = \mathrm{Chill}(\Sigma)$.

4 Twisted actions and central extensions

4.1 Untwisting.

In order to apply the construction of the previous section to define a representation of (a central extension of) the Torelli group of Σ (rather than just the smaller Chillingworth subgroup), we will need a certain "untwisting" trick. This may either be done explicitly at the level of chain complexes, or it may be done already at the level of spaces equipped with local systems, *before* passing to chains. In this section, we explain the latter point of view. We first describe a general algebraic trick for "untwisting" representations of groups, and then augment it to a fibrewise version, which may be applied to local systems on spaces.

Let G, K be groups and let M be a right $\mathbb{Z}[K]$-module. Suppose that G has a *twisted* left action on M, in the sense that there are actions

$$\alpha\colon G \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(M)$$
$$\beta\colon G \longrightarrow \mathrm{Aut}(K)$$

such that $\alpha(g)(m.h) = \alpha(g)(m).\beta(g)(h)$ for all $g \in G$, $h \in K$ and $m \in M$. Moreover, suppose that the action β of G on K is by *inner* automorphisms, and define a central extension of G by the following pullback (the symbol $\lrcorner$ denotes a pullback square):

$$\begin{tikzcd}
1 \arrow[d] & 1 \arrow[d] & \\
\mathcal{Z}(K) \arrow[r, "\mathrm{id}"] \arrow[d] & \mathcal{Z}(K) \arrow[d] & \\
\widetilde{G} \arrow[r, "\theta"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d] & \\
G \arrow[r, "\beta"] \arrow[d] & \mathrm{Inn}(K) \arrow[d] & \subseteq \mathrm{Aut}(K) \\
1 & 1 &
\end{tikzcd} \tag{31}$$

Lemma 32. *There is a well-defined* ***untwisted*** *left action of $\widetilde{G}$ on M by $\mathbb{Z}[K]$-module automorphisms*

$$\gamma\colon \widetilde{G} \longrightarrow \mathrm{Aut}_{\mathbb{Z}[K]}(M)$$

given by the formula $\gamma(\widetilde{g})(m) = \alpha(\pi(\widetilde{g}))(m).\theta(\widetilde{g})$.

Proof. It is clear that, for fixed $\widetilde{g} \in \widetilde{G}$, the formula $\alpha(\pi(\widetilde{g}))(-).\theta(\widetilde{g})$ defines a $\mathbb{Z}$-module automorphism of M. We therefore just have to check two things:

- The automorphism $\alpha(\pi(\widetilde{g}))(-).\theta(\widetilde{g})$ of M commutes with the right action of K.
- The function $\widetilde{g} \mapsto \alpha(\pi(\widetilde{g}))(-).\theta(\widetilde{g})$ is a group homomorphism.

For the first point, let $\widetilde{g} \in \widetilde{G}$, $m \in M$ and $h \in K$. We have

$$\begin{aligned}
\alpha(\pi(\widetilde{g}))(m.h).\theta(\widetilde{g}) &= \alpha(\pi(\widetilde{g}))(m).\beta(\pi(\widetilde{g}))(h).\theta(\widetilde{g}) \\
&= \alpha(\pi(\widetilde{g}))(m).\theta(\widetilde{g}).h.\theta(\widetilde{g})^{-1}.\theta(\widetilde{g}) \\
&= \alpha(\pi(\widetilde{g}))(m).\theta(\widetilde{g}).h,
\end{aligned}$$

where the first equality holds by our compatibility assumption between the actions α and β, and the second one holds by commutativity of (31). This says precisely that $\alpha(\pi(\widetilde{g}))(-).\theta(\widetilde{g})$ commutes with the right action of K.

For the second point, let $\widetilde{g}_1, \widetilde{g}_2 \in \widetilde{G}$ and $m \in M$. We have

$$\begin{aligned}\alpha(\pi(\widetilde{g}_2))\Big(\alpha(\pi(\widetilde{g}_1))(m).\theta(\widetilde{g}_1)\Big).\theta(\widetilde{g}_2) &= \alpha(\pi(\widetilde{g}_2))\Big(\alpha(\pi(\widetilde{g}_1))(m)\Big).\beta(\pi(\widetilde{g}_2))(\theta(\widetilde{g}_1)).\theta(\widetilde{g}_2)\\ &= \alpha(\pi(\widetilde{g}_2))\Big(\alpha(\pi(\widetilde{g}_1))(m)\Big).\theta(\widetilde{g}_2).\theta(\widetilde{g}_1).\theta(\widetilde{g}_2)^{-1}.\theta(\widetilde{g}_2)\\ &= \alpha(\pi(\widetilde{g}_2\widetilde{g}_1))(m).\theta(\widetilde{g}_2\widetilde{g}_1),\end{aligned}$$

where, again, the first equality holds by our compatibility assumption between the actions α and β, the second one holds by commutativity of (31) and the third one holds since α, π and θ are homomorphisms. This says precisely that $\widetilde{g} \mapsto \alpha(\pi(\widetilde{g}))(-).\theta(\widetilde{g})$ is a group homomorphism. ■

4.2 Untwisting in bundles.

We will need a natural *fibrewise* version of Lemma 32, whose proof is identical (one just has to think of bundles of modules instead of modules). Let X be a space and let $\xi\colon \mathcal{M} \to X$ be a bundle of $\mathbb{Z}[K]$-modules. Suppose that G has a twisted left action on $\mathcal{M}$. Precisely, this means a pair of homomorphisms

$$\begin{aligned}\alpha\colon G &\longrightarrow \mathrm{Homeo}(\mathcal{M})\\ \beta\colon G &\longrightarrow \mathrm{Aut}(K)\end{aligned}$$

such that, for each $g \in G$, $x \in X$, $m \in \xi^{-1}(x)$ and $h \in K$, we have:

- $\alpha(g)$ preserves the fibres of ξ,
- the restriction of $\alpha(g)$ to each fibre is a $\mathbb{Z}$-linear automorphism,
- $\alpha(g)(m.h) = \alpha(g)(m).\beta(g)(h)$.

(By definition, such an action is an *untwisted* left action by automorphisms of bundles of $\mathbb{Z}[K]$-modules exactly when β is the trivial action.)

As before, assume that the action β of G on K is by *inner* automorphisms and define the central extension $\widetilde{G}$ of G as in (31).

Lemma 33. *There is a well-defined* ***untwisted*** *left action of $\widetilde{G}$ on $\xi\colon \mathcal{M} \to X$ by automorphisms of bundles of $\mathbb{Z}[K]$-modules*

$$\gamma\colon \widetilde{G} \longrightarrow \mathrm{Aut}_{\mathbb{Z}[K]}(\xi\colon \mathcal{M} \to X)$$

given by the formula $\gamma(\widetilde{g})(m) = \alpha(\pi(\widetilde{g}))(m).\theta(\widetilde{g})$.

4.3 Rescaling.

Once we have obtained an untwisted representation of $\widetilde{G}$, the following "rescaling" lemma gives (under conditions) a trick to ensure that it descends to an (untwisted) representation of G. It is abstracted from §2 of [13].

Suppose that we have a central extension

$$1 \to \mathbb{Z} \xrightarrow{\iota} \widetilde{G} \xrightarrow{\pi} G \to 1$$

together with a representation ρ of $\widetilde{G}$ on an R-module V. Suppose also that there exists a quotient $q\colon \widetilde{G} \twoheadrightarrow \mathbb{Z}$ such that $q(\iota(1)) = k \neq 0$ and that $\rho(\iota(1)) = \mathrm{id}_V.\lambda$, for an element $\lambda \in R^*$ that admits a k-th root, in other words there exists $\mu \in R^*$ with $\mu^k = \lambda$. Then the following lemma is immediate.

Lemma 34. *Under the above conditions, the representation of $\widetilde{G}$ on the R-module V given by the formula*

$$\widetilde{g} \longmapsto \rho(\widetilde{g}).\mu^{-q(\widetilde{g})}$$

descends to a representation of G.

5 Constructing the representations

We now put everything together to prove Theorems A and D: in §5.2 we construct the twisted representations of $\mathfrak{M}(\Sigma)$ from Theorem A; their untwisted restrictions to the Chillingworth subgroup are described in §5.3. Then in §5.4 we construct the untwisted representations of the Torelli group from Theorem D. The representations of Theorems B, C and E will be constructed in §6. Before all of this, we discuss some general notions of twisted representations in §5.1.

5.1 Twisted representations of groups.

When a group acts (up to homotopy) on a space, it has a well-defined induced representation on the homology of that space. However, this is not true for *twisted* homology, unless the group action preserves the chosen local system on the space. Instead, there is an induced *twisted* representation of the group. There is more than one way to formulate this (see Remark 40); we will take the viewpoint that a twisted representation of G consists of:

- an action of G on a set X, which defines a groupoid $\mathrm{Ac}(G \curvearrowright X)$ [Definition 35];
- a functor $\mathrm{Ac}(G \curvearrowright X) \to \mathrm{Mod}_R$ [Definition 38].

In the setting above, X will be a set of local systems on the underlying space, on which G acts by pullback. In a sense, the group has been "spread out" over several objects according to this action. We will see the details of a concrete example of this in §5.2.

Definition 35. Let G be a group equipped with a left action $a\colon G \to \mathrm{Sym}(X)$. The *action groupoid* $\mathrm{Ac}(G \curvearrowright X)$ is the groupoid whose objects are $\mathrm{im}(a)$, in other words those symmetries of X that are induced by some element of G, and whose morphisms $\sigma \to \tau$ are the elements $a^{-1}(\tau^{-1}\sigma) \subseteq G$. Composition is given by multiplication in the group.

Remark 36. It is sometimes convenient to consider slightly different but isomorphic action groupoids: for signs $\epsilon, \delta \in \{\pm 1\}$, we may define $\mathrm{Ac}^{\epsilon\delta}(G \curvearrowright X)$ to have objects im(a) and set of morphisms $\sigma \to \tau$ equal to $a^{-1}((\tau^{-\epsilon}\sigma^{\epsilon})^{\delta})$. There are then isomorphisms $\mathrm{Ac}(G \curvearrowright X) \cong \mathrm{Ac}^{\epsilon\delta}(G \curvearrowright X)$ given by $\tau \mapsto \tau^{\epsilon}$ on objects and $g \mapsto g^{\delta}$ on morphisms.

Remark 37. The automorphism group of each object is equal to $\ker(a) \subseteq G$. Moreover, the set of all morphisms with fixed target is naturally identified with the whole group G. When $X = G$ acting on itself by left-multiplication, this is sometimes known as the *translation groupoid* of G.

Definition 38. A *twisted representation* of a group G over a ring R is a left G-set X and a functor $\mathrm{Ac}(G \curvearrowright X) \to \mathrm{Mod}_R$, where Mod_R is the category of R-modules. Similarly for any other flavour of representations given by a category $\mathcal{C}$, such as the category of Hilbert spaces (unitary representations), or the category of bundles of R-modules (fibrewise representations over R): a *twisted representation* of G of this flavour is a left G-set X and a functor $\mathrm{Ac}(G \curvearrowright X) \to \mathcal{C}$.

Remark 39. In the previous section, we considered a notion of *twisted action* of a group G. This determines a twisted representation as follows. Recall that a twisted left action of G on a right $\mathbb{Z}[K]$-module M is a $\mathbb{Z}$-linear left action α of G on M together with a left action β of G on K, such that $\alpha(g)(m.h) = \alpha(g)(m).\beta(g)(h)$. We obtain from this a twisted representation of G over $\mathbb{Z}[K]$ by taking $X = K$ with G-action given by β, and we define the functor

$$\mathrm{Ac}(G \curvearrowright K) \longrightarrow \mathrm{Mod}_{\mathbb{Z}[K]}$$

to send the object $\sigma \in \mathrm{im}(\beta) \subseteq \mathrm{Aut}(K)$ to the $\mathbb{Z}[K]$-module $M_{\sigma^{-1}}$. Here we set $M_\sigma = M$ as $\mathbb{Z}$-modules, but the right K-action on M_σ is given by $m \cdot_\sigma h = m.\sigma(h)$, where . denotes the right K-action on M. Morphisms of $\mathrm{Ac}(G \curvearrowright K)$ are elements of G, and the functor is defined on them by α. The compatibility condition between α and β assumed above is exactly what is needed to imply that this gives a well-defined functor into the category of $\mathbb{Z}[K]$-modules (not just $\mathbb{Z}$-modules).

Remark 40. To formulate the notion of a *twisted representation* of a group, one may either break apart the group into a groupoid (as above) or one may enlarge the target category from Mod_R to a larger category that also contains *twisted* R-linear homomorphisms (similarly to the viewpoint of §4). These two viewpoints are related, as explained in Remark 39. The former viewpoint is most convenient for our purposes, whereas the latter viewpoint is more convenient if one wishes to construct twisted representations of *categories* (rather than just groups); see [38, §2]. In particular, the twisted representations (39) of mapping class groups that we construct below also appear in [38, §5.4.1] when $n \geqslant 3$, using this alternative categorical viewpoint.

5.2 A twisted representation of the mapping class group.

Recall from §3 (Propositions 14, 27 and 28) that we have a representation

$$\Psi \colon \mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}(\mathcal{H})$$

such that $\ker(\Psi) = \mathrm{Chill}(\Sigma)$ and $\Psi^{-1}(\mathrm{Inn}(\mathcal{H})) = \mathfrak{T}(\Sigma)$ for $g \geqslant 1$.

The quotient homomorphism $\phi\colon \mathbb{B}_n(\Sigma) \twoheadrightarrow \mathcal{H}$ (§1) corresponds to a regular covering $\widetilde{\mathcal{C}}_n(\Sigma) \to \mathcal{C}_n(\Sigma)$. Let $f \in \mathfrak{M}(\Sigma)$, $f_{\mathcal{H}}$ be its action on the Heisenberg group and $\mathcal{C}_n(f)$ be the action (up to isotopy) on the configuration space $\mathcal{C}_n(\Sigma)$. From Proposition 14 we know that $\mathcal{C}_n(f)_\sharp = f_{\mathbb{B}_n(\Sigma)}$ preserves $\ker(\phi)$ which implies that there exists a unique lift of $\mathcal{C}_n(f)$ fixing the base point

$$\widetilde{\mathcal{C}}_n(f)\colon \widetilde{\mathcal{C}}_n(\Sigma) \longrightarrow \widetilde{\mathcal{C}}_n(\Sigma) \ . \tag{32}$$

The action of $\widetilde{\mathcal{C}}_n(f)$ on the fibre over the base point identified with $\mathcal{H}$ coincides with $f_{\mathcal{H}}$, and for the deck action of $h \in \mathcal{H}$ on $x \in \widetilde{\mathcal{C}}_n(\Sigma)$ we have the twisting formula

$$\widetilde{\mathcal{C}}_n(f)(x \cdot h) = \widetilde{\mathcal{C}}_n(f) \cdot f_{\mathcal{H}}(h) \ .$$

The induced action on the singular complex $\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))$ is twisted $\mathbb{Z}[\mathcal{H}]$-linear, which can be formulated as a $\mathbb{Z}[\mathcal{H}]$-linear isomorphism (up to chain homotopy)

$$\mathcal{S}_*(\widetilde{\mathcal{C}}_n(f))\colon \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))_{f_{\mathcal{H}}^{-1}} \longrightarrow \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \ .$$

Here the subscript on the domain means that the right action of $\mathcal{H}$ is twisted with $f_{\mathcal{H}}^{-1}$, just as in Remark 39. The result for $\mathbb{Z}[\mathcal{H}]$-local homology is a $\mathbb{Z}[\mathcal{H}]$-linear isomorphism

$$\mathcal{C}_n(f)_*\colon H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})_{f_{\mathcal{H}}^{-1}} \longrightarrow H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H}) \ . \tag{33}$$

More generally, if V is a left representation of the Heisenberg group over a ring R, then we obtain an R-linear isomorphism

$$\mathcal{C}_n(f)_*\colon H_*^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_{f_{\mathcal{H}}}V\big) \longrightarrow H_*^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\big) \ , \tag{34}$$

where the left-hand homology group is obtained from the chain complex

$$\mathcal{S}_*\big(\widetilde{\mathcal{C}}_n(\Sigma)\big)_{f_{\mathcal{H}}^{-1}} \underset{\mathbb{Z}[\mathcal{H}]}{\otimes} V \ \cong\ \mathcal{S}_*\big(\widetilde{\mathcal{C}}_n(\Sigma)\big) \underset{\mathbb{Z}[\mathcal{H}]}{\otimes} {}_{f_{\mathcal{H}}}V \ .$$

(Here, "obtained from" means in more detail that we consider the quotients of this chain complex given by the relative singular complexes for all subspaces of $\widetilde{\mathcal{C}}_n(\Sigma)$ of the form $\pi^{-1}(\mathcal{C}_n(\Sigma, \partial^-(\Sigma)) \cup (\mathcal{C}_n(\Sigma) \smallsetminus T))$ for compact subsets $T \subset \mathcal{C}_n(\Sigma)$, where π denotes the covering map $\widetilde{\mathcal{C}}_n(\Sigma) \to \mathcal{C}_n(\Sigma)$, take the homology of each of these quotients and then take the inverse limit of this diagram.)

An equivalent way to see that we obtain (33) and (34) is as follows. For a quotient homomorphism $q\colon \pi_1(\mathcal{C}_n(\Sigma)) \twoheadrightarrow Q$ let us write $\widetilde{\mathcal{C}}_n(\Sigma)^q$ for the corresponding regular Q-covering of $\mathcal{C}_n(\Sigma)$, considered as a space with a right Q-action. In this notation, the lifted action (32) is of the form

$$\widetilde{\mathcal{C}}_n(\Sigma)^{f_{\mathcal{H}} \circ \phi} \longrightarrow \widetilde{\mathcal{C}}_n(\Sigma)^{\phi} \tag{35}$$

and commutes with the right $\mathcal{H}$-action on the source and target. Note that the right $\mathcal{H}$-action on the left-hand space is twisted by $f_{\mathcal{H}}^{-1}$ compared with its right action on the right-hand space. This is because the action of

$$\pi_1(\mathcal{C}_n(\Sigma)) \xrightarrow{\ \phi\ } \mathcal{H} \xrightarrow{\ f_{\mathcal{H}}\ } \mathcal{H} \ni h$$

is given by sending h backwards along $f_{\mathcal{H}}$ and then applying the untwisted action. Thus, applying relative Borel-Moore homology to (35), we obtain (33) with $\mathbb{Z}[\mathcal{H}]$-local coefficients and (34) with V-local coefficients.

Slightly more generally, for $\tau \in \mathrm{Aut}(\mathcal{H})$, the action $\mathcal{C}_n(f)\colon \mathcal{C}_n(\Sigma) \to \mathcal{C}_n(\Sigma)$ lifts to

$$\widetilde{\mathcal{C}}_n(\Sigma)^{\tau \circ f_{\mathcal{H}} \circ \phi} \longrightarrow \widetilde{\mathcal{C}}_n(\Sigma)^{\tau \circ \phi} \tag{36}$$

and, applying relative Borel-Moore homology, we obtain

$$H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})_{f_{\mathcal{H}}^{-1} \circ \tau^{-1}} \longrightarrow H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})_{\tau^{-1}} \tag{37}$$

with $\mathbb{Z}[\mathcal{H}]$-local coefficients and

$$H_*^{BM}\bigl(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_{\tau \circ f_{\mathcal{H}}}V\bigr) \longrightarrow H_*^{BM}\bigl(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_{\tau}V\bigr) \tag{38}$$

with V-local coefficients.

Summarising this discussion, we have shown:

Theorem 41 (Theorem A(b))**.** *Associated to any representation V of $\mathcal{H}$ over R, there is a well-defined twisted representation*

$$\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H}) \longrightarrow \mathrm{Mod}_R \tag{39}$$

in the sense of Definition 38, where each object $\tau\colon \mathcal{H} \to \mathcal{H}$ is sent to the R-module

$$H_n^{BM}\bigl(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_{\tau}V\bigr)$$

and the morphism $f\colon \tau \circ f_{\mathcal{H}} \to \tau$ is sent to the R-linear isomorphism (38)*.*

Remark 42. The functor (39) factors through the category of pairs of spaces equipped with local systems over $\mathbb{Z}[\mathcal{H}]$, which we denote by $\mathrm{Top}^2_{\mathbb{Z}[\mathcal{H}]}$. To see this, we send the object τ of $\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H})$ to the bundle of $\mathbb{Z}[\mathcal{H}]$-modules obtained by applying the free abelian group functor fibrewise to $\widetilde{\mathcal{C}}_n(\Sigma)^{\tau \circ \phi}$, together with the subspace $\mathcal{C}_n(\Sigma, \partial^-(\Sigma)) \subset \mathcal{C}_n(\Sigma)$. (We recall that, under mild conditions that are satisfied here, local systems over $\mathbb{Z}[\mathcal{H}]$ may be thought of as bundles of $\mathbb{Z}[\mathcal{H}]$-modules.) We send the morphism $f\colon \tau \circ f_{\mathcal{H}} \to \tau$ to the result of applying the free abelian group functor fibrewise to (36). This defines a functor

$$\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H}) \longrightarrow \mathrm{Top}^2_{\mathbb{Z}[\mathcal{H}]} \tag{40}$$

and the remainder of the construction then consists in composing (40) with the fibrewise tensor product functor $- \otimes_{\mathbb{Z}[\mathcal{H}]} V\colon \mathrm{Top}^2_{\mathbb{Z}[\mathcal{H}]} \to \mathrm{Top}^2_R$ and relative Borel-Moore homology functor $H_n^{BM}\colon \mathrm{Top}^2_R \to \mathrm{Mod}_R$.

5.3 Restricting to the Chillingworth subgroup.

As mentioned in Remark 37, the automorphism groups of the groupoid $\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H})$ are all isomorphic to the kernel of the action $\Psi\colon \mathfrak{M}(\Sigma) \curvearrowright \mathcal{H}$, which is the Chillingworth subgroup $\mathrm{Chill}(\Sigma)$, by Proposition 27. Restricting (39) to the automorphism group of the object $\mathrm{id}\colon \mathcal{H} \to \mathcal{H}$ of $\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H})$ therefore gives us an untwisted representation

$$\mathrm{Chill}(\Sigma) \longrightarrow \mathrm{Mod}_R$$

of the Chillingworth group. Concretely, the underlying R-module of this representation is the relative V-local Borel-Moore homology module

$$H_*^{BM}\bigl(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\bigr),$$

and each $f \in \mathrm{Chill}(\Sigma)$ is sent to (38) with $\tau = f_{\mathcal{H}} = \mathrm{id}$. Thus we have shown:

Theorem 43. *Associated to any representation V of $\mathcal{H}$ over R, there is a well-defined representation*

$$\mathrm{Chill}(\Sigma) \longrightarrow \mathrm{Aut}_R\bigl(H_n^{BM}\bigl(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\bigr)\bigr), \tag{41}$$

which is a restriction of (39) to a single object.

Remark 44. Recall from §2 that the R-module $H_n^{BM}\bigl(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\bigr)$ is naturally isomorphic to a direct sum of $\binom{2g+n-1}{n}$ copies of V. In particular, if V is a free R-module of rank N, the right-hand side of (41) may be written as $GL_{N\binom{2g+n-1}{n}}(R)$.

5.4 The Torelli group.

We now restrict to the Torelli group $\mathfrak{T}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$. In this case we have $\Psi^{-1}(\mathrm{Inn}(\mathcal{H})) = \mathfrak{T}(\Sigma)$ by Proposition 28. We may therefore pull back the $\mathbb{Z}$-central extension

$$1 \to \mathbb{Z} \cong \mathcal{Z}(\mathcal{H}) \longrightarrow \mathcal{H} \longrightarrow \mathrm{Inn}(\mathcal{H}) \to 1$$

along the homomorphism $\Psi\colon \mathfrak{T}(\Sigma) \to \mathrm{Inn}(\mathcal{H})$ to obtain a $\mathbb{Z}$-central extension

$$1 \to \mathbb{Z} \longrightarrow \widetilde{\mathfrak{T}}(\Sigma) \longrightarrow \mathfrak{T}(\Sigma) \to 1$$

and a homomorphism

$$\widetilde{\Psi}\colon \widetilde{\mathfrak{T}}(\Sigma) \longrightarrow \mathcal{H}$$

lifting Ψ. We will use this to "untwist" the representation (39) on the Torelli group.

Warning 45. Although we are using the tilde notation $\widetilde{\ }$ for this central extension of the Torelli group, we are not (yet) claiming that it is the same as the restriction of the stably universal central extension $\widetilde{\mathfrak{M}}(\Sigma)$ of the mapping class group. However, we will see shortly (Lemma 47 and its proof) that both $\widetilde{\mathfrak{T}}(\Sigma) \to \mathfrak{T}(\Sigma)$ and the restriction of $\widetilde{\mathfrak{M}}(\Sigma) \to \mathfrak{M}(\Sigma)$ to the Torelli group are *trivial* extensions; in particular they coincide.

For an element $h \in \mathcal{H}$, denote by $c_h = h - h^{-1}$ the corresponding inner automorphism $c_h \in \mathrm{Inn}(\mathcal{H})$. One may check that the isomorphism

$$- \cdot h \colon \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))_{c_h} \longrightarrow \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \tag{42}$$

of singular chain complexes given by the right-action of h is $\mathbb{Z}[\mathcal{H}]$-linear. The construction of §5.2, shifted by $f_{\mathcal{H}}$, sends each $f \in \mathfrak{T}(\Sigma)$ to a $\mathbb{Z}[\mathcal{H}]$-linear isomorphism

$$\mathcal{S}_*(\widetilde{\mathcal{C}}_n(f)) \colon \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \longrightarrow \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))_{f_{\mathcal{H}}} \ , \tag{43}$$

where $f_{\mathcal{H}} = \Psi(f)$. For $\widetilde{f} \in \widetilde{\mathfrak{T}}(\Sigma)$ we therefore obtain a $\mathbb{Z}[\mathcal{H}]$-linear automorphism

$$\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \xrightarrow{(43)} \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))_{f_{\mathcal{H}}} \xrightarrow{(42)} \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \ , \tag{44}$$

where we take $h = \widetilde{\Psi}(\widetilde{f})$ in (42). This defines an untwisted action of the central extension $\widetilde{\mathfrak{T}}(\Sigma)$ of the Torelli group on the singular chain complex $\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))$, and thus also on $\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \otimes_{\mathbb{Z}[\mathcal{H}]} V$ for any left $\mathcal{H}$-representation V. Moreover, we may repeat this construction for the *relative* singular chain complex with respect to any subspace of $\mathcal{C}_n(\Sigma)$, and this is compatible with taking inverse limits, and so we obtain an untwisted action of the central extension $\widetilde{\mathfrak{T}}(\Sigma)$ of the Torelli group on the relative V-local Borel-Moore homology

$$H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\big)$$

for any left $\mathcal{H}$-representation V. Thus we have shown that there is a well-defined representation

$$\widetilde{\mathfrak{T}}(\Sigma) \longrightarrow \mathrm{Aut}_R\big(H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\big)\big), \tag{45}$$

where $\widetilde{\mathfrak{T}}(\Sigma)$ is a central extension by $\mathbb{Z}$ of the Torelli group $\mathfrak{T}(\Sigma)$.

Remark 46. We have described the untwisting at the level of the singular chain complex, but it may in fact be done already at the level of spaces equipped with local systems over $\mathbb{Z}[\mathcal{H}]$, as explain in §4; see especially Lemma 33. The verification that the composition (44) really gives a well-defined action of $\widetilde{\mathfrak{T}}(\Sigma)$ is essentially equivalent to the proof of Lemma 32. In particular, the central extension $\widetilde{\mathfrak{T}}(\Sigma)$ of the Torelli group is the pullback in diagram (31) in the special case where $K = \mathcal{H}$ and $G = \mathfrak{T}(\Sigma)$.

To complete the construction, we show that:

Lemma 47. *The central extension $\widetilde{\mathfrak{T}}(\Sigma)$ of $\mathfrak{T}(\Sigma)$ is trivial, i.e. it is isomorphic to the product $\mathfrak{T}(\Sigma) \times \mathbb{Z}$.*

Proof. We begin by showing that it suffices to prove the statement for all sufficiently large g; we will then be able to assume $g \geqslant 3$ in the rest of the proof. For $g < h$, recall the inclusion (25), which restricts to an inclusion of Torelli groups

$$\iota \colon \mathfrak{T}(\Sigma_{g,1}) \hookrightarrow \mathfrak{T}(\Sigma_{h,1}). \tag{46}$$

We claim that the pullback of the central extension $\widetilde{\mathfrak{T}}(\Sigma_{h,1})$ along (46) is $\widetilde{\mathfrak{T}}(\Sigma_{g,1})$. Indeed, the pullback of $\widetilde{\mathfrak{T}}(\Sigma_{h,1})$ is represented by the cocycle $(f,g) \mapsto \mathfrak{d}(\iota(f)) \cdot \mathfrak{d}(\iota(g))$ and $\widetilde{\mathfrak{T}}(\Sigma_{g,1})$ is represented by the cocycle $(f,g) \mapsto \mathfrak{d}(f) \cdot \mathfrak{d}(g)$. These cocycles are equal, by Lemma 24 and the fact that $H_1(\Sigma_{g,1}) \to H_1(\Sigma_{h,1})$ preserves the intersection form. Thus triviality of $\widetilde{\mathfrak{T}}(\Sigma_{h,1})$ will imply triviality of $\widetilde{\mathfrak{T}}(\Sigma_{g,1})$ for any $g < h$.

By [8, Lemma A.1(xiii)] and homological stability [44, Theorem 1.2], the canonical surjection $\mathfrak{M}(\Sigma) \twoheadrightarrow Sp(H)$ induces an isomorphism on $H^2(-;\mathbb{Z})$ when $g \geqslant 3$. It follows that the inclusion $\mathfrak{T}(\Sigma) \hookrightarrow \mathfrak{M}(\Sigma)$ induces the trivial map on $H^2(-;\mathbb{Z})$. This means that every $\mathbb{Z}$-central extension of $\mathfrak{M}(\Sigma)$ becomes trivial when restricted to $\mathfrak{T}(\Sigma)$. We will prove the lemma by showing that $\widetilde{\mathfrak{T}}(\Sigma)$ is the restriction of a $\mathbb{Z}$-central extension defined on the whole mapping class group $\mathfrak{M}(\Sigma)$.

There is a 2-cocycle c on $\mathfrak{M}(\Sigma)$, defined by Morita [34], given by the formula $c(f,g) = \mathfrak{d}(f^{-1}) \cdot \mathfrak{d}(g)$, where $\mathfrak{d} \colon \mathfrak{M}(\Sigma) \to H$ is Morita's crossed homomorphism (see §3.3) and $\cdot$ is the intersection form on H. By general properties of crossed homomorphisms, we have $\mathfrak{d}(f^{-1}) = -f_*^{-1}(\mathfrak{d}(f))$, and so we may rewrite this as

$$c(f,g) = -f_*^{-1}(\mathfrak{d}(f)) \cdot \mathfrak{d}(g) = -\mathfrak{d}(f) \cdot f_*(\mathfrak{d}(g)). \tag{47}$$

Recall from §3, especially equation (23), that the restriction of $\Psi \colon \mathfrak{M}(\Sigma) \to \mathrm{Aut}^+(\mathcal{H}) \cong Sp(H) \ltimes H$ to $\Psi|_{\mathfrak{T}(\Sigma)} \colon \mathfrak{T}(\Sigma) \to \mathrm{Inn}(\mathcal{H}) \cong 2H$ is equal to Morita's crossed homomorphism $\mathfrak{d}$ (restricted to the Torelli group). The central extension $\widetilde{\mathfrak{T}}(\Sigma)$ is therefore represented by the 2-cocycle c' on $\mathfrak{T}(\Sigma)$ given by $c'(f,g) = \Psi(f) \cdot \Psi(g) = \mathfrak{d}(f) \cdot \mathfrak{d}(g)$. Restricted to the Torelli group, Morita's cocycle (47) may be written as $c(f,g) = -\mathfrak{d}(f) \cdot \mathfrak{d}(g)$, since we have $f_* = \mathrm{id}$ for $f \in \mathfrak{T}(\Sigma)$. Thus we have $c' = -c$ on $\mathfrak{T}(\Sigma)$. In particular, since c is the restriction of a 2-cocycle defined on the whole mapping class group, so is c'. ■

Theorem 48 (Theorem D)**.** *Associated to any representation V of $\mathcal{H}$ over R, there is a well-defined representation of the Torelli group*

$$\mathfrak{T}(\Sigma) \longrightarrow \mathrm{Aut}_R\bigl(H_n^{BM}\bigl(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\bigr)\bigr) \tag{48}$$

that lifts the natural projective action of $\mathfrak{T}(\Sigma)$ on this homology module.

Proof. Let us abbreviate $\mathcal{V}_n(V) = H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)$. The homomorphism (45) must send the subgroup $\mathbb{Z} \subset \widetilde{\mathfrak{T}}(\Sigma)$ (the kernel of the central extension) to the centre of $\mathrm{Aut}_R(\mathcal{V}_n(V))$, so it descends to

$$\mathfrak{T}(\Sigma) \longrightarrow \mathrm{PAut}_R(\mathcal{V}_n(V)), \tag{49}$$

where the projective automorphism group $\mathrm{PAut}_R(A)$ of an R-module A is the quotient of $\mathrm{Aut}_R(A)$ by its centre. Note that the centre of $\mathrm{Aut}_R(A)$ is equal to $\{- \cdot \lambda \mid \lambda \in \mathcal{Z}(R^\times)\}$ when A is a free R-module, but may be larger when A is not free. This is the natural projective action of the Torelli group. To lift it to a linear action, we may compose (45) with any section of the central extension $\widetilde{\mathfrak{T}}(\Sigma) \to \mathfrak{T}(\Sigma)$, which exists by Lemma 47. ■

Remark 44 applies also in this setting; in particular, $H_n^{BM}\bigl(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\bigr)$ is a free R-module whenever V is a free R-module.

Remark 49. In [26], Johnson defined a homomorphism $t\colon \mathfrak{T}(\Sigma) \to H = H_1(\Sigma)$, which was implicit in [15] and called it the Chillingworth homomorphism. Its value on a genus k bounding pair generator $\tau_\gamma\tau_\delta^{-1}$ is equal to $2kc$ where c is the class of γ with orientation given by the surface representing the homology between γ and δ. Its image is $2H$ and so we obtain a 2-cocycle Ω on $\mathfrak{T}(\Sigma)$ with values in $\mathbb{Z}$ by the formula $\Omega(f,g) = \frac{1}{4}\omega(t_f, t_g)$, where ω is the symplectic 2-cocycle on H. Under the identifications $H \cong \mathcal{H}/\mathcal{Z}(\mathcal{H}) \cong \mathrm{Inn}(\mathcal{H})$, this is precisely the action $\Psi\colon \mathfrak{M}(\Sigma) \to \mathrm{Aut}(\mathcal{H})$ restricted to the Torelli group (which acts on $\mathcal{H}$ by inner automorphisms by Proposition 28). Thus the central extension $\widetilde{\mathfrak{T}}(\Sigma)$, which was defined abstractly by pulling back the central extension

$$1 \to \mathbb{Z} \cong \mathcal{Z}(\mathcal{H}) \hookrightarrow \mathcal{H} \twoheadrightarrow \mathcal{H}/\mathcal{Z}(\mathcal{H}) \cong \mathrm{Inn}(\mathcal{H}) \to 1$$

along the inner action $\Psi|_{\mathfrak{T}(\Sigma)}\colon \mathfrak{T}(\Sigma) \to \mathrm{Inn}(\mathcal{H})$, see diagram (31), may be described more explicitly as the central extension of the Torelli group associated to Ω. In other words, we have $\widetilde{\mathfrak{T}}(\Sigma) = \mathbb{Z} \times \mathfrak{T}(\Sigma)$ as a set, and $(k,f)(l,g) = (k+l+\Omega(f,g), fg)$. Moreover, we have a lift of the Chillingworth homomorphism

$$\widetilde{t}\colon \widetilde{\mathfrak{T}}(\Sigma) \longrightarrow \mathcal{H}$$

given by the pullback construction (31), which may be described by the formula $\widetilde{t}(k,f) = (4k, t_f)$. This is the homomorphism denoted by $\widetilde{\Psi}$ above.

6 Untwisting on the full mapping class group via Schrödinger

The Heisenberg group $\mathcal{H}$ can be realised as a group of matrices, which gives a faithful finite dimensional representation, defined as follows:

$$\left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right) \longmapsto \begin{pmatrix} 1 & p & \frac{k+p\cdot q}{2} \\ 0 & I_g & q \\ 0 & 0 & 1 \end{pmatrix},$$

where $p = (p_i)$ is a row vector and $q = (q_i)$ is a column vector. This matrix form is often given as the definition of the Heisenberg group so that we may call this representation the tautological one.

Another well-known representation, which is infinite-dimensional and unitary, is the *Schrödinger representation*, which is parametrised by the Planck constant, a non-zero real number $\hbar$. The right action on the Hilbert space $L^2(\mathbb{R}^g)$ is given by the following formula:

$$\left[\Pi_\hbar\left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right)\psi\right](s) = e^{i\hbar\frac{k-p\cdot q}{2}} e^{i\hbar p\cdot s}\psi(s-q). \tag{50}$$

The Schrödinger representation occupies a special place in the representation theory of the Heisenberg group, and in this section we explain how to leverage its properties to construct an untwisted representation on the full mapping class group $\mathfrak{M}(\Sigma)$, after passing to a central extension. For comparison, recall that, in the previous section, we constructed an untwisted representation of the Torelli group $\mathfrak{T}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$ associated to *any* representation V of the Heisenberg group. The difference in this section is that we focus only on the special case where V is the *Schrödinger representation*, but as a consequence we are able to untwist the representation on the full mapping class group.

In §6.1 we first discuss the Schrödinger representation in more detail, as well as the Stone-von Neumann theorem and its consequences. In §6.2 we discuss the metaplectic group and the universal central extension of the mapping class group. We then prove Theorems B and E in §6.3, constructing untwisted representations of the universal central extension of the mapping class group and of the Morita subgroup (without passing to a central extension). Finally, in §6.4 we explain how to adapt our construction to the finite-dimensional analogues of the Schrödinger representation to prove Theorem C.

6.1 The Schrödinger representation and the Stone-von Neumann theorem.

The *continuous Heisenberg group* is defined similarly to the discrete Heisenberg group. As a set it is $\mathbb{R} \times H_1(\Sigma;\mathbb{R})$ with multiplication given by $(s,x).(t,y) = (s+t+\omega(x,y), x+y)$, where ω is the intersection form on $H_1(\Sigma;\mathbb{R})$. We denote it by $\mathcal{H}_{\mathbb{R}}$ and note that the discrete Heisenberg group $\mathcal{H}$ is naturally a subgroup of $\mathcal{H}_{\mathbb{R}}$. As explained in Appendix B, there is a natural inclusion

$$\mathrm{Aut}^+(\mathcal{H}) \hookrightarrow \mathrm{Aut}^+(\mathcal{H}_{\mathbb{R}}),$$

denoted by $\varphi \mapsto \varphi_{\mathbb{R}}$, such that $\varphi_{\mathbb{R}}$ is an extension of φ (see (90)).

As an alternative to the explicit formula (50), the Schrödinger representation may also be defined more abstractly as follows. First note that $\mathcal{H}_{\mathbb{R}}$ may be written as a semi-direct product

$$\mathcal{H}_{\mathbb{R}} = \mathbb{R}\{(0,b_1),\ldots,(0,b_g)\} \ltimes \mathbb{R}\{(1,0),(0,a_1),\ldots,(0,a_g)\},$$

where $a_1,\ldots,a_g,b_1,\ldots,b_g$ form a symplectic basis for $H_1(\Sigma;\mathbb{R})$. Fix a real number $\hbar > 0$. There is a one-dimensional complex unitary representation

$$\mathbb{R}\{(1,0),(0,a_1),\ldots,(0,a_g)\} \longrightarrow \mathbb{S}^1 = U(1)$$

defined by $(t,x) \mapsto e^{\hbar i t/2}$. This may then be induced to a complex unitary representation of the whole group $\mathcal{H}_{\mathbb{R}}$ on the complex Hilbert space $L^2(\mathbb{R}\{(0,b_1),\ldots,(0,b_g)\}) = L^2(\mathbb{R}^g)$. This is the Schrödinger representation of $\mathcal{H}_{\mathbb{R}}$. From now on, let us denote this representation by

$$W = L^2(\mathbb{R}^g) \qquad \text{and} \qquad \rho_W \colon \mathcal{H}_{\mathbb{R}} \longrightarrow U(W). \tag{51}$$

We will usually not make the dependence on $\hbar$ explicit in the notation; in particular we write ρ_W instead of $\rho_{W,\hbar}$. The key properties of ρ_W that we shall need are the following.

Theorem 50 (The *Stone–von Neumann theorem*; [31, page 19])**.**

(a) *The representation* (51) *is irreducible.*

(b) *If H' is a complex Hilbert space and*

$$\rho\colon \mathcal{H}_{\mathbb{R}} \longrightarrow U(H')$$

is a unitary representation such that $\rho(t,0) = e^{\hbar i t/2}.\mathrm{id}_{H'}$ for all $t \in \mathbb{R}$, then there is another Hilbert space H'' and an isomorphism $\kappa\colon H' \to W \otimes H''$ such that, for any $(t,x) \in \mathcal{H}_{\mathbb{R}}$, the following diagram commutes:

$$\begin{tikzcd} H' \arrow[r, "\kappa"] \arrow[d, "\rho(t,x)"'] & W \otimes H'' \arrow[d, "\rho_W(t,x)\otimes \mathrm{id}_{H''}"] \\ H' \arrow[r, "\kappa"] & W \otimes H''. \end{tikzcd}$$

Corollary 51. *If $\rho\colon \mathcal{H}_{\mathbb{R}} \to U(W)$ is an irreducible unitary representation such that $\rho(t,0) = e^{\hbar i t/2}.\mathrm{id}_W$ for all $t \in \mathbb{R}$, then there is a commutative diagram*

$$\begin{tikzcd} \mathcal{H}_{\mathbb{R}} \arrow[r, "\rho_W"] \arrow[dr, "\rho"'] & U(W) \arrow[d, "u.-.u^{-1}"] \\ & U(W) \end{tikzcd}$$

for some element $u \in U(W)$, which is unique up to rescaling by an element of $\mathbb{S}^1$.

Proof. Apply Theorem 50 and note that $\dim(H'') = 1$ since ρ is irreducible. The unitary isomorphism κ together with any choice of unitary isomorphism $W \otimes \mathbb{R} \cong W$ give an element u as claimed. To see uniqueness up to a scalar in $\mathbb{S}^1$, note that any two such elements u differ by an automorphism of the irreducible representation ρ_W, which must therefore be a scalar (in $\mathbb{C}^*$) multiple of the identity, by Schur's lemma. Moreover, since ρ_W is unitary, this scalar must lie in $\mathbb{S}^1 \subset \mathbb{C}^*$. ■

Definition 52. Denote by $PU(W) = U(W)/\mathbb{S}^1$ the *projective unitary group* of the Hilbert space W. Since scalar multiples of the identity are central, this fits into a central extension

$$1 \longrightarrow \mathbb{S}^1 \longrightarrow U(W) \longrightarrow PU(W) \longrightarrow 1. \tag{52}$$

We denote by $\omega_{PU}\colon PU(W) \times PU(W) \to \mathbb{S}^1$ a choice of 2-cocycle corresponding to this central extension; in other words we write $U(W) \cong \mathbb{S}^1 \times PU(W)$ with multiplication given by $(s,g)(t,h) = (s.t.\omega_{PU}(g,h), gh)$.

Definition 53. For an automorphism $\varphi \in \mathrm{Aut}(\mathcal{H}_{\mathbb{R}})$, Corollary 51 applied to the representation $\rho = \rho_W \circ \varphi$ tells us that there is a unique element $u = T(\varphi) \in PU(W)$ such that $\rho_W \circ \varphi = T(\varphi).\rho_W.T(\varphi)^{-1}$. The assignment $\varphi \mapsto T(\varphi)$ defines a group homomorphism

$$T\colon \mathrm{Aut}(\mathcal{H}_{\mathbb{R}}) \longrightarrow PU(W). \tag{53}$$

As shown in Appendix B, the subgroup $\mathrm{Aut}^+(\mathcal{H}_\mathbb{R}) \subseteq \mathrm{Aut}(\mathcal{H}_\mathbb{R})$ splits as the semi-direct product $Sp(H_\mathbb{R}) \ltimes \mathrm{Hom}(H_\mathbb{R}, \mathbb{R})$, where $H_\mathbb{R} = H_1(\Sigma; \mathbb{R})$. Restricting (53) to the subgroup $Sp(H_\mathbb{R}) = Sp_{2g}(\mathbb{R})$, we obtain a projective representation

$$R = T|_{Sp_{2g}(\mathbb{R})} \colon Sp_{2g}(\mathbb{R}) \longrightarrow PU(W). \tag{54}$$

This is the *Shale-Weil projective representation* of the symplectic group. (It is sometimes also called the *Segal-Shale-Weil projective representation*, see for example [31, page 53].) Pulling back the central extension (52) along the homomorphism (54), we then obtain a central extension

$$1 \longrightarrow \mathbb{S}^1 \longrightarrow \overline{Sp}_{2g}(\mathbb{R}) \longrightarrow Sp_{2g}(\mathbb{R}) \longrightarrow 1 \tag{55}$$

and a lifted representation

$$\overline{R} \colon \overline{Sp}_{2g}(\mathbb{R}) \longrightarrow U(W). \tag{56}$$

The group $\overline{Sp}_{2g}(\mathbb{R})$ is sometimes known as the *Mackey obstruction group* of the projective representation (54). Since (55) is pulled back from (52) along R, we may write $\overline{Sp}_{2g}(\mathbb{R}) \cong \mathbb{S}^1 \times Sp_{2g}(\mathbb{R})$ with multiplication given by $(s, g)(t, h) = (s.t.\omega_{Sp}(g, h), gh)$, where

$$\omega_{Sp} = \omega_{PU} \circ (R \times R) \colon Sp_{2g}(\mathbb{R}) \times Sp_{2g}(\mathbb{R}) \longrightarrow PU(W) \times PU(W) \longrightarrow \mathbb{S}^1.$$

6.2 Metaplectic and universal extensions.

Definition 54. The fundamental group of $Sp_{2g}(\mathbb{R})$ is infinite cyclic. It therefore has a unique connected double covering group, called the *metaplectic group*, which we denote by $Mp_{2g}(\mathbb{R})$.

For an explicit construction of $Mp_{2g}(\mathbb{R})$ as an extension of $Sp_{2g}(\mathbb{R})$, see [42, §2].

Proposition 55. *There is an inclusion of central extensions $Mp_{2g}(\mathbb{R}) \hookrightarrow \overline{Sp}_{2g}(\mathbb{R})$.*

Proof. We first show that it suffices to prove the statement for all g sufficiently large; we will then be able to assume that $g \geqslant 4$ in the rest of the proof, which is the stable range for (co)homology of degree at most 2 for $Sp_{2g}(\mathbb{R})$ and $\mathfrak{M}(\Sigma_{g,1})$. For $g < h$ there is an inclusion map $Sp_{2g}(\mathbb{R}) \hookrightarrow Sp_{2h}(\mathbb{R})$ given by extending a symplectic automorphism of $\mathbb{R}^{2g}$ by the identity on $\mathbb{R}^{2h-2g}$. We claim that the pullbacks of $Mp_{2h}(\mathbb{R})$ and of $\overline{Sp}_{2h}(\mathbb{R})$ under this inclusion are $Mp_{2g}(\mathbb{R})$ and of $\overline{Sp}_{2g}(\mathbb{R})$ respectively. For Mp this follows from the fact that the induced map $\pi_1(Sp_{2g}(\mathbb{R})) \cong \mathbb{Z} \to \mathbb{Z} = \pi_1(Sp_{2h}(\mathbb{R}))$ is an isomorphism and the metaplectic double covering corresponds to the unique index-2 subgroup of π_1. For $\overline{Sp}$, note that the Shale-Weil projective representations in genus g and h fit into a commutative square as follows:

$$\begin{tikzcd}
Sp_{2g}(\mathbb{R}) \arrow[r, "R"] \arrow[d, hook] & PU(L^2(\mathbb{R}^g)) \arrow[d] & U(L^2(\mathbb{R}^g)) \arrow[l, two heads] \arrow[d] \\
Sp_{2h}(\mathbb{R}) \arrow[r, "R"] & PU(L^2(\mathbb{R}^h)) & U(L^2(\mathbb{R}^h)) \arrow[l, two heads]
\end{tikzcd} \tag{57}$$

The right-hand side of this diagram arises as follows. We consider $L^2(\mathbb{R}^g)$ as the (closed) subspace of $L^2(\mathbb{R}^h)$ of those L^2-functions that factor through $\mathbb{R}^h = \mathbb{R}^g \times \mathbb{R}^{h-g} \twoheadrightarrow \mathbb{R}^g$. Any closed subspace of a Hilbert space has an orthogonal complement, so we may extend unitary automorphisms by the identity on this complement to obtain a homomorphism $U(L^2(\mathbb{R}^g)) \to U(L^2(\mathbb{R}^h))$, which descends to the projective unitary groups. The right-hand squre of (57) is a pullback square (this is true for any closed subspace of a Hilbert space). By definition, $\overline{Sp}_{2g}(\mathbb{R}) \twoheadrightarrow Sp_{2g}(\mathbb{R})$ is the pullback along the Shale-Weil projective representation of the extension $U(L^2(\mathbb{R}^g)) \twoheadrightarrow PU(L^2(\mathbb{R}^g))$. Commutativity of (57) then implies that the pullback of $\overline{Sp}_{2h}(\mathbb{R})$ along the inclusion is $\overline{Sp}_{2g}(\mathbb{R})$. Thus the existence of an embedding $Mp_{2h}(\mathbb{R}) \hookrightarrow \overline{Sp}_{2h}(\mathbb{R})$ will imply the existence of an embedding $Mp_{2g}(\mathbb{R}) \hookrightarrow \overline{Sp}_{2g}(\mathbb{R})$ for $g < h$. We henceforth assume that $g \geqslant 4$ in this proof (this is only needed in the last paragraph).

First, it is proven in §1.7 of [31] that the cocycle ω_{Sp} takes values in the cyclic subgroup $\mathbb{Z}/8 \subseteq \mathbb{S}^1$, so there is an embedding of central extensions $Sp_{2g}(\mathbb{R})^{(8)} \hookrightarrow \overline{Sp}_{2g}(\mathbb{R})$, for a certain $\mathbb{Z}/8$-central extension $Sp_{2g}(\mathbb{R})^{(8)}$ of $Sp_{2g}(\mathbb{R})$. Moreover, this central extension is classified by $-[\tau].8\mathbb{Z} \in H^2(Sp_{2g}(\mathbb{R}); \mathbb{Z}/8)$, the reduction modulo 8 of the element $-[\tau] \in H^2(Sp_{2g}(\mathbb{R}); \mathbb{Z})$ represented by the negative of the *Maslov cocycle* τ (see formula 1.7.7 on page 70 of [31]).

Second, it is also proven in §1.7 of [31] that there is a function $s\colon Sp_{2g}(\mathbb{R}) \to \mathbb{Z}/4 \subseteq \mathbb{S}^1$ such that $\omega_{Sp}(g,h)^2 = s(g)^{-1}s(h)^{-1}s(gh)$ (formula 1.7.8 on page 70 of [31]). It follows that the subset of $Sp_{2g}(\mathbb{R})^{(8)}$ of those pairs (t,g) for which $t^2 = s(g)$ is a subgroup. The projection onto $Sp_{2g}(\mathbb{R})$ restricted to this subgroup is a double covering, and so this subgroup must be either $Sp_{2g}(\mathbb{R}) \times \mathbb{Z}/2$ or the metaplectic group $Mp_{2g}(\mathbb{R})$.

To finish the proof, we just have to show that it cannot be $Sp_{2g}(\mathbb{R}) \times \mathbb{Z}/2$. Suppose for a contradiction that it is. Then $Sp_{2g}(\mathbb{R})^{(8)}$ admits a section, so it is a trivial extension and we must have $[\tau].8\mathbb{Z} = 0 \in H^2(Sp_{2g}(\mathbb{R}); \mathbb{Z}/8)$. However, the pullback of $[\tau]$ along the projection $\mathfrak{M}(\Sigma) \to Sp_{2g}(\mathbb{R})$, also denoted by $[\tau]$, is $4c_1$, where c_1 is a generator of $H^2(\mathfrak{M}(\Sigma); \mathbb{Z}) \cong \mathbb{Z}$. Thus $[\tau].8\mathbb{Z} \in H^2(\mathfrak{M}(\Sigma); \mathbb{Z}/8) \cong \mathbb{Z}/8$ is non-zero. Hence we must have $[\tau].8\mathbb{Z} \neq 0$ already in $H^2(Sp_{2g}(\mathbb{R}); \mathbb{Z}/8)$. This completes the proof. ■

Definition 56. Recall that, if G is a perfect group, i.e. if $H_1(G;\mathbb{Z}) = 0$, then we have $H^2(G; H_2(G;\mathbb{Z})) \cong \mathrm{Hom}(H_2(G;\mathbb{Z}), H_2(G;\mathbb{Z}))$ by the universal coefficient theorem, and the $H_2(G;\mathbb{Z})$-central extension of G corresponding to the identity map is the *universal central extension* of G. For $G = \mathfrak{M}(\Sigma = \Sigma_{g,1})$, we have that G is perfect when $g \geqslant 3$ and we have $H_2(G;\mathbb{Z}) \cong \mathbb{Z}$ when $g \geqslant 4$. In particular, for $g \geqslant 4$, let us denote by

$$1 \to \mathbb{Z} \longrightarrow \widetilde{\mathfrak{M}}(\Sigma) \longrightarrow \mathfrak{M}(\Sigma) \to 1$$

the *universal central extension of* $\mathfrak{M}(\Sigma)$.

As explained in the introduction, the inclusion map $\mathfrak{M}(\Sigma_{g,1}) \hookrightarrow \mathfrak{M}(\Sigma_{h,1})$ induces isomorphisms on first and second (co)homology for $h \geqslant g \geqslant 4$ (see [23] or [44]), so the pullback of $\widetilde{\mathfrak{M}}(\Sigma_{h,1})$ along this inclusion is $\widetilde{\mathfrak{M}}(\Sigma_{g,1})$. For any $g \geqslant 1$, we may therefore define the *stably universal central extension* $\widetilde{\mathfrak{M}}(\Sigma_{g,1})$ of $\mathfrak{M}(\Sigma_{g,1})$ to be the pullback of $\widetilde{\mathfrak{M}}(\Sigma_{h,1})$ for any $h \geqslant \max(g,4)$.

Definition 57. The *metaplectic mapping class group* $\widehat{\mathfrak{M}}(\Sigma)$ is the double covering group of the mapping class group $\mathfrak{M}(\Sigma)$ pulled back from the double covering $Mp_{2g}(\mathbb{R}) \to Sp_{2g}(\mathbb{R})$ of the symplectic group by the metaplectic group along the map

$$\mathfrak{M}(\Sigma) \twoheadrightarrow Sp_{2g}(\mathbb{Z}) \hookrightarrow Sp_{2g}(\mathbb{R}).$$

Lemma 58. *The metaplectic mapping class group $\widehat{\mathfrak{M}}(\Sigma)$ is isomorphic, as a $\mathbb{Z}/2$-central extension of $\mathfrak{M}(\Sigma)$, to the reduction modulo two of the stably universal central extension of $\mathfrak{M}(\Sigma)$.*

Proof. First, we note that it suffices to prove this statement for all sufficiently large g: this is because, for $g < h$, the pullbacks of $\widetilde{\mathfrak{M}}(\Sigma_{h,1})$ and of $\widehat{\mathfrak{M}}(\Sigma_{h,1})$ along the inclusion $\mathfrak{M}(\Sigma_{g,1}) \hookrightarrow \mathfrak{M}(\Sigma_{h,1})$ are $\widetilde{\mathfrak{M}}(\Sigma_{g,1})$ and of $\widehat{\mathfrak{M}}(\Sigma_{g,1})$ respectively. For $\widetilde{\mathfrak{M}}$ this is explained in Definition 56, whereas for $\widehat{\mathfrak{M}}$ it follows from the fact that the pullback of $Mp_{2h}(\mathbb{R})$ along $Sp_{2g}(\mathbb{R}) \hookrightarrow Sp_{2h}(\mathbb{R})$ is $Mp_{2g}(\mathbb{R})$, which was explained during the proof of Propostiion 55. Thus we may assume that $g \geqslant 4$, so that we are in the stable range for the (co)homology of $\mathfrak{M}(\Sigma)$ and $Sp_{2g}(\mathbb{R})$ in degrees at most 2.

By [8, Lemma A.1 (i) and (xiv)] and homological stability [44, Theorem 1.2], the canonical surjection $\mathfrak{M}(\Sigma) \to Sp_{2g}(\mathbb{Z})$ induces an isomorphism on $H^2(-;\mathbb{Z}/2)$, and moreover we have $H^2(\mathfrak{M}(\Sigma);\mathbb{Z}/2) \cong H^2(Sp_{2g}(\mathbb{Z});\mathbb{Z}/2) \cong \mathbb{Z}/2$. The metaplectic extension $Mp_{2g}(\mathbb{Z}) \to Sp_{2g}(\mathbb{Z})$ is a non-trivial central extension, so it represents the unique non-trivial element of $H^2(Sp_{2g}(\mathbb{Z});\mathbb{Z}/2)$. So the metaplectic mapping class group $\widehat{\mathfrak{M}}(\Sigma)$ is the unique non-trivial $\mathbb{Z}/2$-central extension of $\mathfrak{M}(\Sigma)$. Let us denote by $\widetilde{\mathfrak{M}}(\Sigma)^{(2)}$ the reduction modulo two of the universal central extension $\widetilde{\mathfrak{M}}(\Sigma)$ of $\mathfrak{M}(\Sigma)$. This is a $\mathbb{Z}/2$-central extension, and it will suffice to show that it is non-trivial (since it must then be isomorphic to $\widehat{\mathfrak{M}}(\Sigma)$). Now, by universality of $\widetilde{\mathfrak{M}}(\Sigma)$, there is a morphism of central extensions $\widetilde{\mathfrak{M}}(\Sigma) \to \widehat{\mathfrak{M}}(\Sigma)$, which must factor as

$$\widetilde{\mathfrak{M}}(\Sigma) \longrightarrow \widetilde{\mathfrak{M}}(\Sigma)^{(2)} \longrightarrow \widehat{\mathfrak{M}}(\Sigma),$$

since the target is a $\mathbb{Z}/2$-central extension. If $\widetilde{\mathfrak{M}}(\Sigma)^{(2)}$ were a trivial extension, it would admit a section, and therefore so would $\widehat{\mathfrak{M}}(\Sigma)$, by composition with the right-hand map above. But $\widehat{\mathfrak{M}}(\Sigma)$ is a non-trivial extension, and hence so is $\widetilde{\mathfrak{M}}(\Sigma)^{(2)}$. ∎

6.3 Constructing the unitary representations.

We now put together everything from §6.1 and §6.2 to prove Theorems B and E. From the previous two subsections, we have the following diagram:

$$\begin{tikzcd}
\mathfrak{M}(\Sigma) \arrow[r, "\Phi=(\mathfrak{s},\mathfrak{d})"] & \mathrm{Aut}^+(\mathcal{H}) \arrow[r] & \mathrm{Aut}^+(\mathcal{H}_{\mathbb{R}}) \arrow[r, "T"] & PU(W) \\
& Sp(H) \ltimes H \arrow[r] \arrow[u, "\cong"] & Sp(H_{\mathbb{R}}) \ltimes H_{\mathbb{R}} \arrow[u, "\cong"] & \\
\mathrm{Mor}(\Sigma) \arrow[r, "\mathfrak{s}"] \arrow[uu] & Sp(H) \arrow[r] \arrow[u] & Sp(H_{\mathbb{R}}), \arrow[u] \arrow[uur, "R"'] &
\end{tikzcd} \tag{58}$$

where unmarked arrows denote inclusions. For $g \geqslant 4$, by universality of $\widetilde{\mathfrak{M}}(\Sigma)$, there is a morphism of central extensions

$$\begin{tikzcd} \widetilde{\mathfrak{M}}(\Sigma) \arrow[r] \arrow[d, "\pi"'] & U(W) \arrow[d] \\ \mathfrak{M}(\Sigma) \arrow[r] & PU(W) \end{tikzcd} \tag{59}$$

where the bottom horizontal arrow is the composition along the top of (58). Moreover, this extends to all $g \geqslant 1$ as follows. Consider the commutative diagram[2]

$$\begin{tikzcd} \mathfrak{M}(\Sigma_{g,1}) \arrow[r, "(\mathfrak{s},\mathfrak{d})"] \arrow[d, hook] & Sp_{2g}(\mathbb{R}) \ltimes \mathbb{R}^{2g} \arrow[r, "T"] \arrow[d] & PU(W) \arrow[d] & U(W) \arrow[l, two heads] \arrow[d] \\ \mathfrak{M}(\Sigma_{h,1}) \arrow[r, "(\mathfrak{s},\mathfrak{d})"] & Sp_{2h}(\mathbb{R}) \ltimes \mathbb{R}^{2h} \arrow[r, "T"] & PU(W) & U(W) \arrow[l, two heads] \end{tikzcd} \tag{60}$$

where the right-hand square arises as explained below diagram (57). Commutativity of the left-hand square follows from Lemma 24 and commutativity of the middle square follows from the defining property of T (Definition 53). Let us write $\overline{\mathfrak{M}}(\Sigma_{g,1})$ for the pullback of $U(W) \to PU(W)$ along $T \circ (\mathfrak{s}, \mathfrak{d})$, and similarly for $\overline{\mathfrak{M}}(\Sigma_{h,1})$. Then $\overline{\mathfrak{M}}(\Sigma_{g,1})$ is the pullback of $\overline{\mathfrak{M}}(\Sigma_{h,1})$ along the inclusion of mapping class groups. From Definition 56, we also have that $\widetilde{\mathfrak{M}}(\Sigma_{g,1})$ is the pullback of $\widetilde{\mathfrak{M}}(\Sigma_{h,1})$ along the inclusion.

If we now take $h \geqslant 4$, then $\widetilde{\mathfrak{M}}(\Sigma_{h,1})$ is by definition the *universal* central extension, so there is a unique morphism of central extensions $\widetilde{\mathfrak{M}}(\Sigma_{h,1}) \to \overline{\mathfrak{M}}(\Sigma_{h,1})$. Pulling back along the inclusion, we obtain a canonical morphism of central extensions $\widetilde{\mathfrak{M}}(\Sigma_{g,1}) \to \overline{\mathfrak{M}}(\Sigma_{g,1})$, even though $\widetilde{\mathfrak{M}}(\Sigma_{g,1})$ is not universal for $g \leqslant 3$. This gives us the desired morphism of central extensions (59).

By Proposition 55, there are morphisms of central extensions

$$\begin{tikzcd} \widehat{\mathrm{Mor}}(\Sigma) \arrow[r] \arrow[d, "\widehat{\pi}"'] & Mp(H_{\mathbb{R}}) \arrow[rr] \arrow[dr] & & \overline{Sp}(H_{\mathbb{R}}) \arrow[r] \arrow[dl] & U(W) \arrow[d] \\ \mathrm{Mor}(\Sigma) \arrow[rr] & & Sp(H_{\mathbb{R}}) \arrow[rr, "R"] & & PU(W) \end{tikzcd} \tag{61}$$

where the two quadrilaterals are pullbacks and the top middle horizontal map is the inclusion of Proposition 55. The composition along the bottom of (61) is the composition along the bottom of (58), and $\widehat{\mathrm{Mor}}(\Sigma)$ denotes the restriction of the metaplectic central extension $\widehat{\mathfrak{M}}(\Sigma)$ of the mapping class group to the Morita subgroup $\mathrm{Mor}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$.

Notation 59. We denote by

$$S \colon \widetilde{\mathfrak{M}}(\Sigma) \longrightarrow U(W)$$

[2] We freely pass between the different notations $Sp_{2g}(\mathbb{R}) = Sp(H_{\mathbb{R}})$ and $\mathbb{R}^{2g} = H_{\mathbb{R}}$, and similarly for the integral versions, depending on whether or not we wish to emphasise the genus g.

the top horizontal map of (59), and by

$$\widehat{S}\colon \widehat{\mathrm{Mor}}(\Sigma) \longrightarrow U(W)$$

the composition along the top of (61).

By commutativity of (58) and the fact (Lemma 58) that the metaplectic mapping class group is the reduction modulo two of the stably universal central extension of the mapping class group, these two maps are related by the following commutative diagram:

$$\begin{tikzcd}
\widetilde{\mathrm{Mor}}(\Sigma) \arrow[r] \arrow[d] & \widetilde{\mathfrak{M}}(\Sigma) \arrow[r, "S"] \arrow[dd, "\pi"'] & U(W) \arrow[dd] \\
\widehat{\mathrm{Mor}}(\Sigma) \arrow[d, "\widehat{\pi}"'] \arrow[urr, "\widehat{S}"'] & & \\
\mathrm{Mor}(\Sigma) \arrow[r] & \mathfrak{M}(\Sigma) \arrow[r] & PU(W)
\end{tikzcd} \tag{62}$$

where the right-hand square is (59), the left-hand square is a pullback and the lower quadrilateral is the outer rectangle of (61).

Notation 60. By abuse of notation, we write

$$\rho_W\colon \mathcal{H} \longrightarrow U(W)$$

for the restriction of the Schrödinger representation (51) to the subgroup $\mathcal{H} \subset \mathcal{H}_{\mathbb{R}}$.

A consequence of Definition 53 and the above commutative diagrams is the following.

Lemma 61. *For $g \in \widetilde{\mathfrak{M}}(\Sigma)$ and $h \in \mathcal{H}$, we have the following equation in $U(W)$:*

$$S(g).\rho_W(h).S(g)^{-1} = \rho_W(\Phi(\pi(g))(h)). \tag{63}$$

Similarly, for $g \in \widehat{\mathrm{Mor}}(\Sigma)$ and $h \in \mathcal{H}$, we have the following equation in $U(W)$:

$$\widehat{S}(g).\rho_W(h).\widehat{S}(g)^{-1} = \rho_W(\Phi(\widehat{\pi}(g))(h)). \tag{64}$$

We now use this to construct *untwisted* unitary representations of the universal central extension $\widetilde{\mathfrak{M}}(\Sigma)$ of $\mathfrak{M}(\Sigma)$ and the metaplectic double covering $\widehat{\mathrm{Mor}}(\Sigma)$ of $\mathrm{Mor}(\Sigma)$ on the homology of configuration spaces with coefficients in the Schrödinger representation.

Let $\widetilde{\mathcal{C}}_n(\Sigma) \to \mathcal{C}_n(\Sigma)$ denote the connected covering of $\mathcal{C}_n(\Sigma)$ corresponding to the kernel of the surjective homomorphism $\pi_1(\mathcal{C}_n(\Sigma)) \twoheadrightarrow \mathcal{H}$. This is a principal $\mathcal{H}$-bundle. Taking free abelian groups fibrewise, we obtain

$$\mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)] \longrightarrow \mathcal{C}_n(\Sigma), \tag{65}$$

which is a bundle of right $\mathbb{Z}[\mathcal{H}]$-modules. Via the Schrödinger representation ρ_W, the Hilbert space W becomes a left $\mathbb{Z}[\mathcal{H}]$-module, and we may take a fibrewise tensor product to obtain

$$\mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)] \otimes_{\mathbb{Z}[\mathcal{H}]} W \longrightarrow \mathcal{C}_n(\Sigma), \tag{66}$$

which is a bundle of Hilbert spaces. There is a natural action of the mapping class group $\mathfrak{M}(\Sigma)$ (up to homotopy) on the base space $\mathcal{C}_n(\Sigma)$, and the induced action on $\pi_1(\mathcal{C}_n(\Sigma))$ preserves the kernel of the surjection $\pi_1(\mathcal{C}_n(\Sigma)) \twoheadrightarrow \mathcal{H}$ (Proposition 14), so that there is a well-defined twisted action of $\mathfrak{M}(\Sigma)$ on the bundle (65), in the following sense. There are homomorphisms

$$\begin{aligned} \alpha &\colon \mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}\bigl(\mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)] \longrightarrow \mathcal{C}_n(\Sigma)\bigr) \\ \Phi &\colon \mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}(\mathcal{H}) \end{aligned}$$

such that, for any $g \in \mathfrak{M}(\Sigma)$, $h \in \mathcal{H}$ and $m \in \mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)]$, we have

$$\alpha(g)(m.h) = \alpha(g)(m).\Phi(g)(h). \tag{67}$$

In other words, Φ measures the failure of α to be an action by fibrewise $\mathbb{Z}[\mathcal{H}]$-module automorphisms. In the above, the target of α is the group of $\mathbb{Z}$-module automorphisms of the bundle (65), in other words the group of self-homeomorphisms of the total space $\mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)]$ that preserve the fibres of the projection and that are $\mathbb{Z}$-linear (but not necessarily $\mathbb{Z}[\mathcal{H}]$-linear) on each fibre.

Theorem 62. *The stably universal central extension $\widetilde{\mathfrak{M}}(\Sigma)$ of $\mathfrak{M}(\Sigma)$ acts on (66) by Hilbert space bundle automorphisms*

$$\gamma \colon \widetilde{\mathfrak{M}}(\Sigma) \longrightarrow U\Bigl(\mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)] \otimes_{\mathbb{Z}[\mathcal{H}]} W \longrightarrow \mathcal{C}_n(\Sigma)\Bigr)$$

via the formula

$$\gamma(g)(m \otimes v) = \alpha(\pi(g))(m) \otimes S(g)(v) \tag{68}$$

for all $g \in \widetilde{\mathfrak{M}}(\Sigma)$, $m \in \mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)]$ and $v \in W$.

Proof. This is similar in spirit to the proofs of Lemmas 32 and 33. The key property that needs to be verified is the following. Since we are taking the (fibrewise) tensor product over $\mathbb{Z}[\mathcal{H}]$, we have that $m.h \otimes v = \rho_W(h)(v)$ for any $h \in \mathcal{H}$, $m \in \mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)]$ and $v \in W$. (Note that we denote the right $\mathcal{H}$-action on the fibres of $\mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)]$ simply by juxtaposition, whereas the left $\mathcal{H}$-action on W is the Schrödinger representation, denoted by ρ_W.) We therefore have to verify that, for each fixed $g \in \widetilde{\mathfrak{M}}(\Sigma)$, the formula (68) gives the same answer when applied to $m.h \otimes v$ or to $m \otimes \rho_W(h)(v)$. To see this, we calculate:

$$\begin{aligned}
\gamma(g)(m.h \otimes v) &= \alpha(\pi(g))(m.h) \otimes S(g)(v) && \text{by definition} \\
&= \alpha(\pi(g))(m).\Phi(\pi(g))(h) \otimes S(g)(v) && \text{by eq. (67)} \\
&= \alpha(\pi(g))(m) \otimes \rho_W(\Phi(\pi(g))(h))(S(g)(v)) && \text{since } \otimes \text{ is over } \mathbb{Z}[\mathcal{H}] \\
&= \alpha(\pi(g))(m) \otimes S(g) \circ \rho_W(h) \circ S(g)^{-1}(S(g)(v)) && \text{by eq. (63) [Lemma 61]} \\
&= \alpha(\pi(g))(m) \otimes S(g)(\rho_W(h)(v)) && \text{simplifying} \\
&= \gamma(g)(m \otimes \rho_W(h)(v)). && \text{by definition}
\end{aligned}$$

This tells us that the formula (68) gives a well-defined bundle automorphism of (66) for each fixed $g \in \widetilde{\mathfrak{M}}(\Sigma)$. It is then routine to verify that this bundle automorphism is $\mathbb{R}$-linear and unitary on fibres – i.e. it is an automorphism of bundles of Hilbert spaces – and that γ is a group homomorphism. ∎

Theorem 63. *The metaplectic double cover* $\widehat{\mathrm{Mor}}(\Sigma)$ *of* $\mathrm{Mor}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$ *acts on* (66) *by Hilbert space bundle automorphisms*

$$\gamma \colon \widehat{\mathrm{Mor}}(\Sigma) \longrightarrow U\Big(\mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)] \otimes_{\mathbb{Z}[\mathcal{H}]} W \longrightarrow \mathcal{C}_n(\Sigma)\Big)$$

via the formula

$$\gamma(g)(m \otimes v) = \alpha(\widehat{\pi}(g))(m) \otimes \widehat{S}(g)(v) \tag{69}$$

for all $g \in \widehat{\mathrm{Mor}}(\Sigma)$*,* $m \in \mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)]$ *and* $v \in W$*.*

Proof. The proof is exactly the same as above, using formula (64) of Lemma 61 instead of formula (63). ∎

Theorem 64 (Theorem B)**.** *The action of the mapping class group on the Borel-Moore homology of the configuration space* $\mathcal{C}_n(\Sigma)$ *with coefficients in the Schrödinger representation induces a well-defined complex unitary representation of the stably universal central extension* $\widetilde{\mathfrak{M}}(\Sigma)$ *of the mapping class group* $\mathfrak{M}(\Sigma)$*:*

$$\widetilde{\mathfrak{M}}(\Sigma) \longrightarrow U\big(H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); W)\big) \tag{70}$$

lifting a natural projective unitary representation of $\mathfrak{M}(\Sigma)$ *on the same space.*

Proof. This is an immediate consequence of Theorem 62. In more detail, according to that theorem, we have a well-defined functor from the group $\widetilde{\mathfrak{M}}(\Sigma)$ to the category of spaces equipped with bundles of Hilbert spaces. Moreover, elements of the mapping class group fix the boundary of Σ pointwise, so the action of the mapping class group on $\mathcal{C}_n(\Sigma)$ preserves the subspace $\mathcal{C}_n(\Sigma, \partial^-(\Sigma))$. Thus we in fact have a functor from $\widetilde{\mathfrak{M}}(\Sigma)$ to the category of *pairs* of spaces equipped with bundles of Hilbert spaces. On the other hand, relative twisted Borel-Moore homology $H_n^{BM}(-)$ is a functor from the category of pairs of spaces equipped with bundles of Hilbert spaces (and bundle maps whose underlying map of spaces is proper) to the category of Hilbert spaces. Composing these two functors, we obtain the desired unitary representation of $\widetilde{\mathfrak{M}}(\Sigma)$.

This automatically descends to a projective unitary representation on $\mathfrak{M}(\Sigma)$ since it sends the kernel of the central extension $\widetilde{\mathfrak{M}}(\Sigma) \to \mathfrak{M}(\Sigma)$ into the centre of the unitary group, which is the kernel of the projection onto the projective unitary group. ∎

Theorem 65. *Restricting to the Morita subgroup of the mapping class group, its action on the Borel-Moore homology of the configuration space* $\mathcal{C}_n(\Sigma)$ *with coefficients in the Schrödinger representation induces a well-defined complex unitary representation of its metaplectic double cover:*

$$\widehat{\mathrm{Mor}}(\Sigma) \longrightarrow U\big(H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); W)\big). \tag{71}$$

Moreover, (71) *is the reduction modulo two of the restriction of* (70)*, in the sense that the following diagram commutes:*

$$\begin{tikzcd} \widetilde{\mathrm{Mor}}(\Sigma) \arrow[r, hook] \arrow[d, two heads, "\mathrm{mod}\ 2"'] & \widetilde{\mathfrak{M}}(\Sigma) \arrow[r, "(70)"] & U\big(H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); W)\big) \\ \widehat{\mathrm{Mor}}(\Sigma) \arrow[urr, "(71)"'] & & \end{tikzcd} \tag{72}$$

Proof. The first part is an immediate consequence of Theorem 63. The relation between (70) and (71) follows from the commutative diagram (62). ∎

To complete the proof of Theorem E, we prove:

Lemma 66. *The metaplectic* $\mathbb{Z}/2$*-central extension* $\widehat{\mathrm{Mor}}(\Sigma) \to \mathrm{Mor}(\Sigma)$ *is trivial, i.e. it is isomorphic to the product* $\mathrm{Mor}(\Sigma) \times \mathbb{Z}/2$*.*

Proof. We first note that it suffices to prove this statement for all sufficiently large g. This is because the inclusion of mapping class groups $\mathfrak{M}(\Sigma_{g,1}) \hookrightarrow \mathfrak{M}(\Sigma_{h,1})$ restricts to an inclusion of Morita subgroups $\mathrm{Mor}(\Sigma_{g,1}) \hookrightarrow \mathrm{Mor}(\Sigma_{h,1})$ (Corollary 25), and the pullback of $\widehat{\mathrm{Mor}}(\Sigma_{h,1})$ along this inclusion is $\widehat{\mathrm{Mor}}(\Sigma_{g,1})$, for any $g < h$. This last fact follows from the fact that the pullback of $Mp_{2h}(\mathbb{R})$ along $Sp_{2g}(\mathbb{R}) \hookrightarrow Sp_{2h}(\mathbb{R})$ is $Mp_{2g}(\mathbb{R})$, which was explained during the proof of Propostiion 55. We now assume that $g \geqslant 4$ for the rest of the proof.

Recall from the proof of Proposition 55 that there is an embedding of central extensions $Mp_{2g}(\mathbb{R}) \hookrightarrow Sp_{2g}(\mathbb{R})^{(8)}$. Pulling back along the projection $\mathfrak{M}(\Sigma) \to Sp_{2g}(\mathbb{R})$, we get an embedding of central extensions $\widehat{\mathfrak{M}}(\Sigma) \hookrightarrow \mathfrak{M}(\Sigma)^{(8)}$, where $\mathfrak{M}(\Sigma)^{(8)}$ is classified by $-[\tau].8\mathbb{Z} \in H^2(\mathfrak{M}(\Sigma); \mathbb{Z}/8) \cong \mathbb{Z}/8$. Now, $H^2(\mathfrak{M}(\Sigma); \mathbb{Z}) \cong \mathbb{Z}$, generated by the first Chern class c_1, and we have $[\tau] = 4c_1$. The intersection cocycle $c \colon \mathfrak{M}(\Sigma) \times \mathfrak{M}(\Sigma) \to \mathbb{Z}$ of Morita [34] is given by $c(f,g) = \mathfrak{d}(f^{-1}) \cdot \mathfrak{d}(g)$, where $\mathfrak{d} \colon \mathfrak{M}(\Sigma) \to H$ is Morita's crossed homomorphism, and we have $[c] = 12c_1$ in $H^2(\mathfrak{M}(\Sigma); \mathbb{Z})$. Thus, in particular, we have $3[\tau] = [c]$. Since $\mathrm{Mor}(\Sigma) = \ker(\mathfrak{d})$, Morita's cocycle c vanishes on $\mathrm{Mor}(\Sigma)$, and so after restricting to the Morita subgroup we have $3[\tau] = [c] = 0 \in H^2(\mathrm{Mor}(\Sigma); \mathbb{Z})$. Reducing modulo 8 we therefore also have $3[\tau].8\mathbb{Z} = 0 \in H^2(\mathrm{Mor}(\Sigma); \mathbb{Z}/8)$. But this cohomology group is a $\mathbb{Z}/8$-module, and 3 is invertible modulo 8, so we may divide by 3 and deduce that $[\tau].8\mathbb{Z} = 0 \in H^2(\mathrm{Mor}(\Sigma); \mathbb{Z}/8)$. Hence the restriction $\mathrm{Mor}(\Sigma)^{(8)}$ of $\mathfrak{M}(\Sigma)^{(8)}$ to the Morita subgroup is a trivial extension. From the embedding $\widehat{\mathrm{Mor}}(\Sigma) \hookrightarrow \mathrm{Mor}(\Sigma)^{(8)}$, it follows that $\widehat{\mathrm{Mor}}(\Sigma)$ is also a trivial extension. ∎

Remark 67. The embedding $\widehat{\mathrm{Mor}}(\Sigma) \hookrightarrow \mathrm{Mor}(\Sigma)^{(8)}$ is essential to the above proof. The extension $\widehat{\mathrm{Mor}}(\Sigma)$ is classified by $c_1.2\mathbb{Z} \in H^2(\mathrm{Mor}(\Sigma); \mathbb{Z}/2)$, by Lemma 58. The vanishing of Morita's cocycle c on $\mathrm{Mor}(\Sigma)$ implies that we have $12c_1 = [c] = 0 \in H^2(\mathrm{Mor}(\Sigma); \mathbb{Z})$, but this only implies the tautology that $12c_1.2\mathbb{Z} = 0 \in H^2(\mathrm{Mor}(\Sigma); \mathbb{Z}/2)$ after reduction modulo two, from which we cannot deduce that $c_1.2\mathbb{Z} = 0$.

Theorem 68 (Theorem E)**.** *The action of the Morita subgroup of the mapping class group on the Borel-Moore homology of the configuration space $\mathcal{C}_n(\Sigma)$ with coefficients in the Schrödinger representation induces a well-defined complex unitary representation*

$$\mathrm{Mor}(\Sigma) \longrightarrow U\big(H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); W)\big) \tag{73}$$

that lifts the natural projective unitary representation of $\mathrm{Mor}(\Sigma)$ *on this Hilbert space.*

Proof. Let us abbreviate $\mathcal{V}_n = H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); W)$. We first note that the homomorphism (71) must send the kernel of the central extension to the centre of $U(\mathcal{V}_n)$, and so it descends to

$$\mathrm{Mor}(\Sigma) \longrightarrow PU(\mathcal{V}_n).$$

This is the natural projective action of $\mathrm{Mor}(\Sigma)$. To lift it to a linear action, we compose (71) with any section of the central extension $\widehat{\mathrm{Mor}}(\Sigma) \to \mathrm{Mor}(\Sigma)$, which exists by Lemma 66. ■

Remark 69. We are not forced to take Borel-Moore homology, or relative homology, in our constructions. The core of the construction is to obtain an action of (a subgroup or an extension of) the mapping class group $\mathfrak{M}(\Sigma)$ on $\mathcal{C}_n(\Sigma)$ equipped with a certain local system. Once we have this, we may apply any (twisted) homology theory we like, possibly relative to a subspace of $\mathcal{C}_n(\Sigma)$ that is invariant under the action of $\mathfrak{M}(\Sigma)$, such as $\partial^-\mathcal{C}_n(\Sigma) = \mathcal{C}_n(\Sigma, \partial^-(\Sigma))$ or $\partial\mathcal{C}_n(\Sigma) = \mathcal{C}_n(\Sigma, \partial\Sigma)$. In particular, we may simply take ordinary (twisted) homology, in which case the right-hand side of diagram (72) becomes $U(H_n(\mathcal{C}_n(\Sigma); W))$. We have chosen to take Borel-Moore (twisted) homology relative to the subspace $\partial^-\mathcal{C}_n(\Sigma)$ in Theorems 64 and 68, since this homology group admits an easily-described basis, as shown in §2.

6.4 Finite dimensional Schrödinger representations.

For an integer $N \geqslant 2$, the finite-dimensional Schrödinger representation is an action of the discrete Heisenberg group $\mathcal{H}$ on the Hilbert space $W_N = L^2((\mathbb{Z}/N)^g)$, which may be defined as follows:

$$\left[\varpi_N\left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right)\psi\right](s) = e^{i\pi\frac{k-p\cdot q}{N}} e^{i\frac{2\pi}{N}p\cdot s}\psi(s-q). \tag{74}$$

Note that this matches the generic formula with $\hbar = \frac{2\pi}{N}$. It may also be constructed by composing the natural quotient

$$\mathcal{H} = \mathbb{Z}^g \ltimes \mathbb{Z}^{g+1} \longrightarrow\!\!\!\!\rightarrow (\mathbb{Z}/N)^g \ltimes \big(\mathbb{Z}/2N \times (\mathbb{Z}/N)^g\big)$$

with the representation of the right-hand group induced from the one-dimensional representation $\mathbb{Z}/2N \times (\mathbb{Z}/N)^g \twoheadrightarrow \mathbb{Z}/2N \hookrightarrow \mathbb{S}^1$, where the second map is $t \mapsto \exp\left(\frac{\pi i t}{N}\right)$.

We may adapt the above construction using W_N in place of W, when N is even, using the analogue of the Stone-von Neumann theorem for W_N proven in [21, Theorem 2.4] (see also [22, Theorem 3.2] and [20, Theorem 2.6]). As before, we obtain from this a (now finite-dimensional) Shale-Weil projective representation of the symplectic group which lifts to a linear representation on the metaplectic group. Using this to untwist the action on the (stably) universal central extension of the mapping class group, we obtain:

Theorem 70 (Theorem C)**.** *The action of the mapping class group on the Borel-Moore homology of the configuration space $\mathcal{C}_n(\Sigma)$ with coefficients in W_N induces a well-defined complex unitary representation of the stably universal central extension $\widetilde{\mathfrak{M}}(\Sigma)$ of the mapping class group $\mathfrak{M}(\Sigma)$ on the $\binom{2g+n-1}{n}N^g$-dimensional complex Hilbert space*

$$\mathcal{V}_{N,n} = H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); W_N\big), \tag{75}$$

lifting a natural projective unitary representation $\mathfrak{M}(\Sigma) \to PU(\mathcal{V}_{N,n})$.

As described in [19], the Shale-Weil representation may be realised geometrically by theta functions, and it may also be interpreted and extended as a $U(1)$-TQFT. An alternative exposition may be found in [18]; see for example the statement for the resolution of the projective ambiguity in Chapter 3, Theorem 4.1.

7 Relation to the Moriyama and Magnus representations

In this section we study the kernels of the representations that we have constructed, and prove Proposition F. The proof will use:

- a theorem of Moriyama [36], which identifies each $\mathfrak{J}(i)$ with the kernel of a certain homological representation of $\mathfrak{M}(\Sigma)$;
- a theorem of Suzuki [41], which identifies the Magnus kernel with the kernel of a certain twisted homological representation of $\mathfrak{M}(\Sigma)$ (a homological interpretation of the Magnus representation, which was originally defined via Fox calculus);

together with a study of the connections between our representations and those of Moriyama and Suzuki.

7.1 The Moriyama representation.

Moriyama [36] studied the action of the mapping class group $\mathfrak{M}(\Sigma)$ on the homology group $H_n^{BM}(\mathcal{F}_n(\Sigma'); \mathbb{Z})$ with trivial coefficients, where Σ' denotes Σ minus a point on its boundary and $\mathcal{F}_n(-)$ denotes the ordered configuration space. On the other hand, our construction (39) (Theorem 41) may be re-interpreted as a twisted representation

$$\mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}^{\mathrm{tw}}_{\mathbb{Z}[\mathcal{H}]}\Big(H_n^{BM}(\mathcal{C}_n(\Sigma'); \mathbb{Z}[\mathcal{H}])\Big). \tag{76}$$

We pause to explain this re-interpretation. We must first of all explain the twisted automorphism group on the right-hand side of (76). Let us write $\mathrm{Mod}_\bullet$ for the category whose objects are pairs (R, M) of a ring R and a right R-module M, and whose morphisms are pairs $(\theta \colon R \to R', \varphi \colon M \to M')$ such that $\varphi(mr) = \varphi(m)\theta(r)$. The automorphism group of (R, M) in $\mathrm{Mod}_\bullet$ is written $\mathrm{Aut}_R^{\mathrm{tw}}(M)$; note that this is generally larger than the automorphism group $\mathrm{Aut}_R(M)$ of M in Mod_R.

If we set $V = \mathbb{Z}[\mathcal{H}]$, then (39) is a functor of the form $\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H}) \to \mathrm{Mod}_{\mathbb{Z}[\mathcal{H}]}$. But any functor of the form $\mathrm{Ac}(G \curvearrowright K) \to \mathrm{Mod}_{\mathbb{Z}[K]}$ corresponds to a homomorphism $G \to \mathrm{Aut}^{\mathrm{tw}}_{\mathbb{Z}[K]}(M)$, where the $\mathbb{Z}[K]$-module M is the image of the object $\mathrm{id} \in \mathrm{Ac}(G \curvearrowright K)$. (Compare Remark 39, which describes the reverse procedure.) Thus (39) corresponds to a homomorphism

$$\mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}^{\mathrm{tw}}_{\mathbb{Z}[\mathcal{H}]}\Big(H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathbb{Z}[\mathcal{H}])\Big).$$

Finally, removing a point (equivalently, removing the closed interval $\partial^-(\Sigma)$) from the boundary of Σ corresponds, on Borel-Moore homology of configuration spaces $\mathcal{C}_n(\Sigma)$, to taking homology relative to the subspace $\mathcal{C}_n(\Sigma, \partial^-(\Sigma))$ of configurations having at least one point in the interval. Thus $H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathbb{Z}[\mathcal{H}])$ and $H_n^{BM}(\mathcal{C}_n(\Sigma'); \mathbb{Z}[\mathcal{H}])$ are isomorphic as $\mathbb{Z}[\mathcal{H}]$-modules, and we obtain (76).

When $n = 2$, Moriyama's representation is a quotient of ours: there is a quotient of groups $\mathcal{H} \twoheadrightarrow \mathbb{Z}/2 = \mathfrak{S}_2$ given by sending $\sigma \mapsto \sigma$ and $a_i, b_i \mapsto 1$, which induces a quotient of twisted $\mathfrak{M}(\Sigma)$-representations

$$H_2^{BM}(\mathcal{C}_2(\Sigma'); \mathbb{Z}[\mathcal{H}]) \longrightarrow\!\!\!\!\rightarrow H_2^{BM}(\mathcal{C}_2(\Sigma'); \mathbb{Z}[\mathfrak{S}_2]) \cong H_2^{BM}(\mathcal{F}_2(\Sigma'); \mathbb{Z}), \tag{77}$$

where the isomorphism on the right-hand side follows from Shapiro's lemma. (Shapiro's lemma holds for arbitrary coverings with ordinary homology, and for *finite* coverings with Borel-Moore homology.) It follows that the kernel of our representation is a subgroup of the kernel of $H_2^{BM}(\mathcal{F}_2(\Sigma'); \mathbb{Z})$, which was proven by Moriyama to be the Johnson kernel $\mathfrak{J}(2)$. In §8 we will compute the action of a genus-1 separating twist $T_\gamma \in \mathfrak{J}(2)$ on $H_2^{BM}(\mathcal{C}_2(\Sigma'); \mathbb{Z}[\mathcal{H}])$, and in particular show that it is (very) non-trivial; see Theorem 79. Thus the kernel of $H_2^{BM}(\mathcal{C}_2(\Sigma'); \mathbb{Z}[\mathcal{H}])$ is strictly smaller than $\mathfrak{J}(2)$.

For any $n \geqslant 2$, we have a quotient of twisted $\mathfrak{M}(\Sigma)$-representations

$$H_n^{BM}(\mathcal{C}_n(\Sigma'); \mathbb{Z}[\mathcal{H}]) \longrightarrow\!\!\!\!\rightarrow H_n^{BM}(\mathcal{C}_n(\Sigma'); \mathbb{Z}).$$

By Shapiro's lemma and the universal coefficient theorem (together with the fact that the integral Borel-Moore homology of $\mathcal{C}_n(\Sigma')$ is free abelian), there are isomorphisms

$$H_n^{BM}(\mathcal{F}_n(\Sigma'); \mathbb{Z}) \cong H_n^{BM}(\mathcal{C}_n(\Sigma'); \mathbb{Z}[\mathfrak{S}_n]) \cong H_n^{BM}(\mathcal{C}_n(\Sigma'); \mathbb{Z}) \otimes \mathbb{Z}[\mathfrak{S}_n].$$

Thus the kernel of the representation $H_n^{BM}(\mathcal{C}_n(\Sigma'); \mathbb{Z})$ is the same as the kernel of the representation $H_n^{BM}(\mathcal{F}_n(\Sigma'); \mathbb{Z})$, since $\mathfrak{M}(\Sigma)$ acts trivially on $\mathfrak{S}_n$. (This is also shown in [37].) The latter kernel was proven by Moriyama to be the nth term $\mathfrak{J}(n)$ of the Johnson filtration.

Summarising this discussion, we have:

Proposition 71. *The kernel of the twisted $\mathfrak{M}(\Sigma)$-representation (76) is contained in the nth term $\mathfrak{J}(n)$ of the Johnson filtration. When $n = 2$ it is moreover a proper subgroup of the Johnson kernel $\mathfrak{J}(2)$.*

7.2 The Magnus representation.

The kernel of our representation (76) is also contained in the kernel of the Magnus representation. This may be seen as follows. The $\mathfrak{M}(\Sigma)$-equivariant surjection $\mathcal{H} \twoheadrightarrow H$ induces a quotient of twisted $\mathfrak{M}(\Sigma)$-representations

$$H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z}[\mathcal{H}]) \longrightarrow\!\!\!\!\!\rightarrow H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z}[H]). \tag{78}$$

By a similar argument as above, the kernel of the representation $H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z}[H])$ is the same as the kernel of the representation $H_n^{BM}(\mathcal{F}_n(\Sigma');\mathbb{Z}[H])$. Moreover, it is shown in [37] that there is an inclusion of $\mathfrak{M}(\Sigma)$-representations

$$\left[H_1^{BM}(\mathcal{F}_1(\Sigma');\mathbb{Z}[H])\right]^{\otimes n} \lhook\joinrel\longrightarrow H_n^{BM}(\mathcal{F}_n(\Sigma');\mathbb{Z}[H]), \tag{79}$$

where $H_1^{BM}(\mathcal{F}_1(\Sigma');\mathbb{Z}[H])$ is the *Magnus representation* of $\mathfrak{M}(\Sigma)$. (This uses [41], which gives a homological interpretation of the Magnus representation, which was originally defined via Fox calculus.) The maps of representations (78) and (79) imply that

$$\begin{aligned} \ker\left[H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z}[\mathcal{H}])\right] \;\subseteq \ker\left[H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z}[H])\right] \\ = \;\ker\left[H_n^{BM}(\mathcal{F}_n(\Sigma');\mathbb{Z}[H])\right] \;\subseteq\; \ker(\mathrm{Magnus}). \end{aligned}$$

Combining this with Proposition 71 and writing $\mathrm{Mag}(\Sigma)$ for the kernel of the Magnus representation, we have:

Proposition 72. *The kernel of* (76) *is contained in* $\mathfrak{J}(n) \cap \mathrm{Mag}(\Sigma)$.

It is known [40, §6] that the kernel of the Magnus representation does not contain $\mathfrak{J}(n)$ for any $n \geqslant 1$, so this implies that the kernel of (76) is strictly contained in $\mathfrak{J}(n)$.

7.3 Other related representations.

Recently, the representations of $\mathfrak{M}(\Sigma)$ on the ordinary (rather than Borel-Moore) homology of the configuration space $\mathcal{F}_n(\Sigma)$ has been studied[3] by Bianchi, Miller and Wilson [10]: they prove that, for each n and i, the kernel of the $\mathfrak{M}(\Sigma)$-representation $H_i(\mathcal{F}_n(\Sigma);\mathbb{Z})$ contains $\mathfrak{J}(i)$, and is in general strictly *larger* than $\mathfrak{J}(i)$. They conjecture that the kernel of the $\mathfrak{M}(\Sigma)$-representation on the total homology $H_*(\mathcal{F}_n(\Sigma);\mathbb{Z})$ is equal to the subgroup generated by $\mathfrak{J}(n)$ and the Dehn twist around the boundary.

The $\mathfrak{M}(\Sigma)$-representation $H_i(\mathcal{C}_n(\Sigma);\mathbb{F})$ for certain field coefficients $\mathbb{F}$ has been completely computed. For $\mathbb{F} = \mathbb{F}_2$ it has been computed in [9, Theorem 3.2] and is *symplectic*, i.e. it restricts to the trivial action on the Torelli group $\mathfrak{T}(\Sigma) = \mathfrak{J}(1)$. For $\mathbb{F} = \mathbb{Q}$ it has been computed in [39, Theorem 1.4] and is not symplectic, but it restricts to the trivial action on the Johnson kernel $\mathfrak{J}(2)$.

[3]This is equivalent to studying the homology of $\mathcal{F}_n(\Sigma')$ since the inclusion $\mathcal{F}_n(\Sigma') \hookrightarrow \mathcal{F}_n(\Sigma)$ is a homotopy equivalence. On the other hand, for *Borel-Moore* homology, this would not be equivalent, since the inclusion is not a *proper* homotopy equivalence.

8 Computations for $n = 2$

In this section we will do some computations in the case $n = 2$, when V is the regular representation $\mathbb{Z}[\mathcal{H}]$ of the Heisenberg group $\mathcal{H}$. The main goal is to obtain in this case an explicit formula for the action of a Dehn twist along a genus 1 separating curve. When the surface has genus 1 this is displayed in Figure 9; in general, the formula is given by Theorem 79. One may compare these calculations to the calculations of An and Ko [1, page 274], although they consider representations of surface braid groups whereas we consider representations of mapping class groups.

We will start with the case where the surface itself has genus 1, where we first compute the action of the Dehn twists T_a, T_b, along the standard essential curves a, b. Since T_a and T_b act non-trivially on the local system $\mathbb{Z}[\mathcal{H}]$, they do not act by automorphisms, but give isomorphisms in the category of spaces with local systems, which, after taking homology with local coefficients, give isomorphisms in the category of $\mathbb{Z}[\mathcal{H}]$-modules. We refer to [17, Chapter 5] for functoriality results concerning homology with local coefficients. The upshot is a twisted action of the full mapping class group $\mathfrak{M}(\Sigma)$. As described in §5.1, a *twisted action* (over a ring R) of a group G is a functor $\mathrm{Ac}(G \curvearrowright X) \to \mathrm{Mod}_R$, where $\mathrm{Ac}(G \curvearrowright X)$ is the *action groupoid* associated to an action of G on some set X. In the present setting, we have $G = \mathfrak{M}(\Sigma)$, $X = \mathcal{H}$ and $R = \mathbb{Z}[\mathcal{H}]$, so the twisted representation is of the form

$$\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H}) \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathcal{H}]}. \tag{80}$$

We briefly recall from §5.2 some of the relevant details of the construction of this twisted representation. Let $f \in \mathfrak{M}(\Sigma)$ and let $f_{\mathcal{H}}$ be its action on the Heisenberg group. Then the Heisenberg homology $H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})$ is defined from the regular covering space $\widetilde{\mathcal{C}}_n(\Sigma)$ associated with the quotient $\phi\colon \mathbb{B}_n(\Sigma) \twoheadrightarrow \mathcal{H}$. As explained in §5.2, at the level of homology there is a twisted functoriality and, in particular, associated with f, we get a right $\mathbb{Z}[\mathcal{H}]$-linear isomorphism

$$\mathcal{C}_n(f)_*\colon H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})_{f_{\mathcal{H}}^{-1}} \longrightarrow H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})\ .$$

Our choice for twisting on the source with $f_{\mathcal{H}}^{-1}$ rather than on the target with $f_{\mathcal{H}}$ will slightly simplify the writing of the matrix. Note also that when working with coefficients in a left $\mathbb{Z}[\mathcal{H}]$-representation V the twisting on the right by $f_{\mathcal{H}}^{-1}$ will correspond to twisting the action on V by $f_{\mathcal{H}}$. More generally, for any $\tau \in \mathrm{Aut}(\mathcal{H})$, we have a *shifted* isomorphism

$$(\mathcal{C}_n(f)_*)_\tau\colon H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})_{f_{\mathcal{H}}^{-1}\circ\tau} \longrightarrow H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})_\tau\ .$$

In terms of the functor (80) on the action groupoid, the above map $(\mathcal{C}_n(f)_*)_\tau$ is the image of the morphism $f\colon \tau^{-1} \circ f_{\mathcal{H}} \to \tau^{-1}$. If f, g are two mapping classes, the composition formula (functoriality of (80)) states the following:

$$\mathcal{C}_n(g \circ f)_* = \mathcal{C}_n(g)_* \circ (\mathcal{C}_n(f)_*)_{g_{\mathcal{H}}^{-1}}\ .$$

We will need to compute compositions in specific bases. Note that a basis B for a right $\mathbb{Z}[\mathcal{H}]$-module M is also a basis for the twisted module M_τ, $\tau \in \mathrm{Aut}(\mathcal{H})$.

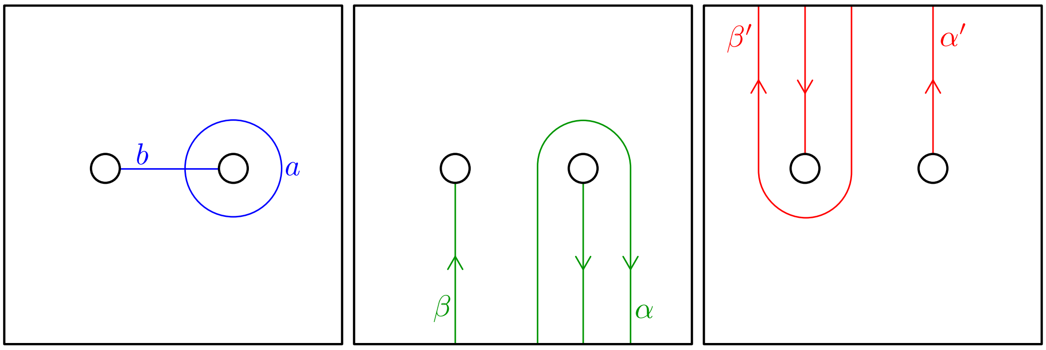


Figure 6: The closed curves a, b and the arcs α, β, α', β'.

Lemma 73. *Let M, M' be free right $\mathbb{Z}[\mathcal{H}]$-modules with fixed bases B, B' and let $\tau \in \operatorname{Aut}(\mathcal{H})$. If a $\mathbb{Z}[\mathcal{H}]$-linear map $F\colon M \to M'$ has matrix $\operatorname{Mat}(F)$ in the bases B, B', then the matrix of the shifted $\mathbb{Z}[\mathcal{H}]$-linear map $F_\tau\colon M_\tau \to M'_\tau$ is $\tau^{-1}(\operatorname{Mat}(F))$.*

The action of τ^{-1} on the matrix is given by its action on each individual coefficient.
Proof. We note that the maps F and F_τ are equal as maps of $\mathbb{Z}$-modules. Let $B = (e_j)_{j\in J}$, $B' = (f_i)_{i\in I}$, $\operatorname{Mat}(F) = (m_{i,j})_{i\in I, j\in J}$. Then for coefficients $h_j \in \mathcal{H}$, $j \in J$, we have

$$\begin{aligned} F_\tau\left(\sum_j e_j \cdot_\tau h_j\right) &= F\left(\sum_j e_j \tau(h_j)\right) \\ &= \sum_{i,j} f_i m_{ij} \tau(h_j) \\ &= \sum_{i,j} f_i \cdot_\tau \tau^{-1}(m_{ij}) h_j, \end{aligned}$$

which gives the stated result. ■

8.1 Genus one

Here we consider the genus 1 case with $n = 2$ configuration points. Let a, b be simple closed curves representing the symplectic basis of $H_1(\Sigma)$ previously denoted (a_1, b_1). We will use the same notation a, b for the curves, their homology classes and their lifts in $\mathcal{H}$ which were previously $\tilde{a}$, $\tilde{b}$. The corresponding Dehn twists are denoted by T_a, T_b. The homology $H_2^{BM}(\mathcal{C}_2(\Sigma), \mathcal{C}_2(\Sigma, \partial^-(\Sigma)); \mathcal{H})$ is a free module of rank 3 over $\mathbb{Z}[\mathcal{H}]$. A basis was described in Theorem 11. Here we replace γ_1, γ_2 by α, β depicted in Figure 6, and the basis is denoted by $w(\alpha) = E_{(2,0)}$, $w(\beta) = E_{(0,2)}$, $v(\alpha, \beta) = E_{(1,1)}$. In more detail, $w(\alpha)$ is represented by the cycle in the 2-point configuration space given by the subspace where both points lie on the arc α. Similarly, $w(\beta)$ is given by the subspace where both points lie on β and $v(\alpha, \beta)$ is given by the subspace where exactly one point lies on each of these arcs.

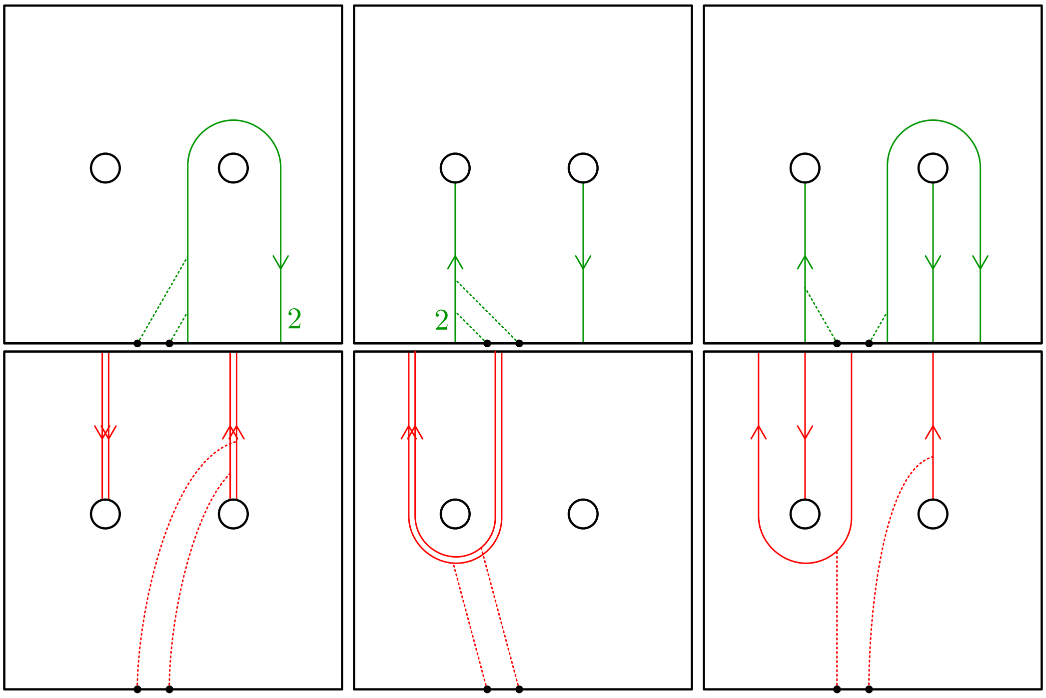


Figure 7: Tethers.

In fact, we have to be even more careful to specify these elements precisely, since the preceding description only determines them *up to the action of the deck transformation group* $\mathcal{H}$, because we have just described cycles in the configuration space $\mathcal{C}_2(\Sigma)$, whereas cycles for the Heisenberg-twisted homology are cycles in the covering space $\widetilde{\mathcal{C}}_2(\Sigma)$. To specify such a lifting of the cycles in $\mathcal{C}_2(\Sigma)$ that we have described, we first choose once and for all a base configuration c_0 contained in $\partial\Sigma$ and a lift of c_0 to $\widetilde{\mathcal{C}}_2(\Sigma)$. A lift of a cycle to $\widetilde{\mathcal{C}}_2(\Sigma)$ is therefore determined by a choice of a path (called a "tether") in $\mathcal{C}_2(\Sigma)$ from a point in the cycle to c_0. For $w(\alpha)$, $w(\beta)$ and $v(\alpha,\beta)$, we choose these tethers as illustrated in the top row of Figure 7.

By Poincaré duality, and the fact that $\mathcal{C}_2(\Sigma)$ is a connected, oriented 4-manifold with boundary $\mathcal{C}_2(\Sigma,\partial\Sigma) = \{c \in \mathcal{C}_2(\Sigma) \mid c \cap \partial\Sigma \neq \varnothing\}$, we have a non-degenerate pairing

$$\langle -,-\rangle \colon H_2^{BM}(\mathcal{C}_2(\Sigma),\partial^-;\mathbb{Z}[\mathcal{H}]) \otimes H_2(\mathcal{C}_2(\Sigma),\partial^+;\mathbb{Z}[\mathcal{H}]) \longrightarrow \mathbb{Z}[\mathcal{H}], \tag{81}$$

where $\partial^\pm$ is an abbreviation of $\mathcal{C}_2(\Sigma,\partial^\pm(\Sigma))$, and we note that the boundary $\partial\mathcal{C}_2(\Sigma) = \mathcal{C}_2(\Sigma,\partial\Sigma)$ decomposes as $\partial^+ \cup \partial^-$, corresponding to the decomposition of the boundary of the surface $\partial\Sigma = \partial^+(\Sigma) \cup \partial^-(\Sigma)$. (Formally, it is a *manifold triad*.) There are natural elements of $H_2(\mathcal{C}_2(\Sigma),\partial^+;\mathbb{Z}[\mathcal{H}])$ that are dual to $w(\alpha)$, $w(\beta)$ and $v(\alpha,\beta)$ with respect to this pairing, which we denote by $\overline{w}(\alpha')$, $\overline{w}(\beta')$ and $v(\alpha',\beta')$ respectively. The element $v(\alpha',\beta')$ is defined exactly as above: it is given by the subspace of 2-point configurations where one point lies on each of the arcs α' and β' of Figure 6. The element $\overline{w}(\alpha')$ is defined as follows: first replace the arc α' with two parallel copies α'_1 and α'_2 (as in the bottom-left of Figure 7), and then $\overline{w}(\alpha')$ is given by the subspace of 2-point configurations where one point lies on each of α'_1 and α'_2. The element $\overline{w}(\beta')$ is defined exactly analogously. Again, in order to specify these elements precisely, we have to choose tethers, which are illustrated in the bottom row of Figure 7.

A practical description of the pairing (81) is as follows. Let $x = w(\gamma)$ or $v(\gamma, \delta)$ for disjoint arcs γ, δ with endpoints on $\partial^-(\Sigma)$, and choose a tether for x, namely a path t_x from c_0 to a point in x. Similarly, let $y = \overline{w}(\epsilon)$ or $v(\epsilon, \zeta)$ for disjoint arcs ϵ, ζ with endpoints on $\partial^+(\Sigma)$, and choose a tether t_y for y. Suppose that the arcs $\gamma \sqcup \delta$ intersect the arcs $\epsilon \sqcup \zeta$ transversely. Then the pairing (81) is given by the formula

$$\big\langle [x, t_x], [y, t_y] \big\rangle = \sum_{p=\{p_1,p_2\} \in x \cap y} \mathrm{sgn}(p_1).\mathrm{sgn}(p_2).\mathrm{sgn}(\ell_p).\phi(\ell_p), \tag{82}$$

where $\ell_p \in \mathbb{B}_2(\Sigma)$ is the loop in $\mathcal{C}_2(\Sigma)$ given by concatenating:

- the tether t_x from c_0 to a point in x,
- a path in x to the intersection point p,
- a path in y from p to the endpoint of the tether t_y,
- the reverse of the tether t_y back to c_0,

$\mathrm{sgn}(\ell_p) \in \{+1, -1\}$ is the sign of the induced permutation in $\mathfrak{S}_2$ and $\mathrm{sgn}(p_i) \in \{+1, -1\}$ is given by the sign convention in Figure 8. (In fact, there should be an extra global -1 sign on the right-hand side of (82), which we have suppressed for simplicity. Thus (82) is really a formula for $-$(81). This global sign ambiguity does not affect our calculations, since all we need is *a* non-degenerate pairing of the form (81), and any non-degenerate pairing multiplied by a unit is again a non-degenerate pairing. This extra global sign also appears in Bigelow's formula [12, page 475, ten lines above Lemma 2.1]. See Appendix C for further explanations of these signs.)

With this description of (81), it is easy to verify that the matrix

$$\begin{pmatrix} \langle [w(\alpha)], [\overline{w}(\alpha')] \rangle & \langle [w(\alpha)], [\overline{w}(\beta')] \rangle & \langle [w(\alpha)], [v(\alpha', \beta')] \rangle \\ \langle [w(\beta)], [\overline{w}(\alpha')] \rangle & \langle [w(\beta)], [\overline{w}(\beta')] \rangle & \langle [w(\beta)], [v(\alpha', \beta')] \rangle \\ \langle [v(\alpha,\beta)], [\overline{w}(\alpha')] \rangle & \langle [v(\alpha,\beta)], [\overline{w}(\beta')] \rangle & \langle [v(\alpha,\beta)], [v(\alpha', \beta')] \rangle \end{pmatrix} \in \mathrm{Mat}_{3,3}(\mathbb{Z}[\mathcal{H}])$$

is the identity; this is the precise sense in which these two 3-tuples of elements are "dual" to each other.[4]

Theorem 74. *With respect to the ordered basis* $(w(\alpha), w(\beta), v(\alpha, \beta))$*:*

(a) *The matrix for the isomorphism*

$$\mathcal{T}_a = \mathcal{C}_2(T_a)_* \colon H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}])_{(T_a^{-1})_{\mathcal{H}}} \longrightarrow H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}])$$

is

$$M_a = \begin{pmatrix} 1 & u^2 a^{-2} b^2 & (u^{-1} - 1) a^{-1} b \\ 0 & 1 & 0 \\ 0 & -a^{-1} b & 1 \end{pmatrix}.$$

[4] Since we know that $w(\alpha)$, $w(\beta)$ and $v(\alpha, \beta)$ form a basis for the $\mathbb{Z}[\mathcal{H}]$-module $H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}])$, it follows that the elements $\overline{w}(\alpha')$, $\overline{w}(\beta')$ and $v(\alpha', \beta')$ are $\mathbb{Z}[\mathcal{H}]$-linearly independent in the $\mathbb{Z}[\mathcal{H}]$-module $H_2(\mathcal{C}_2(\Sigma), \partial^+; \mathbb{Z}[\mathcal{H}])$, although they do not necessarily span it.

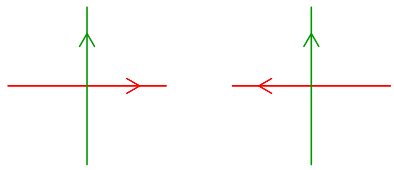


Figure 8: Sign convention for intersections between cycles representing elements of the homology groups $H_n^{BM}(\mathcal{C}_n(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}])$ and $H_n(\mathcal{C}_n(\Sigma), \partial^+; \mathbb{Z}[\mathcal{H}])$.

(b) *The matrix for the isomorphism*

$$\mathcal{T}_b = \mathcal{C}_2(T_b)_* \colon H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}])_{(T_b^{-1})_{\mathcal{H}}} \longrightarrow H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}])$$

is

$$M_b = \begin{pmatrix} u^{-2}b^2 & 0 & 0 \\ -u^{-1} & 1 & 1-u^{-1} \\ -u^{-1}b & 0 & b \end{pmatrix}.$$

Proof. Let us simplify the notation for the basis and the corresponding dual homology classes by

$$(e_1, e_2, e_3) = (w(\alpha), w(\beta), v(\alpha, \beta)) \qquad (e_1', e_2', e_3') = (\overline{w}(\alpha'), \overline{w}(\beta'), v(\alpha', \beta')).$$

Using the non-degenerate pairing (81) and elementary linear algebra, we have that

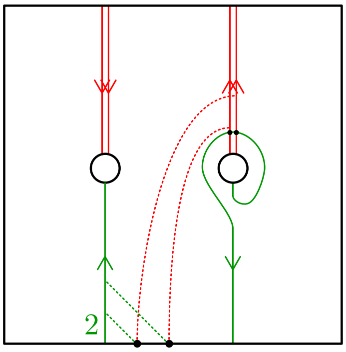
$$\mathcal{C}_2(f)_*(e_i) = \sum_{j=1}^{3} e_j . \langle \mathcal{C}_2(f)_*(e_i), e_j' \rangle$$

for any $f \in \mathfrak{M}(\Sigma)$. Computing the matrices M_a and M_b therefore consists in computing $\langle \mathcal{T}_a(e_i), e_j' \rangle$ and $\langle \mathcal{T}_b(e_i), e_j' \rangle$ for $i, j \in \{1, 2, 3\}$. We will explain how to compute two of these 18 elements of $\mathbb{Z}[\mathcal{H}]$, the remaining 16 being left as exercises for the reader. In each case the idea is the same: apply the Dehn twist to the explicit cycle (described above) representing the homology class e_i, and then use the formula (82) to compute the pairing.

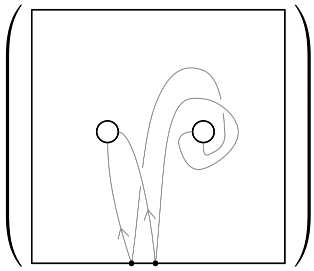
We begin by computing $\langle \mathcal{T}_a(e_2), e_1' \rangle = \langle \mathcal{T}_a(w(\beta)), \overline{w}(\alpha') \rangle$, the top-middle entry of M_a.

$$\langle \mathcal{T}_a(w(\beta)), \overline{w}(\alpha') \rangle = \langle w(T_a(\beta)), \overline{w}(\alpha') \rangle$$

$=$



$= (-1).(-1).(+1).\phi$



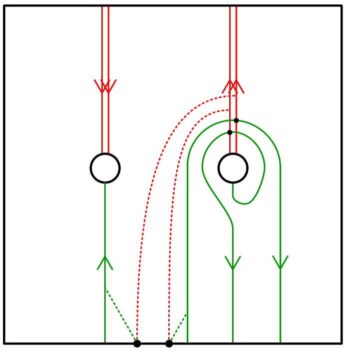
$$= \phi(a^{-1} b \sigma^{-1} a^{-1} b \sigma)$$
$$= a^{-1} b a^{-1} b$$
$$= u^2 a^{-2} b^2.$$

We next calculate $\langle \mathcal{T}_a(e_3), e_1' \rangle = \langle \mathcal{T}_a(v(\alpha, \beta)), \overline{w}(\alpha') \rangle$, the top-right entry of M_a. This is slightly more complicated, since in this case there are two intersection points in the configuration space $\mathcal{C}_2(\Sigma)$, so we obtain a Heisenberg polynomial (i.e. element of $\mathbb{Z}[\mathcal{H}]$)

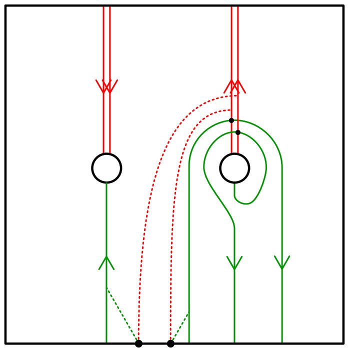
with two terms.

$$\langle \mathcal{T}_a(v(\alpha,\beta)), \overline{w}(\alpha')\rangle = \langle v(\alpha, T_a(\beta)), \overline{w}(\alpha')\rangle$$

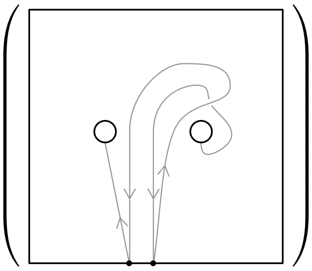
$$=$$



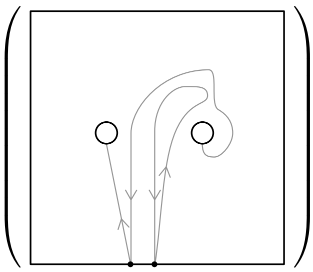
$$+$$



$$= (-1).(+1).(-1).\phi$$



$$+ (+1).(-1).(+1).\phi$$



$$\begin{aligned} &= \phi(\sigma^{-1}a^{-1}b) - \phi(a^{-1}b) \\ &= u^{-1}a^{-1}b - a^{-1}b \\ &= (u^{-1}-1)a^{-1}b. \end{aligned}$$

The other 16 entries of the matrices M_a and M_b may be computed analogously. ∎

Notation 75. To shorten the notation in the following, we will use the abbreviation

$$A := H_2^{BM}(\mathcal{C}_2(\Sigma), \mathcal{C}_2(\Sigma, \partial^-(\Sigma)); \mathcal{H}) = H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}]).$$

Remark 76 (*Verifying the braid relation.*)**.** Recall that $\mathfrak{M}(\Sigma_{1,1})$ is generated by T_a and T_b subject to the single relation $T_aT_bT_a = T_bT_aT_b$. It must therefore be the case that the isomorphism

$$A_{(T_aT_bT_a)^{-1}_{\mathcal{H}}} \xrightarrow{(\mathcal{T}_a)_{(T_aT_b)^{-1}_{\mathcal{H}}}} A_{(T_aT_b)^{-1}_{\mathcal{H}}} \xrightarrow{(\mathcal{T}_b)_{(T_a)^{-1}_{\mathcal{H}}}} A_{(T_a)^{-1}_{\mathcal{H}}} \xrightarrow{\mathcal{T}_a} A$$

is equal to the isomorphism

$$A_{(T_bT_aT_b)_{\mathcal{H}}^{-1}} \xrightarrow{(\mathcal{T}_b)_{(T_bT_a)_{\mathcal{H}}^{-1}}} A_{(T_bT_a)_{\mathcal{H}}^{-1}} \xrightarrow{(\mathcal{T}_a)_{(T_b)_{\mathcal{H}}^{-1}}} A_{(T_b)_{\mathcal{H}}^{-1}} \xrightarrow{\mathcal{T}_b} A$$

in other words, using Lemma 73, we must have the following equality of matrices:

$$M_a.(T_a)_{\mathcal{H}}(M_b).(T_aT_b)_{\mathcal{H}}(M_a) = M_b.(T_b)_{\mathcal{H}}(M_a).(T_bT_a)_{\mathcal{H}}(M_b), \tag{83}$$

where M_a and M_b are as in Theorem 74 and the automorphisms $(T_a)_{\mathcal{H}}, (T_b)_{\mathcal{H}} \in \mathrm{Aut}(\mathcal{H})$ are extended linearly to automorphisms of $\mathbb{Z}[\mathcal{H}]$ and thus to automorphisms of matrices over $\mathbb{Z}[\mathcal{H}]$. Indeed, one may calculate that both sides of (83) are equal to

$$\begin{pmatrix} 0 & u^2a^{-2}b^2 & 0 \\ -u^{-1} & 1+(u^{-3}-u^{-2})a^{-1}-u^{-5}a^{-2} & (1-u^{-1})(1+u^{-3}a^{-1}) \\ 0 & -a^{-1}b-u^{-1}a^{-2}b & u^{-1}a^{-1}b \end{pmatrix}. \tag{84}$$

Remark 77 (*The Dehn twist around the boundary.*)**.** In a similar way, we may compute the matrix M_∂ for the action $\mathcal{T}_\partial$ of the Dehn twist T_∂ around the boundary of $\Sigma_{1,1}$. We note that T_∂ lies in the Chillingworth subgroup of $\mathfrak{M}(\Sigma_{1,1})$, so its action on $\mathcal{H}$ is trivial and the action $\mathcal{T}_\partial$ is an *automorphism*

$$\mathcal{T}_\partial \colon A \longrightarrow A.$$

However, to compute its matrix M_∂, it is convenient to decompose $\mathcal{T}_\partial$ into isomorphisms as follows. Write $g = T_aT_bT_a = T_bT_aT_b$, so that $T_\partial = g^4$. Then $\mathcal{T}_\partial$ decomposes as

$$A = A_{g_{\mathcal{H}}^{-4}} \xrightarrow{(\mathcal{T}_g)_{g_{\mathcal{H}}^{-3}}} A_{g_{\mathcal{H}}^{-3}} \xrightarrow{(\mathcal{T}_g)_{g_{\mathcal{H}}^{-2}}} A_{g_{\mathcal{H}}^{-2}} \xrightarrow{(\mathcal{T}_g)_{g_{\mathcal{H}}^{-1}}} A_{g_{\mathcal{H}}^{-1}} \xrightarrow{\mathcal{T}_g} A$$

where $\mathcal{T}_g$ denotes the action of g, given by the matrix (84) above. The matrix M_∂ may therefore be obtained by multiplying together four copies of (84), shifted by the actions of id, $g_{\mathcal{H}}$, $g_{\mathcal{H}}^2$ and $g_{\mathcal{H}}^3$ respectively. This may be implemented in Sage to show that M_∂ is equal to the matrix displayed in Figure 9. More details of these Sage calculations are given in Appendix D.

One may verify explicitly by hand that, if we set $a = b = u^2 = 1$ in the matrix $M_\partial =$ (Figure 9), it simplifies to the identity matrix. This is expected, since applying this specialisation to our representation recovers the second Moriyama representation (as discussed in §7; see in particular the quotient (77) of $\mathfrak{M}(\Sigma)$-representations), whose kernel is the Johnson kernel $\mathfrak{J}(2)$ by [36], which contains T_∂.

8.2 Higher genus

For arbitrary genus $g \geqslant 1$, we view the surface $\Sigma = \Sigma_{g,1}$ as the quotient of the punctured rectangle depicted in Figure 10, where the $2g$ holes are identified in pairs by reflection.

$$\begin{pmatrix}
\substack{u^{-8}b^2+u^{-4}a^{-2}-ua^{-2}b^2+(u^{-1}-u^{-2})a^{-2}b+\\ (u^{-3}-u^{-4})a^{-1}b^2+(u^{-4}-u^{-5})a^{-1}b} &
\substack{(u^2+1-2u^{-1}+u^{-2}+u^{-4})a^{-2}b^2-ua^{-2}b^4+\\ (-u^2+u+u^{-1}-u^{-2})a^{-2}b^3-u^{-3}a^{-2}+\\ (-1+u^{-1}+u^{-3}-u^{-4})a^{-2}b} &
\substack{(-1+2u^{-1}-u^{-2}-u^{-4}+u^{-5})a^{-2}b+\\ (u-1)a^{-2}b^3+(u^2-u-u^{-1}+2u^{-2}-u^{-3})a^{-2}b^2+\\ (-u^{-3}+u^{-4})a^{-1}b+(u^{-4}-u^{-5})a^{-1}b^3+\\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}b^2+\\ (-u^{-3}+u^{-4})a^{-2}} \\
\substack{-u^{-1}-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-2}a^2+\\ (u^{-1}-u^{-2}-u^{-4}+u^{-5})a+u^{-6}a^{-2}+\\ (u^{-3}-u^{-4}-u^{-6}+u^{-7})a^{-1}} &
\substack{1+u^{-2}-u^{-3}+u^{-6}+u^{-6}a^{-2}b^2-u^{-1}b^2+\\ (u^{-3}-u^{-4})a^{-1}b^2+(-1+u^{-1}+u^{-3}-u^{-4})b+\\ (u^{-2}-2u^{-3}+u^{-4}+u^{-6}-u^{-7})a^{-1}b-u^{-5}a^{-2}+\\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}+(u^{-5}-u^{-6})a^{-2}b} &
\substack{(-u^{-6}+u^{-7})a^{-2}b+\\ (u^{-1}-u^{-2}-u^{-4}+2u^{-5}-u^{-6})b+\\ (-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-8})a^{-1}b+\\ 1-u^{-1}+u^{-2}-3u^{-3}+2u^{-4}+u^{-6}-u^{-7}+\\ (-u^{-2}+2u^{-3}-u^{-4}+u^{-5}-2u^{-6}+u^{-7})a^{-1}\\ +(u^{-2}-u^{-3})ab+(-1+u^{-1}+u^{-3}-u^{-4})a+\\ (-u^{-5}+u^{-6})a^{-2}} \\
\substack{-u^{-6}ab+(-u^{-3}+u^{-4}-u^{-7})b-u^{-4}+\\ (u^{-1}-u^{-4}+u^{-5})a^{-1}b+u^{-2}a^{-2}b+\\ (-u^{-3}+u^{-6})a^{-1}+u^{-5}a^{-2}} &
\substack{(-1-u^{-2}+2u^{-3}-u^{-6})a^{-1}b+u^{-1}a^{-1}b^3+\\ u^{-2}a^{-2}b^3+(1-u^{-1}-u^{-3}+u^{-4})a^{-1}b^2+\\ (u^{-1}-u^{-2}+u^{-5})a^{-2}b^2+(-u^{-1}+u^{-4}-u^{-5})a^{-2}b+\\ (u^{-2}-u^{-5})a^{-1}-u^{-4}a^{-2}} &
\substack{u^{-3}+(u^{-2}-u^{-3}-u^{-5}+u^{-6})a^{-1}+\\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-1}b^2+\\ (-u^{-2}+u^{-3})a^{-2}b^2+\\ (-1+u^{-1}+2u^{-3}-3u^{-4}+u^{-7})a^{-1}b+\\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-2}b+(-u^{-4}+u^{-5})b^2+\\ (u^{-2}-u^{-3}-u^{-5}+u^{-6})b+(-u^{-4}+u^{-5})a^{-2}}
\end{pmatrix}$$

Figure 9: The action of the Dehn twist around the boundary of $\Sigma_{1,1}$.

The arcs α_i, β_i for $i \in \{1, \ldots, g\}$ form a symplectic basis for the first homology of Σ relative to the lower edge of the rectangle. Following Theorem 11, a basis for the free $\mathbb{Z}[\mathcal{H}]$-module $H_2^{BM}(\mathcal{C}_2(\Sigma), \mathcal{C}_2(\Sigma, \partial^-(\Sigma)); \mathcal{H})$ is given by the homology classes represented by the 2-cycles

- $w(\epsilon)$ for $\epsilon \in \{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g\}$,
- $v(\delta, \epsilon)$ for $\delta, \epsilon \in \{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g\}$ with $\delta < \epsilon$

where we use the ordering $\alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_g < \beta_g$. Here $w(\epsilon)$ denotes the subspace of configurations where both points lie on ϵ and $v(\delta, \epsilon)$ denotes the subspace of configurations where one point lies on each of δ and ϵ. As in the genus 1 setting, we have to be more careful to specify these elements precisely; this is done by choosing, for each of the 2-cycles listed above, a path (called a "tether") in $\mathcal{C}_2(\Sigma)$ from a point in the cycle to c_0, the base configuration, which is contained in the bottom edge of the rectangle. Note that the space of configurations of two points in the bottom edge of the rectangle is contractible, so it is equivalent to choose a path in $\mathcal{C}_2(\Sigma)$ from a point in the cycle to *any* configuration contained in the bottom edge of the rectangle.

For cycles of the form $w(\epsilon)$, we may choose tethers exactly as in the genus 1 setting: see the top-left and top-middle of Figure 7. For cycles of the form $v(\alpha_i, \beta_i)$, we may also choose tethers exactly as in the genus 1 setting: see the top-right of Figure 7. For other cycles of the form $v(\delta, \epsilon)$, we choose tethers as illustrated in Figure 11.

Exactly as in the genus 1 setting, there is a non-degenerate pairing (81) defined via Poincaré duality for the 4-manifold-with-boundary $\mathcal{C}_2(\Sigma)$. Associated to the collection of arcs α'_i, β'_i illustrated in Figure 10 there are elements of $H_2(\mathcal{C}_2(\Sigma), \partial^+; \mathcal{H})$:

- $\overline{w}(\epsilon)$ for $\epsilon \in \{\alpha'_1, \beta'_1, \alpha'_2, \beta'_2, \ldots, \alpha'_g, \beta'_g\}$,

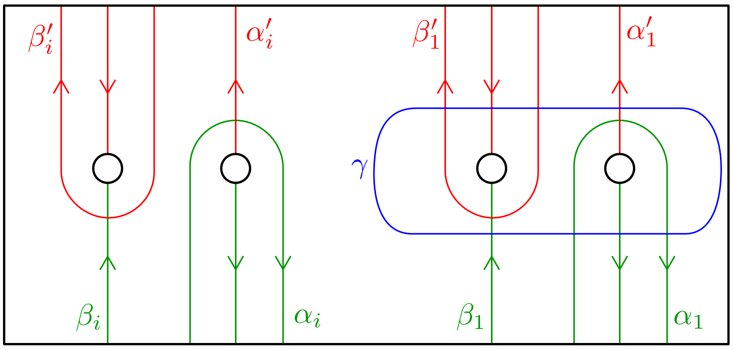


Figure 10: The arcs $\alpha_i, \beta_i, \alpha_i', \beta_i'$ and the closed genus-one-separating curve γ.

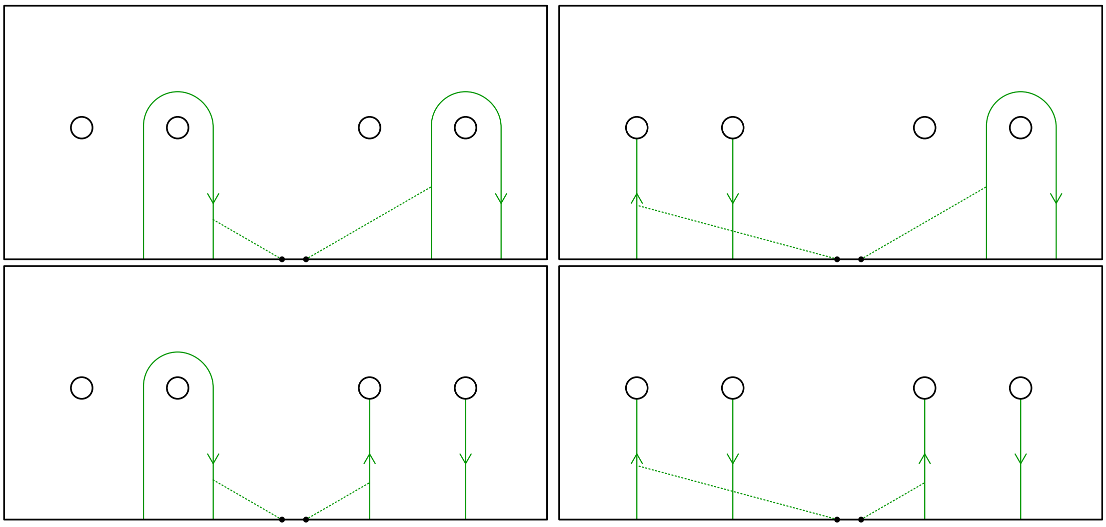


Figure 11: More tethers.

- $v(\delta, \epsilon)$ for $\delta, \epsilon \in \{\alpha_1', \beta_1', \alpha_2', \beta_2', \ldots, \alpha_g', \beta_g'\}$ with $\delta < \epsilon$

where we use the ordering $\alpha_1' < \beta_1' < \alpha_2' < \cdots < \alpha_g' < \beta_g'$. Here, $\overline{w}(\epsilon)$ is the subspace of configurations where one point lies on each of ϵ^+ and ϵ^-, where ϵ^+, ϵ^- are two parallel, disjoint copies of ϵ. As above, we specify these elements precisely by choosing tethers (paths in $\mathcal{C}_2(\Sigma)$ from a point on the cycle to a configurations contained in the bottom edge of the rectangle). For elements of the form $\overline{w}(\epsilon)$ or $v(\alpha_i', \beta_i')$, we choose these exactly as in the genus 1 setting; see the bottom row of Figure 7. For other elements of the form $v(\delta, \epsilon)$, we choose them as illustrated in Figure 12.

Remark 78. These choices of tethers may seem a little arbitrary, and indeed they are; however, any different choice would have the effect simply of changing the chosen basis for the Heisenberg homology $H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathcal{H})$ by rescaling each basis vector by a unit of $\mathbb{Z}[\mathcal{H}]$. This would have the effect of conjugating the matrices that we calculate by an invertible diagonal matrix.

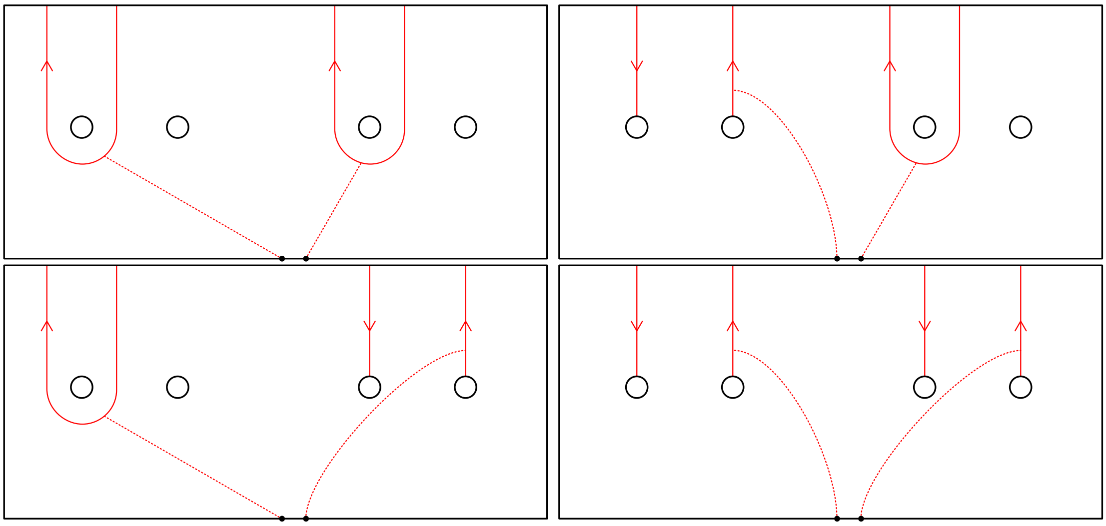


Figure 12: Even more tethers.

The geometric formula (82) for the non-degenerate pairing $\langle - , - \rangle$ holds exactly as in the genus 1 setting, and one may easily verify using this formula that the bases

$$\begin{aligned} \mathcal{B} &= \{w(\epsilon), v(\delta,\epsilon) \mid \delta < \epsilon \in \{\alpha_1,\ldots,\beta_g\}\} \\ \mathcal{B}' &= \{\overline{w}(\epsilon), v(\delta,\epsilon) \mid \delta < \epsilon \in \{\alpha_1',\ldots,\beta_g'\}\} \end{aligned} \tag{85}$$

for $H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathcal{H})$ and $H_2(\mathcal{C}_2(\Sigma), \partial^+; \mathcal{H})$ respectively are dual with respect to this pairing. Choose a total ordering of $\mathcal{B}$ as follows:

- $w(\alpha_1)$, $w(\beta_1)$, $v(\alpha_1,\beta_1)$,
- $v(\alpha_1,\epsilon)$ for $\epsilon = \alpha_2,\beta_2,\ldots,\alpha_g,\beta_g$,
- $v(\beta_1,\epsilon)$ for $\epsilon = \alpha_2,\beta_2,\ldots,\alpha_g,\beta_g$,
- followed by all other basis elements in any order,

and similarly for $\mathcal{B}'$. Denote by γ the genus-1 separating curve in Σ pictured in Figure 10.

Theorem 79. *With respect to the ordered bases (85), the matrix for the automorphism $\mathcal{T}_\gamma = \mathcal{C}_2(T_\gamma)_*$ of $H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathcal{H})$ is given in block form as*

$$M_\gamma = \begin{pmatrix} \Lambda & 0 & 0 & 0 \\ 0 & p.I & r.I & 0 \\ 0 & q.I & s.I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \tag{86}$$

where Λ is the 3×3 matrix depicted in Figure 9, the middle two columns and rows each have width/height $2g-2$ and the Heisenberg polynomials $p,q,r,s \in \mathbb{Z}[\mathcal{H}]$ are:

- $p = -a^{-1}b + u^{-2}b + u^{-2}a^{-1}$,
- $q = 1 - a + u^{-2} - u^{-2}a^{-1}$,

- $r = a^{-1}(-b + b^2 + u^{-2} - u^{-2}b)$,
- $s = 1 - b + u^{-2} + u^{-2}a^{-1}b - u^{-2}a^{-1}$,

where we are abbreviating the elements $a_1, b_1 \in \mathcal{H}$ as a, b respectively.

Proof. As in the proof of Theorem 74, this reduces to computing $\langle \mathcal{T}_\gamma(e_i), e'_j \rangle$ as e_i and e'_j run through the ordered bases (85).

First note that the basis elements come in three types: those entirely supported in the genus-1 subsurface containing γ (the first three), those supported partially in this subsurface and partially in the complementary genus-$(g-1)$ subsurface (the next $4g-4$) and those supported entirely outside of the genus-1 subsurface (the rest). The Dehn twist T_γ does not mix these two complementary subsurfaces, so M_γ is a block matrix with respect to this partition.

The top-left 3×3 matrix involves only the basis elements $w(\alpha_1)$, $w(\beta_1)$, $v(\alpha_1, \beta_1)$ and their duals, and so the calculation of this submatrix is identical to the calculation in genus 1, which is given by the matrix in Figure 9.

The bottom-right submatrix involves only basis elements supported outside of the genus-1 subsurface containing γ, so the effect of $\mathcal{T}_\gamma$ is the identity on these elements.

It remains to calculate the middle $(4g-4) \times (4g-4)$ submatrix, which records the effect of $\mathcal{T}_\gamma$ on $v(\alpha_1, \epsilon)$ and $v(\beta_1, \epsilon)$ for $\epsilon \in \{\alpha_2, \ldots, \beta_g\}$. Since $\epsilon \cap \gamma = \varnothing$, we must have

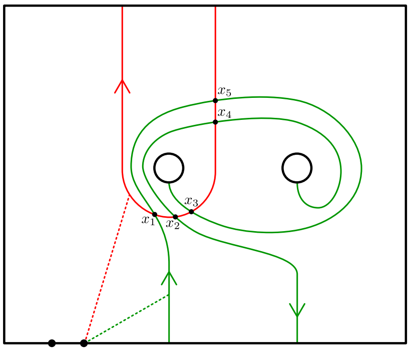
$$\mathcal{T}_\gamma(v(\alpha_1, \epsilon)) = p_\epsilon . v(\alpha_1, \epsilon) + q_\epsilon . v(\beta_1, \epsilon)$$
$$\mathcal{T}_\gamma(v(\beta_1, \epsilon)) = r_\epsilon . v(\alpha_1, \epsilon) + s_\epsilon . v(\beta_1, \epsilon)$$

for some $p_\epsilon, q_\epsilon, r_\epsilon, s_\epsilon \in \mathbb{Z}[\mathcal{H}]$. Precisely, we have

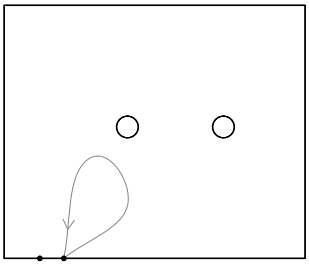
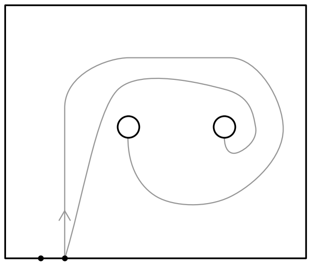
$$p_\epsilon = \big\langle v(T_\gamma(\alpha_1), \epsilon), v(\alpha'_1, \epsilon') \big\rangle \qquad q_\epsilon = \big\langle v(T_\gamma(\alpha_1), \epsilon), v(\beta'_1, \epsilon') \big\rangle$$
$$r_\epsilon = \big\langle v(T_\gamma(\beta_1), \epsilon), v(\alpha'_1, \epsilon') \big\rangle \qquad s_\epsilon = \big\langle v(T_\gamma(\beta_1), \epsilon), v(\beta'_1, \epsilon') \big\rangle,$$

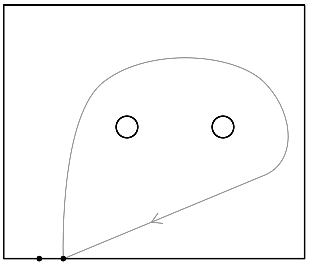
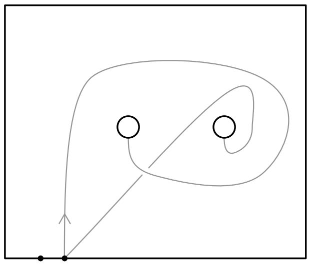
where ϵ' denotes the dual of ϵ, and we have again used the fact that $\epsilon \cap \gamma = \varnothing$ to rewrite $\mathcal{T}_\gamma(v(\alpha_1, \epsilon)) = v(T_\gamma(\alpha_1), T_\gamma(\epsilon)) = v(T_\gamma(\alpha_1), \epsilon)$ and similarly for $\mathcal{T}_\gamma(v(\alpha_1, \epsilon))$. From these formulas and (82) it is clear that $p_\epsilon, q_\epsilon, r_\epsilon, s_\epsilon$ do not in fact depend on ϵ. Indeed, when computing these values of the non-degenerate pairing, we may ignore one of the two configuration points (the one that starts on the left in the base configuration and which travels via the arcs ϵ and ϵ'), since it contributes neither to the signs nor to the loops ℓ_p in the formula (82). We will compute $s_\epsilon = s$, leaving the computation of the other three polynomials as exercises for the reader. In the following computations, as mentioned above, we ignore one of the two configuration points, since it does not contribute anything non-trivial to the formula (82).

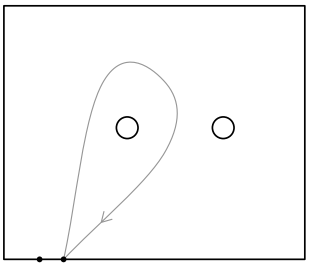
$$s = \langle v(T_\gamma(\beta_1), \epsilon), v(\beta_1', \epsilon') \rangle$$

$=$ 

(5 intersection points: $x_1, \ldots, x_5$)

$= \phi\Bigg($  $\Bigg) - \phi\Bigg($  $\Bigg)$

$+ \phi\Bigg($  $\Bigg) + \phi\Bigg($  $\Bigg)$

$- \phi\Bigg($  $\Bigg)$

$$\begin{aligned} &= \phi(\ ) - \phi(\sigma^{-1}b^{-1}aba^{-1}bab^{-1}a^{-1}b\sigma) + \phi(\sigma^{-1}ab^{-1}a^{-1}b\sigma) \\ &\quad + \phi(\sigma^{-1}a^{-1}bab^{-1}a^{-1}b\sigma) - \phi(\sigma^{-1}b^{-1}a^{-1}b\sigma) \\ &= 1 - b + u^{-2} + u^{-2}a^{-1}b - u^{-2}a^{-1}. \end{aligned}$$

■

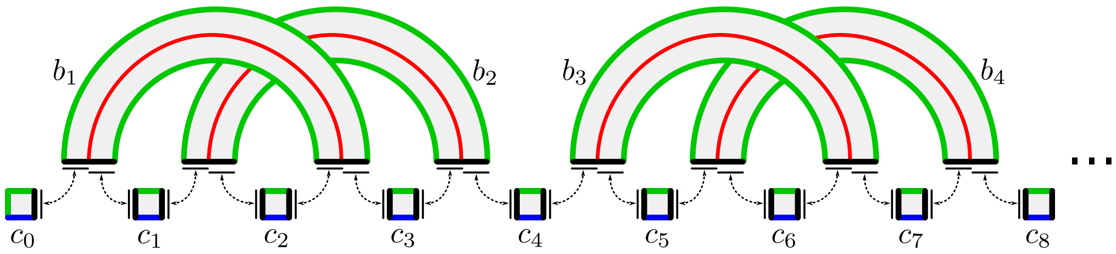


Figure 13: A model for Σ

Appendix A: a deformation retraction through Lipschitz embeddings

Here we will prove Lemma 13. We have a model for (Σ, Γ) by gluing $2g$ bands $b_j = [-1,1] \times [-l,l]$, $1 \leqslant j \leqslant 2g$ and $4g+1$ squares $c_\nu = [0,1] \times [0,1]$, $0 \leqslant j \leqslant 4g$ according to the identifications depicted in Figure 13. We obtain a deformation retraction h which is defined on each band by the formula $h_t(u,v) = ((1-t)u, v)$ and on each square by $h_t(u,v) = (u, (1-t)v)$. It remains to show that for an appropriate metric d the map h_t, $0 \leqslant t < 1$, is a 1-Lipschitz embedding. On each band and square we use the standard Euclidean metric. Then for points $x, y \in \Sigma$, the distance $d(x,y)$ is defined as the shortest length of a path from x to y. It is convenient to assume that l is big enough so that no shortest path can go across a handle. Then d is a metric which is flat outside $4g$ boundary points where the curvature is concentrated. Then we have that h_t, $0 \leqslant t < 1$, is a 1-Lipshitz embedding in each band or square from which we deduce that h_t, $0 \leqslant t < 1$, is globally a 1-Lipschitz embedding.

Appendix B: automorphisms of the Heisenberg group

In this appendix we prove Lemma 16. Denote $H^* = \mathrm{Hom}(H, \mathbb{Z})$. There is an obvious action of $\mathrm{Aut}(H) = GL(H)$ (and hence of $Sp(H) \subseteq \mathrm{Aut}(H)$) on H^* by pre-composition, and we consider the semi-direct product $Sp(H) \ltimes H^*$ with respect to this action. There is a well-defined homomorphism

$$Sp(H) \ltimes H^* \longrightarrow \mathrm{Aut}^+(\mathcal{H}) \tag{87}$$

given by sending $(g \colon H \to H, f \colon H \to \mathbb{Z})$ to the automorphism of $\mathcal{H} = \mathbb{Z} \times H$ that sends $(1,0)$ to itself and $(0,x)$ to $(f(x), g(x))$ for each $x \in H$. This fits into a commutative diagram of the form

$$\begin{tikzcd}
1 \arrow[d] & 1 \arrow[d] \\
H^* \arrow[r] \arrow[d, hook] & \mathrm{Aut}^H(\mathcal{H}) \arrow[d, hook] \\
Sp(H) \ltimes H^* \arrow[r, "(87)"] \arrow[d, two heads] & \mathrm{Aut}^+(\mathcal{H}) \arrow[d] \\
Sp(H) \arrow[r, "\mathrm{incl}"] \arrow[d] \arrow[u, bend left] & \mathrm{Aut}(H) \\
1 &
\end{tikzcd}$$

whose columns are exact, and where $\mathrm{Aut}^H(\mathcal{H})$ denotes the automorphisms of $\mathcal{H}$ that send $u = (1, 0)$ to itself and induce the identity on $H = \mathcal{H}/\mathcal{Z}(\mathcal{H})$. It is easy to verify by hand that (1) the top horizontal map $H^* \to \mathrm{Aut}^H(\mathcal{H})$ is bijective and (2) the image of the vertical map $\mathrm{Aut}^+(\mathcal{H}) \to \mathrm{Aut}(H)$ is contained in $Sp(H)$. These two facts imply that, if we replace the bottom-right group $\mathrm{Aut}(H)$ with $Sp(H)$, the diagram above becomes a map of short exact sequences

$$\begin{tikzcd}
1 \arrow[d] & 1 \arrow[d] \\
H^* \arrow[r, "\cong"] \arrow[d, hook] & \mathrm{Aut}^H(\mathcal{H}) \arrow[d, hook] \\
Sp(H) \ltimes H^* \arrow[r, "(87)"] \arrow[d, two heads] & \mathrm{Aut}^+(\mathcal{H}) \arrow[d, two heads] \\
Sp(H) \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[u, bend left] & Sp(H) \arrow[d] \\
1 & 1
\end{tikzcd}$$

and so the five-lemma implies that (87) is an isomorphism. We record this as:

Lemma 80. *The homomorphism* (87) *is an isomorphism.*

We note that the inverse of (87) may be described as follows. By commutativity of the bottom square of the diagram above, the homomorphism

$$\mathrm{pr}_1 \circ (87)^{-1} \colon \mathrm{Aut}^+(\mathcal{H}) \longrightarrow Sp(H) \ltimes H^* \twoheadrightarrow Sp(H)$$

coincides with the natural projection $\mathrm{Aut}^+(\mathcal{H}) \to Sp(H)$. The function (crossed homomorphism)

$$\mathrm{pr}_2 \circ (87)^{-1} \colon \mathrm{Aut}^+(\mathcal{H}) \longrightarrow Sp(H) \ltimes H^* \twoheadrightarrow H^*$$

is given by sending an automorphism φ to $\mathrm{pr}_1(\varphi(0,-))\colon H \hookrightarrow \mathcal{H} \to \mathcal{H} \twoheadrightarrow \mathbb{Z}$. Putting these together, we recover precisely the description of the homomorphism $\mathrm{Aut}^+(\mathcal{H}) \to Sp(H) \ltimes H^*$ given just before the statement of Lemma 16. Thus Lemma 80 is equivalent to Lemma 16 and we have (22) = $(87)^{-1}$.

Remark 81. The arguments in this appendix apply more generally to any 2-nilpotent group. Suppose that G is a 2-nilpotent group, specifically the central extension

$$1 \to Z \longrightarrow G \longrightarrow H \to 1$$

of an abelian group H associated to a given 2-cocycle $\omega\colon H \times H \to Z$. Then there is a natural isomorphism

$$\Upsilon\colon \mathrm{Aut}_\omega(H) \ltimes \mathrm{Hom}(H, Z) \cong \mathrm{Aut}^Z(G), \tag{88}$$

where $\mathrm{Aut}^Z(G) \subseteq \mathrm{Aut}(G)$ is the group of automorphisms of G that restrict to the identity on Z, and $\mathrm{Aut}_\omega(H) \subseteq \mathrm{Aut}(H)$ is the group of automorphisms of H that preserve the 2-cocycle ω. The isomorphism (88) is given explicitly by

$$\Upsilon(g, f)(z, h) = (z + f(h), g(h)), \tag{89}$$

where we are writing $G = Z \times H$ with its product twisted by ω. This specialises to (87) when $H = H_1(\Sigma)$, $Z = \mathbb{Z}$ and ω is the intersection form on $H_1(\Sigma)$.

Remark 82. We note that the isomorphism (88) is really a *homomorphism*, and not an anti-homomorphism, as one may check using the explicit formula (89) and the definition of the semi-direct product. If we had written the semi-direct product on the left-hand side of (88) in the more usual way as $\rtimes$ (swapping the two factors), then this formula would *not* have defined a homomorphism. This is because in general, for a group Γ acting on another group Λ, the swap map $(g, f) \mapsto (f, g)$ is not an isomorphism $\Gamma \ltimes \Lambda \cong \Lambda \rtimes \Gamma$, but an isomorphism $\Gamma \ltimes \Lambda \cong (\Lambda^{\mathrm{op}} \rtimes \Gamma^{\mathrm{op}})^{\mathrm{op}}$. For this reason, we have consistently written all semi-direct products as $\ltimes$ throughout the paper.

In particular, we may consider the continuous Heisenberg group $\mathcal{H}_{\mathbb{R}}$ (used in §6): this is the central extension of $H_{\mathbb{R}} := H_1(\Sigma; \mathbb{R}) \cong \mathbb{R}^{2g}$ by $\mathbb{R}$ corresponding to the intersection form ω on $H_{\mathbb{R}}$. By the discussion above, we have natural isomorphisms

$$\mathrm{Aut}^+(\mathcal{H}) \cong Sp(H) \ltimes \mathrm{Hom}(H, \mathbb{Z})$$
$$\mathrm{Aut}^+(\mathcal{H}_{\mathbb{R}}) \cong Sp(H_{\mathbb{R}}) \ltimes \mathrm{Hom}(H_{\mathbb{R}}, \mathbb{R}),$$

where, as in the discrete case, $\mathrm{Aut}^+(\mathcal{H}_{\mathbb{R}})$ denotes the subgroup of $\mathrm{Aut}(\mathcal{H}_{\mathbb{R}})$ of automorphisms that act by the identity on the central copy of $\mathbb{R}$. There are natural inclusions $Sp(H) \hookrightarrow Sp(H_{\mathbb{R}})$ and $\mathrm{Hom}(H, \mathbb{Z}) \hookrightarrow \mathrm{Hom}(H_{\mathbb{R}}, \mathbb{R})$ given by tensoring $- \otimes_{\mathbb{Z}} \mathbb{R}$ (they are injective since $\mathbb{R}$ is torsion-free, hence flat over $\mathbb{Z}$). Together with the natural isomorphisms above, they induce a natural inclusion

$$\mathrm{Aut}^+(\mathcal{H}) \longrightarrow \mathrm{Aut}^+(\mathcal{H}_{\mathbb{R}}), \quad \varphi \mapsto \varphi_{\mathbb{R}} \tag{90}$$

having the property that, for any $\varphi \in \mathrm{Aut}^+(\mathcal{H})$, the automorphism $\varphi_{\mathbb{R}}\colon \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ sends $\mathcal{H} \subset \mathcal{H}_{\mathbb{R}}$ onto itself and restricts to $\varphi\colon \mathcal{H} \to \mathcal{H}$.

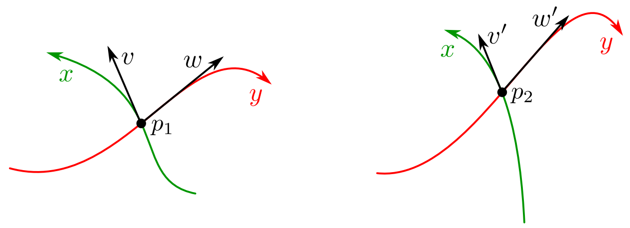


Figure 14: Choices of tangent vectors from the computation of the sign of the intersection of x and y at $p = \{p_1, p_2\} \in \mathcal{C}_2(\Sigma)$.

Appendix C: signs in the intersection pairing formula

Here we explain the signs appearing in the formula (82) for the intersection pairing on the homology of 2-point configuration spaces, including the extra global -1 sign that was suppressed in (82) (see the comment in the paragraph below the formula).

We take the viewpoint that an orientation o of a d-dimensional smooth manifold M is given by a consistent choice of vector $o(p) \in \Lambda^d T_p M$ for all $p \in M$. We either choose a metric on the bundle $\Lambda^d TM$ and require $o(p)$ to be a unit vector with respect to this metric, or we consider $o(p)$ up to rescaling by positive real numbers.

Let us fix an orientation o_Σ for the surface Σ. This determines an orientation $o_{\mathcal{C}_2(\Sigma)}$ of the configuration space $\mathcal{C}_2(\Sigma)$ by setting

$$o_{\mathcal{C}_2(\Sigma)}(\{p_1, p_2\}) = o_\Sigma(p_1) \wedge o_\Sigma(p_2).$$

Recall that we have 2-dimensional submanifolds x and y of $\mathcal{C}_2(\Sigma)$ that intersect transversely, and let $p = \{p_1, p_2\}$ be a point of $x \cap y$. Let v, w be the tangent vectors at p_1 and let v', w' be the tangent vectors at p_2 illustrated in Figure 14. We have

$$\begin{aligned} v \wedge w &= \operatorname{sgn}(p_1).o_\Sigma(p_1) \\ v' \wedge w' &= \operatorname{sgn}(p_2).o_\Sigma(p_2), \end{aligned}$$

where $\operatorname{sgn}(p_i)$ is the sign of the intersection of the arcs in Σ underlying x and y at p_i. Similarly, we have

$$o_x(p) \wedge o_y(p) = \operatorname{sgn}(p).o_{\mathcal{C}_2(\Sigma)}(p),$$

where $\operatorname{sgn}(p)$ is the sign that we are trying to compute: the sign of the intersection of x and y in the configuration space. The orientations of x and y depend on the tethers t_x, t_y that have been chosen. Precisely, we have

$$o_x(p) = \begin{Bmatrix} v \wedge v' & (*) \\ v' \wedge v & (\dagger) \end{Bmatrix} \qquad\qquad o_y(p) = \begin{Bmatrix} w \wedge w' & (*) \\ w' \wedge w & (\dagger) \end{Bmatrix},$$

where the possibilities $((*), (*))$ or $((\dagger), (\dagger))$ occur if $\operatorname{sgn}(\ell_p) = +1$ and the possibilities $((*), (\dagger))$ or $((\dagger), (*))$ occur if $\operatorname{sgn}(\ell_p) = -1$. We therefore have

$$o_x(p) \wedge o_y(p) = \operatorname{sgn}(\ell_p).(v \wedge v') \wedge (w \wedge w').$$

Putting this together with the formulas above, we obtain

$$\begin{aligned}(v \wedge w) \wedge (v' \wedge w') &= \mathrm{sgn}(p_1).\mathrm{sgn}(p_2).o_\Sigma(p_1) \wedge o_\Sigma(p_2) \\ &= \mathrm{sgn}(p_1).\mathrm{sgn}(p_2).o_{\mathcal{C}_2(\Sigma)}(p) \\ &= \mathrm{sgn}(p_1).\mathrm{sgn}(p_2).\mathrm{sgn}(p).o_x(p) \wedge o_y(p) \\ &= \mathrm{sgn}(p_1).\mathrm{sgn}(p_2).\mathrm{sgn}(p).\mathrm{sgn}(\ell_p).(v \wedge v') \wedge (w \wedge w') \\ &= -\mathrm{sgn}(p_1).\mathrm{sgn}(p_2).\mathrm{sgn}(p).\mathrm{sgn}(\ell_p).(v \wedge w) \wedge (v' \wedge w'),\end{aligned}$$

and hence we have

$$\mathrm{sgn}(p) = -\mathrm{sgn}(p_1).\mathrm{sgn}(p_2).\mathrm{sgn}(\ell_p).$$

Appendix D: Sage computations

Here we give the worksheet of the Sage computations used in the calculation of the matrix M_∂ displayed in Figure 9 (cf. Remark 77 on page 56).

In [1]:
```
load("HeisLatex_.sage") #available on demand
```

In [2]:
```
# R is the center of Heisenberg group ring
R.<u>= LaurentPolynomialRing(ZZ,1)
```

In [3]:
```
# H is Heisenberg group ring
H = Heis(base=R, category=Rings())
```

In [32]:
```
a=H(dict({(1,0):1}))   #generator (0,a)
b=H(dict({(0,1):1}))
am=H(dict({(-1,0):1})) #inverse generators
bm=H(dict({(0,-1):1}))
```

In [33]:
```
a*b-u^2*b*a #check relation
```

Out[33]: 0

In [34]:
```
# a->a , b -> a^-1b (T_a action on H)
def Ha(h:HeisEl):
    d0=h.d
    h1=H()
    for k in d0:
        i=k[0]
        j=k[1]
        h1+= H({(i-j,j):d0[k]*u^(j*(j-1))})
    return h1
def MHa(M): # same on matrices
    M1=matrix(H,3)
    for i in range(3):
        for j in range(3):
            M1[i,j]=Ha(M[i,j])
    return M1
```

In [35]:
```
def Hb(h:HeisEl): #T_b action on H
    d0=h.d
    h1=H()
    for k in d0:
        i=k[0]
        j=k[1]
        h1+= H({(i,i+j):d0[k]*u^(-i*(i-1))})
    return h1
def MHb(M): # same on matrices
    M1=matrix(H,3)
    for i in range(3):
        for j in range(3):
            M1[i,j]=Hb(M[i,j])
    return M1
```

In [36]:
```
def Hab(h):       #other actions
    return Ha(Hb(h))
def Hba(h):
    return Hb(Ha(h))
def Haba(h):
    return Ha(Hba(h))
def Hbab(h):
    return Hb(Hab(h))
def Hs(h):
    return(Haba(Haba(h)))
```

In [37]:
```
def MHab(M):    #same on matrices
    return MHa(MHb(M))
def MHba(M):
    return MHb(MHa(M))
def MHaba(M):
    return MHa(MHba(M))
def MHbab(M):
    return MHb(MHab(M))
def MHs(M):
    return(MHaba(MHaba(M)))
```

In [38]:
```
Ma=matrix([[H(1),u^2*am^(2)*b^2,(H(u^(-1))-H(1))*am*b],[H(0),H(1),H(0)],[H(0),H(-1)*am*b,H(1)]])
```

In [39]:
```
%display latex
Ma #Ta action
```

Out[39]:
$$\begin{pmatrix} 1 & u^2a^{-2}b^2 & (-1+u^{-1})a^{-1}b^1 \\ 0 & 1 & 0 \\ 0 & -a^{-1}b^1 & 1 \end{pmatrix}$$

In [40]:
```
Mb=matrix([[H(u^(-2))*b^2,H(0),H(0)],[H(-u^(-1)),H(1),H(1-u^(-1))],[H(-u^(-1))*b,H(0),b]])
```

In [41]:
```
Mb #Tb action
```

Out[41]:
$$\begin{pmatrix} u^{-2}b^2 & 0 & 0 \\ -u^{-1} & 1 & 1-u^{-1} \\ -u^{-1}b^1 & 0 & b^1 \end{pmatrix}$$

In [42]:
```
MHa(Mb) # Ta shifted action of Tb
```

Out[42]:
$$\begin{pmatrix} a^{-2}b^2 & 0 & 0 \\ -u^{-1} & 1 & 1-u^{-1} \\ -u^{-1}a^{-1}b^1 & 0 & a^{-1}b^1 \end{pmatrix}$$

In [43]:
```
MHb(Ma) #Tb shifted action of Ta
```

Out[43]:
$$\begin{pmatrix} 1 & u^{-4}a^{-2} & (-u^{-2}+u^{-3})a^{-1} \\ 0 & 1 & 0 \\ 0 & -u^{-2}a^{-1} & 1 \end{pmatrix}$$

In [44]:
```
MHab(Ma) #TaTb shifted action of Ta
```

Out[44]:
$$\begin{pmatrix} 1 & u^{-4}a^{-2} & (-u^{-2}+u^{-3})a^{-1} \\ 0 & 1 & 0 \\ 0 & -u^{-2}a^{-1} & 1 \end{pmatrix}$$

In [45]:
```
MHba(Mb) #TbTa shifted action of Tb
```
Out[45]:
$$\begin{pmatrix} u^{-6}a^{-2} & 0 & 0 \\ -u^{-1} & 1 & 1-u^{-1} \\ -u^{-3}a^{-1} & 0 & u^{-2}a^{-1} \end{pmatrix}$$

In [46]:
```
X=Ma*MHa(Mb)*MHab(Ma) #action of TaTbTa
X
```
Out[46]:
$$\begin{pmatrix} 0 & u^2a^{-2}b^2 & 0 \\ -u^{-1} & -u^{-5}a^{-2}+1+(-u^{-2}+u^{-3})a^{-1} & (u^{-3}-u^{-4})a^{-1}+1-u^{-1} \\ 0 & -a^{-1}b^1-u^{-1}a^{-2}b^1 & u^{-1}a^{-1}b^1 \end{pmatrix}$$

In [47]:
```
Y=Mb*MHb(Ma)*MHba(Mb) #action of TbTaTb
Y
```
Out[47]:
$$\begin{pmatrix} 0 & u^2a^{-2}b^2 & 0 \\ -u^{-1} & -u^{-5}a^{-2}+1+(-u^{-2}+u^{-3})a^{-1} & 1-u^{-1}+(u^{-3}-u^{-4})a^{-1} \\ 0 & -u^{-1}a^{-2}b^1-a^{-1}b^1 & u^{-1}a^{-1}b^1 \end{pmatrix}$$

In [48]:
```
X-Y #check braid relation TaTbTa=TbTaTb
```
Out[48]:
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In [49]:
```
Z=X*MHaba(X) #action of (TaTbTa)^2
```

In [50]:
```
Z[:,0] # first colomn
```
Out[50]:
$$\left(\begin{array}{r} -ua^{-2}b^2 \\ u^{-6}a^{-2}+-u^{-1}+(u^{-3}-u^{-4})a^{-1} \\ u^{-1}a^{-1}b^1+u^{-2}a^{-2}b^1 \end{array}\right)$$

In [51]:
```
Z[:,1]
```
Out[51]:
$$\left(\begin{array}{r} -u^{-3}a^{-2}+u^2a^{-2}b^2+(-1+u^{-1})a^{-2}b^1 \\ -u^{-5}b^{-2}+-u^{-5}a^{-2}+1+(-u^{-2}+u^{-3})a^{-1}+(-u^{-2}+u^{-3})b^{-1}+(-u^{-5}+u^{-6})a^{-1}b^{-1} \\ u^{-5}a^{-1}b^{-1}-a^{-1}b^1+-u^{-1}a^{-2}b^1+(u^{-2}-u^{-3})a^{-1}+-u^{-4}a^{-2} \end{array}\right)$$

In [52]:
```
Z[:,2]
```
Out[52]:
$$\left(\begin{array}{r} (u^{-1}-u^{-2})a^{-2}b^1+(u^2-u)a^{-2}b^2 \\ (u^{-3}-u^{-4})b^{-1}+(u^{-6}-u^{-7})a^{-1}b^{-1}+(-u^{-5}+u^{-6})a^{-2}+1-u^{-1}+(-u^{-2}+2u^{-3}-u^{-4})a^{-1} \\ (-u^{-3}+u^{-4})a^{-1}+u^{-5}a^{-2}+(-1+u^{-1})a^{-1}b^1+(-u^{-1}+u^{-2})a^{-2}b^1 \end{array}\right)$$

In [53]:
```
ZZ=Z*MHs(Z) #action of Tc=(TaTbTa)^4
```

In [54]:
```
ZZ[:,0]
```
Out[54]:
$$\left(\begin{array}{r} u^{-8}b^2+u^{-4}a^{-2}+-ua^{-2}b^2+(u^{-1}-u^{-2})a^{-2}b^1+(u^{-3}-u^{-4})a^{-1}b^2+(u^{-4}-u^{-5})a^{-1}b^1 \\ -u^{-1}-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-2}a^2+(u^{-1}-u^{-2}-u^{-4}+u^{-5})a^1+u^{-6}a^{-2}+(u^{-3}-u^{-4}-u^{-6}+u^{-7})a^{-1} \\ -u^{-6}a^1b^1+(-u^{-3}+u^{-4}-u^{-7})b^1+-u^{-4}+(u^{-1}-u^{-4}+u^{-5})a^{-1}b^1+u^{-2}a^{-2}b^1+(-u^{-3}+u^{-6})a^{-1}+u^{-5}a^{-2} \end{array}\right)$$

In [55]:
```
ZZ[:,1]
```
Out[55]:
$$\left(\begin{array}{c} (u^2+1-2u^{-1}+u^{-2}+u^{-4})a^{-2}b^2+-ua^{-2}b^4+(-u^2+u+u^{-1}-u^{-2})a^{-2}b^3+-u^{-3}a^{-2}+(-1+u^{-1}+u^{-3}-u^{-4})a^{-2}b^1 \\ 1+u^{-2}-u^{-3}+u^{-6}+u^{-6}a^{-2}b^2+-u^{-1}b^2+(u^{-3}-u^{-4})a^{-1}b^2+(-1+u^{-1}+u^{-3}-u^{-4})b^1 \\ +(u^{-2}-2u^{-3}+u^{-4}+u^{-6}-u^{-7})a^{-1}b^1+-u^{-5}a^{-2}+(-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}+(u^{-5}-u^{-6})a^{-2}b^1 \\ (-1-u^{-2}+2u^{-3}-u^{-6})a^{-1}b^1+u^{-1}a^{-1}b^3+u^{-2}a^{-2}b^3+(1-u^{-1}-u^{-3}+u^{-4})a^{-1}b^2+(u^{-1}-u^{-2}+u^{-5})a^{-2}b^2 \\ +(-u^{-1}+u^{-4}-u^{-5})a^{-2}b^1+(u^{-2}-u^{-5})a^{-1}+-u^{-4}a^{-2} \end{array}\right)$$

In [56]:
```
ZZ[:,2]
```
Out[56]:
$$\left(\begin{array}{c} (-1+2u^{-1}-u^{-2}-u^{-4}+u^{-5})a^{-2}b^1+(u-1)a^{-2}b^3+(u^2-u-u^{-1}+2u^{-2}-u^{-3})a^{-2}b^2+(-u^{-3}+u^{-4})a^{-1}b^1 \\ +(u^{-4}-u^{-5})a^{-1}b^3+(-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}b^2+(-u^{-3}+u^{-4})a^{-2} \\ (-u^{-6}+u^{-7})a^{-2}b^1+(u^{-1}-u^{-2}-u^{-4}+2u^{-5}-u^{-6})b^1+(-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-8})a^{-1}b^1+1-u^{-1}+u^{-2}-3u^{-3} \\ +2u^{-4}+u^{-6}-u^{-7}+(-u^{-2}+2u^{-3}-u^{-4}+u^{-5}-2u^{-6}+u^{-7})a^{-1}+(u^{-2}-u^{-3})a^1b^1+(-1+u^{-1}+u^{-3}-u^{-4})a^1 \\ +(-u^{-5}+u^{-6})a^{-2} \\ u^{-3}+(u^{-2}-u^{-3}-u^{-5}+u^{-6})a^{-1}+(-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-1}b^2+(-u^{-2}+u^{-3})a^{-2}b^2 \\ +(-1+u^{-1}+2u^{-3}-3u^{-4}+u^{-7})a^{-1}b^1+(-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-2}b^1+(-u^{-4}+u^{-5})b^2+(u^{-2}-u^{-3}-u^{-5}+u^{-6})b^1 \\ +(-u^{-4}+u^{-5})a^{-2} \end{array}\right)$$

In [57]:
```
ZZ*Ma-Ma*MHa(ZZ) # check that Tc is central
```
Out[57]:
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In [58]:
```
ZZ*Mb-Mb*MHb(ZZ)
```
Out[58]:
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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