BIG MAPPING CLASS GROUPS WITH UNCOUNTABLE INTEGRAL HOMOLOGY

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Abstract. We prove that, for any infinite-type surface S, the integral homology of the pure mapping class group $\mathrm{PMap}(S)$ and of the Torelli group $\mathcal{T}(S)$ is uncountable in every positive degree. By our results in [PW] and other known computations, such a statement cannot be true for the full mapping class group $\mathrm{Map}(S)$ for all infinite-type surfaces S. However, we are still able to prove that the integral homology of $\mathrm{Map}(S)$ is uncountable in all positive degrees for many infinite-type surfaces S. The key hypothesis is, roughly, that the space of ends E of the surface S contains a limit point of topologically distinguished points. This includes all surfaces with countable end spaces that have a unique point of maximal Cantor-Bendixson rank, which is a successor ordinal. We also observe an order-10 element in the first homology of the pure mapping class group of any surface of genus 2 with non-empty boundary, answering a recent question of George Domat.

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Introduction

There has been a recent wave of interest in *big mapping class groups* (mapping class groups of infinite-type surfaces); see [AV20] for a survey. In [PW], the authors recently computed the homology of a large family of big mapping class groups, namely the families of 1-*holed* or *punctured binary tree surfaces* (see the introduction of [PW] for this terminology). Specifically, the mapping class group of every 1-holed binary tree surface is *acyclic* and the homology of the mapping class group of every punctured binary tree surface is periodic with $\mathbb{Z}$ in every even degree and zero in every odd degree. One instance of this result says that the mapping class group $\mathrm{Map}(\mathbb{D}^2 \smallsetminus \mathcal{C})$ is acyclic and that $H_i(\mathrm{Map}(\mathbb{R}^2 \smallsetminus \mathcal{C}))$ is $\mathbb{Z}$ for i even and zero for i odd, where $\mathcal{C}$ is a Cantor set embedded in the interior of the disc. In particular, in all of these examples, the homology groups $H_i(\mathrm{Map}(S))$ are finitely generated for each i.

In this paper we prove a contrasting result: for many infinite-type surfaces S, the group $H_i(\mathrm{Map}(S))$ is uncountable for all $i > 0$. Moreover, we prove that for *any* infinite-type surface S, the integral homology of the pure mapping class group $\mathrm{PMap}(S)$ and of the Torelli group $\mathcal{T}(S)$ is uncountable in every positive degree.

Our proofs are built on ideas from [APV20, Dom, MT]. In [APV20], Aramayona, Patel and Vlamis determined $H^1(\mathrm{PMap}(S))$ for any infinite-type surface S. Along the way they proved that, when S has infinitely many non-planar ends, its pure mapping class group $\mathrm{PMap}(S)$ admits a split-surjection onto the *Baer-Specker group* $\mathbb{Z}^{\mathbb{N}}$. Later, Domat proved that big mapping class groups are never perfect [Dom]. In fact, he showed that $H_1(\mathrm{PMap}(S))$ is uncountable for many infinite-type surfaces S and the first homology of the Torelli group $H_1(\mathcal{T}(S))$ is uncountable for all infinite-type surfaces S. Malestein and Tao [MT] were able to further push the results of Domat and prove that $H_1(\mathrm{Map}(S))$ is uncountable for a certain class of surfaces S, including $S = \mathbb{R}^2 \smallsetminus \mathbb{Z}$.

Uncountable homology. Given a surface S, recall that the pure mapping class group $\mathrm{PMap}(S)$ of a surface is the subgroup of mapping classes that fix the ends of S pointwise. The Torelli group $\mathcal{T}(S)$ is the kernel of the natural homomorphism $\mathrm{Map}(S) \to \mathrm{Aut}(H_1(S))$. Our first theorem works for any infinite-type surface.

Theorem A (Theorems 5.1 and 5.5). *Let S be an infinite-type surface. Then the integral homology groups*

$$H_i(\mathrm{PMap}(S)) \quad \textit{and} \quad H_i(\mathcal{T}(S))$$

are uncountable for every $i \geq 1$. In fact they each contain $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ in every degree, where $\mathfrak{c}$ denotes the cardinality of the continuum.

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In order to state our result for the full mapping class groups, we first recall some background about ends of surfaces. More details are given in §1 and §2. Every surface S has a space of ends E, which is a compact, separable, totally disconnected topological space. The key hypothesis in our main theorem is a condition on the structure of the space E.

Definition 0.1. For points $x, y \in E$, we write $x \sim y$ and say that x is *similar* to y if and only if there are open neighbourhoods U, V of x, y respectively such that (U, x) and (V, y) are homeomorphic as based spaces. A point $x \in E$ is *topologically distinguished* if it is not equivalent to any other point of E under this equivalence relation.

Definition 0.2. For a topological space E, write $\Upsilon^+(E) = E\omega + 1$, where $E\omega$ means a countably infinite disjoint union of copies of E and $X + 1$ means the one-point compactification of X.

Theorem B. *Let S be a connected surface of finite genus and with finitely many boundary components, whose space of ends E is of the form $E = E_1 \sqcup \Upsilon^+(E_2)$, where E_2 has a topologically distinguished point x and no point of E_1 is similar to x. Then the integral homology group*

$$H_i(\mathrm{Map}(S))$$

is uncountable for every $i \geq 1$. In fact, there is an injective homomorphism of graded abelian groups

$$\Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \longrightarrow H_*(\mathrm{Map}(S)),$$

where Λ^* denotes the exterior algebra on an abelian group.

Remark 0.3. In the course of the proof of Theorem B, we also prove the same statement with S replaced by the Loch Ness monster surface L; see Proposition 4.1.

Remark 0.4. All *countable* end spaces of surfaces are of the form $E = \omega^\alpha.n + 1$ for a countable ordinal α and a positive integer n. This end space satisfies the hypotheses of Theorem B if and only if $n = 1$ and α is a *successor* ordinal. This suggests the following question:

Question 0.5. *Let $E = \omega^\alpha.n+1$ for a countable ordinal α and positive integer n and consider a surface S of finite genus whose space of ends is homeomorphic to E. Is the homology of* $\mathrm{Map}(S)$ *uncountable in all positive degrees when $n > 1$ or α is a limit ordinal?*

Remark 0.6. Without the hypothesis on the structure of the space of ends E of S, the conclusion of Theorem B is false. For example, as mentioned above, we prove in [PW] that

$$H_i(\mathrm{Map}(\mathbb{R}^2 \smallsetminus \mathcal{C})) \cong \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

Remark 0.7. The hypotheses of this paper and the hypotheses of [PW] are in some sense opposite, with opposite conclusions. In [PW] we consider 1-holed binary tree surfaces, whose end spaces are *Cantor compactifications* $(E\omega)^{\mathcal{C}}$ (see [PW, §1.2] for the definition), which are highly self-similar (in particular $(E\omega)^{\mathcal{C}} \cong \mathcal{C}$ if $E = \varnothing$ or $E = \mathcal{C}$, which is homogeneous), and we prove that $H_i(\mathrm{Map}(S)) = 0$ for all $i > 0$. On the other hand, in this paper we consider surfaces S whose end spaces E satisfy the "self-similarity-breaking" hypothesis of Theorem B (roughly: E has a limit point of topologically distinguished points), and conclude that $H_i(\mathrm{Map}(S))$ is uncountable for all $i > 0$.

Non-trivial torsion. So far, the elements that we have constructed in the homology of big mapping class groups all have infinite order. It would be interesting also to find some torsion elements. In fact, the following question was asked by Domat in [Dom, Question 11.3].

Question 0.8. *Let S be an infinite-type surface. Are there torsion elements in $H_1(\overline{\mathrm{PMap}_c(S)})$?*

Recall that $\mathrm{PMap}_c(S)$ denotes the compactly-supported mapping class group of S and $\overline{\mathrm{PMap}_c(S)}$ is its closure in $\mathrm{Map}(S)$ in the compact-open topology. We recall that $\overline{\mathrm{PMap}_c(S)} \subseteq \mathrm{PMap}(S)$ coincides with $\mathrm{PMap}(S)$ if and only if S has at most one non-planar end [PV18, Theorem 4]. Our third result answers Domat's question in the positive.

Theorem C. *Let S be an infinite-type surface of genus 2 with non-empty boundary. Then the homology groups $H_1(\mathrm{PMap}(S)) = H_1(\overline{\mathrm{PMap}_c(S)})$ and $H_1(\mathrm{Map}(S))$ both contain an order-10 element.*

Remark 0.9. By comparing the stable homology of (orientable, finite-type) mapping class groups with rational coefficients [MW07] and with mod-p coefficients [Gal04], one sees that there are also many torsion elements in the integral homology of mapping class groups in the stable range. Using this and Lemma 7.2, one may find many higher-degree torsion elements in the homology of big mapping class groups of surfaces with finite genus.

In a sense, our answer to Domat's question is "cheating", since we simply show that a certain torsion element in the homology of the mapping class group of a finite-type subsurface of S injects into the homology of the mapping class group of S. Together with our uncountability results (Theorems A and B) above, this suggests two refinements of Domat's question:

Question 0.10. *Let S be an infinite-type surface. Do the homology groups $H_1(\overline{\mathrm{PMap}_c(S)})$ or $H_1(\mathrm{PMap}(S))$ contain torsion elements that are not supported on any finite-type subsurface of S?*

Question 0.11. *Let S be an infinite-type surface. Do the homology groups $H_1(\overline{\mathrm{PMap}_c(S)})$ or $H_1(\mathrm{PMap}(S))$ contain an uncountable torsion subgroup?*

Notice that a positive answer to Question 0.11 would imply a positive answer to Question 0.10, since torsion admitting finite-type support can only account for countably many torsion elements.

Outline. We begin with two sections of background: §1 on infinite-type surfaces and big mapping class groups and §2 on notions of *topologically distinguished points*. In §3 we prove a basic lemma that often allows us to lift uncountability of degree-one homology to higher degrees. The first step in the proof of Theorem B is carried out in §4, where we prove uncountability for the homology of the Loch Ness monster surface, building on results of Domat [Dom]. We then adapt these arguments in §5 to prove Theorem A. In §6 we then apply a covering space argument, inspired by a technique of Malestein and Tao [MT], to complete the proof of Theorem B. This is the step in which we essentially use the hypothesis on the structure of the end space of the surface. We then prove Theorem C in §7 and discuss some related open questions in §8. In Appendix A we collect some basic facts about abelian groups that are needed in several of our proofs.

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1. Surfaces, ends and mapping class groups

1.1. **Infinite-type surfaces.** All surfaces will be assumed to be second countable, connected, orientable and to have compact boundary. If the fundamental group of S is finitely generated, we say that S has *finite type*, otherwise it has *infinite type*. The classification of surfaces of possibly infinite type was proven by von Kerékjártó [vK23] and Richards [Ric63]. Recall that an *end* of a surface S is an element of the set

$$\mathrm{Ends}(S) = \varprojlim \pi_0(S \setminus K), \tag{1.1}$$

where the inverse limit is taken over all compact subsets $K \subset S$. The *Freudenthal compactification* of S is the union

$$\overline{S} = S \sqcup \mathrm{Ends}(S)$$

equipped with the topology generated by $U \sqcup \{e \in \mathrm{Ends}(S) \mid e < U\}$ for all open subsets $U \subseteq S$. Here $e < U$ means that there is a compact subset $K \subset S$ such that U contains the component of $S \setminus K$ hit by e under the canonical map $\mathrm{Ends}(S) \to \pi_0(S \setminus K)$. The induced subspace topology on $\mathrm{Ends}(S)$ coincides with the limit topology induced from the discrete topology on each term in the inverse system. With this topology, $\mathrm{Ends}(S)$ is homeomorphic to a closed subset of the Cantor set. We call an end $e \in \mathrm{Ends}(S)$ *planar* if it has a neighbourhood (in the topology of $\overline{S}$) that embeds into the plane, otherwise we call it *non-planar*. The (closed) subspace of non-planar ends is denoted by $\mathrm{Ends}_{np}(S) \subseteq \mathrm{Ends}(S)$.

Theorem 1.1 ([Ric63, §4.5])**.** *Let S_1, S_2 be two surfaces of genus $g_1, g_2 \in \mathbb{N} \cup \{\infty\}$ and with $b_1, b_2 \in \mathbb{N}$ boundary components. Then $S_1 \cong S_2$ if and only if $g_1 = g_2$, $b_1 = b_2$ and there is a homeomorphism of pairs of spaces*

$$(\mathrm{Ends}(S_1), \mathrm{Ends}_{np}(S_1)) \;\cong\; (\mathrm{Ends}(S_2), \mathrm{Ends}_{np}(S_2)).$$

Conversely, given $g \in \mathbb{N} \cup \{\infty\}$, $b \in \mathbb{N}$ and a pair $X \subseteq Y$ of closed subsets of the Cantor set, where we require that $g = \infty$ if and only if $X \neq \varnothing$, there exists a surface S of genus g with b boundary components such that $(\mathrm{Ends}(S), \mathrm{Ends}_{np}(S)) \cong (Y, X)$.

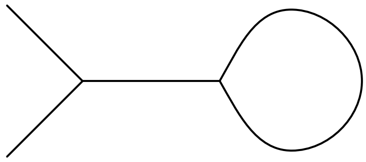

FIGURE 2.1. The 3-valent vertices of this graph are globally topologically distinguished but not topologically distinguished, since they are similar (but not globally similar) to each other.

1.2. **Mapping class groups.** For a surface S, the *mapping class group* of S is the group of isotopy classes of orientation-preserving diffeomorphisms of S fixing the boundary of S pointwise, i.e.

$$\mathrm{Map}(S) := \pi_0(\mathrm{Diff}^+(S, \partial S)).$$

The *pure mapping class group* $\mathrm{PMap}(S)$ of S is the subgroup of $\mathrm{Map}(S)$ consisting of all elements whose induced action on $\mathrm{Ends}(S)$ is the identity. These groups fit into the following short exact sequence.

Proposition 1.2. *Let S be any surface. Then there is a short exact sequence of groups*

$$1 \to \mathrm{PMap}(S) \longrightarrow \mathrm{Map}(S) \longrightarrow \mathrm{Homeo}(\mathrm{Ends}(S), \mathrm{Ends}_{np}(S)) \to 1,$$

where $\mathrm{Homeo}(\mathrm{Ends}(S), \mathrm{Ends}_{np}(S))$ *is the group of homeomorphisms of the pair* $(\mathrm{Ends}(S), \mathrm{Ends}_{np}(S))$.

2. Topologically distinguished points

We now recall from the introduction the notion of *topologically distinguished points* (Definition 0.1) and compare it to a weaker notion of *globally topologically distinguished points.*

Definition 2.1. Let E be a topological space. Two points $x, y \in E$ are called *similar* if there are open neighbourhoods U and V of x and y respectively and a homeomorphism $U \cong V$ taking x to y. This is an equivalence relation on E. A point $x \in E$ is called *topologically distinguished* if its equivalence class under this relation is $\{x\}$, in other words it is similar only to itself.

Definition 2.2. Let E be a topological space. Two points $x, y \in E$ are called *globally similar* if there is a homeomorphism $\varphi \in \mathrm{Homeo}(E)$ with $\varphi(x) = y$. This is an equivalence relation on E. A point $x \in E$ is called *globally topologically distinguished* if its equivalence class under this relation is $\{x\}$, in other words it is similar only to itself. Equivalently, $x \in E$ is globally topologically distinguished if it is a fixed point of the action of $\mathrm{Homeo}(E)$ on E.

Remark 2.3. We record two immediate observations:
- If x and y are globally similar then they are similar.
- If x is topologically distinguished then it is globally topologically distinguished.

The converses of these two statements are false in general. For example, the two vertices of valence 3 in the graph pictured in Figure 2.1 are similar but not globally similar; also, both of them are globally topologically distinguished but not topologically distinguished. However, for zero-dimensional (Hausdorff) spaces the converse does hold:

Lemma 2.4. *Suppose that E is Hausdorff and zero-dimensional, i.e. it has a basis for its topology consisting of clopen subsets. Then two points $x, y \in E$ are similar if and only if they are globally similar. Thus $x \in E$ is topologically distinguished if and only if it is globally topologically distinguished.*

Proof. The second statement follows from the first one, so we only have to prove the first statement, that $x, y \in E$ are similar if and only if they are globally similar. One implication is obvious; we will prove the opposite implication. So let us assume that $x, y \in E$ are similar and choose open neighbourhoods U and V of x and y respectively and a homeomorphism $\varphi \colon U \to V$ taking x to y. Assume that $x \neq y$ (otherwise the result is obvious). Since E is zero-dimensional, we may assume, by shrinking them if necessary, that U and V are clopen. Since E is Hausdorff, we may assume, by shrinking them if necessary, that U and V are disjoint. We may therefore extend φ to a homeomorphism $\bar{\varphi} \in \mathrm{Homeo}(E)$ by:
- $\bar{\varphi}(e) = \varphi(e)$ for $e \in U$;
- $\bar{\varphi}(e) = \varphi^{-1}(e)$ for $e \in V$;
- $\bar{\varphi}(e) = e$ for $e \in E \smallsetminus (U \sqcup V)$.

This bijection is continuous since $\{U, V, E \smallsetminus (U \sqcup V)\}$ is an open cover of E and $\bar{\varphi}$ is continuous when restricted to each of these subsets. Its inverse is continuous for the same reason, so it is a homeomorphism of E taking x to y. Thus x and y are globally similar. □

Remark 2.5. Ends spaces of surfaces are always Hausdorff and zero-dimensional, so Lemma 2.4 implies that *topologically distinguished* and *globally topologically distinguished* are the same for end spaces.

Lemma 2.6. *If a space E has a topologically distinguished point, then $E\omega+1$ has a globally topologically distinguished point. In fact, the point at infinity is globally topologically distinguished.*

Proof. Let ∞ denote the point at infinity of the one-point compactification $E\omega + 1$ of $E\omega = \bigsqcup_\omega E$. Let $\varphi \in \mathrm{Homeo}(E\omega+1)$. We just need to show that $\varphi(\infty) = \infty$, since it will then follow that ∞ is a globally topologically distinguished point of $E\omega+1$. Suppose for a contradiction that $\varphi(\infty) \neq \infty$. Write $E_i = E$ for each $i \in \mathbb{N}$, and identify $E\omega = \bigsqcup_{i\in\mathbb{N}} E_i$. By assumption, $\varphi(\infty) \in E_j$ for some $j \in \mathbb{N}$. Let $x \in E$ be a topologically distinguished point. Every open neighbourhood U of $\infty \in E\omega + 1$ contains infinitely many points that are similar to x, since, by definition of the one-point compactification, U must contain E_i for infinitely many i. Since φ is a homeomorphism, it must also be true that every open neighbourhood of $\varphi(\infty) \in E\omega + 1$ contains infinitely many points that are similar to x. But E_j is an open neighbourhood of $\varphi(\infty) \in E\omega + 1$ and it contains only one point that is similar to x, a contradiction. □

Corollary 2.7. *Suppose that E is Hausdorff and zero-dimensional. If E has a topologically distinguished point, then the point at infinity of $E\omega + 1$ is topologically distinguished.*

Proof. By Lemma 2.6, the point at infinity of $E\omega+1$ is globally topologically distinguished. Hausdorffness and zero-dimensionality of E imply Hausdorffness and zero-dimensionality of $E\omega+1$, so Lemma 2.4 then implies that the point at infinity of $E\omega + 1$ is topologically distinguished. □

Remark 2.8. There is another, a priori different, equivalence relation on topological spaces, defined by [MRb]. They define, for points $x, y \in E$:

$$x \leq y \iff \forall \text{ open neighbourhoods } U \ni y,\ \exists z \in U : z \sim x,$$

where $z \sim x$ means that z and x are similar in the sense of Definition 2.1. This is a pre-order on E, so it induces an equivalence relation

$$x \approx y \iff x \leq y \text{ and } y \leq x$$

on E and a poset structure on the quotient $E/\!\approx$. Clearly $x \sim y$ implies $x \approx y$. Also, if we now assume that E is the end space of a surface Σ, it is not hard to see (using Lemma 2.4) that $x \sim y$ if and only if there is a homeomorphism of Σ taking x to y. Theorem 1.2 of [MRb] says that if $x \approx y$ then there is a homeomorphism of Σ taking x to y. It follows that $\sim$ and $\approx$ are the same equivalence relation on E if it is the end space of a surface. In [MRa], the authors often consider the condition that "Σ has a unique maximal end", i.e. there is a unique maximal equivalence class $[x] \in E/\!\approx$ and the equivalence class $[x]$ has size 1. The condition that we require in this paper is however much weaker, namely that "Σ has a topologically distinguished end", i.e. there is an equivalence class $[x] \in E/\!\approx$ of size 1 (but it need not be maximal in the poset structure of $E/\!\approx$).

3. Lifting uncountability to higher degree homology

We prove in this section a key lemma, which we use several times to lift uncountability of homology in degree one to higher degrees.

Notation 3.1. Let us fix some notation that will be used throughout the rest of the paper.

- For an abelian group A, denote by $\Lambda^*(A)$ the exterior algebra on A.
- We denote by $\mathfrak{c}$ the cardinality of the continuum.

Lemma 3.2. *Let G be a group, $\alpha\colon G \twoheadrightarrow G/[G,G] = H_1(G)$ the quotient map and $\iota\colon \bigoplus_{\mathfrak{c}} \mathbb{Z} \to G$ a homomorphism. Suppose that there is an embedding $f\colon \bigoplus_{\mathfrak{c}} \mathbb{Q} \hookrightarrow H_1(G)$ such that the diagram*

$$\begin{array}{ccc} \bigoplus_{\mathfrak{c}} \mathbb{Z} & \xrightarrow{\iota} & G \\ \big\downarrow & & \big\downarrow \alpha \\ \bigoplus_{\mathfrak{c}} \mathbb{Q} & \xhookrightarrow{f} & H_1(G), \end{array} \tag{3.1}$$

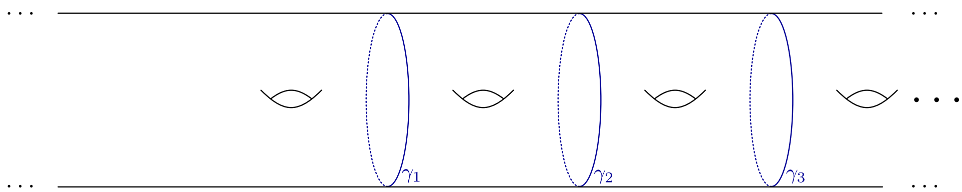

FIGURE 4.1. The once-punctured Loch Ness monster surface.

commutes, where $\bigoplus_{\mathfrak{c}} \mathbb{Z} \hookrightarrow \bigoplus_{\mathfrak{c}} \mathbb{Q}$ *is the canonical inclusion. Then there is an injective homomorphism of graded abelian groups*

$$\Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \longhookrightarrow H_*(G).$$

Proof. By Lemma A.1, the embedding f admits a retraction. Hence the canonical inclusion

$$\bigoplus_{\mathfrak{c}} \mathbb{Z} \longhookrightarrow \bigoplus_{\mathfrak{c}} \mathbb{Q} \tag{3.2}$$

factors through G. It follows that the induced homomorphism of graded abelian groups

$$H_*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \longrightarrow H_*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Q}\Big) \tag{3.3}$$

factors through $H_*(G)$. Now, the integral homology of a torsion-free abelian group A is naturally isomorphic to the exterior algebra $\Lambda^*(A)$ (see [Bro82, Theorem V.6.4(ii)]), so we have homomorphisms of graded abelian groups

$$\Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \longrightarrow H_*(G) \longrightarrow \Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Q}\Big) \tag{3.4}$$

whose composition is injective by Lemma A.3. In particular the first map must be injective. □

4. Uncountability for the Loch Ness monster surface

The first step in the proof of Theorem B is the following proposition, which is the same statement for the Loch Ness monster surface. In §6 we will use this to deduce Theorem B.

Proposition 4.1. *Denote by L the Loch Ness monster surface. Then there is an injective homomorphism of graded abelian groups*

$$\Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \longhookrightarrow H_*(\mathrm{Map}(L)).$$

Denote by L' the once-punctured Loch Ness monster surface, viewed as the bi-infinite cylinder $\mathbb{R} \times S^1$ with infinitely many handles attached to it via performing a connected sum with a torus along a small disc centred at the points $(2n, *) \in \mathbb{R} \times S^1$ for each $n \in \mathbb{N}$. For each integer $i \geq 1$, denote by γ_i the simple closed curve $\{2i-1\} \times S^1 \subset L'$. See Figure 4.1. Following [Dom, §8.3], choose an infinite subset $\Lambda_a \subseteq \mathbb{N}$ for each $a \in \mathbb{R}$ such that $\Lambda_a \cap \Lambda_b$ is finite for $a \neq b$ (for example identify $\mathbb{N}$ with $\mathbb{Q}$ and choose Λ_a to be a sequence of rational numbers converging to a). Since the curves γ_i are pairwise disjoint, we may consider the infinite product of Dehn twists

$$f_a = \prod_{i \in \Lambda_a} (T_{\gamma_i})^{i!} \in \overline{\mathrm{PMap}}_c(L').$$

(The precise form of the exponents in this product is not important; it is just important that for each fixed integer k, all but finitely many of the exponents are divisible by k.) These elements pairwise commute, so they determine a group homomorphism

$$\bigoplus_{a \in \mathbb{R}} \mathbb{Z} \longrightarrow \overline{\mathrm{PMap}}_c(L'). \tag{4.1}$$

Domat then proves the following:

Proposition 4.2 ([Dom, §8.3])**.** *The composition of* (4.1) *with the abelianisation of the right-hand side sends the generator of each copy of* $\mathbb{Z}$ *to a divisible element of* $H_1(\overline{\mathrm{PMap}}_c(L'))$*. In other words, there is a (necessarily unique) homomorphism* $(*)$ *completing the diagram*

$$\begin{tikzcd} \bigoplus_{a\in\mathbb{R}} \mathbb{Z} \arrow[r, "(4.1)"] \arrow[d, hook] & \overline{\mathrm{PMap}}_c(L') \arrow[d, two heads] \\ \bigoplus_{a\in\mathbb{R}} \mathbb{Q} \arrow[r, "(*)"] & H_1(\overline{\mathrm{PMap}}_c(L')). \end{tikzcd} \tag{4.2}$$

Moreover, the homomorphism $(*)$ *is injective.*

We now use this, together with Lemma 3.2, to prove Proposition 4.1.

Proof of Proposition 4.1. Since L' has at most one non-planar end, we have $\overline{\mathrm{PMap}}_c(L') = \mathrm{PMap}(L')$, by [PV18, Theorem 4]. We also have $\mathrm{PMap}(L') = \mathrm{Map}(L')$ since L' has only two punctures, which cannot be interchanged since exactly one of them is non-planar. Thus we may rewrite $\overline{\mathrm{PMap}}_c(L')$ as $\mathrm{Map}(L')$ in (4.2).

In order to replace L' with L, we use the Birman exact sequence, which takes the form

$$1 \to \pi_1(L) \longrightarrow \mathrm{Map}(L') \longrightarrow \mathrm{Map}(L) \to 1. \tag{4.3}$$

Since abelianisation is a right-exact functor, it follows that the kernel of $H_1(\mathrm{Map}(L')) \to H_1(\mathrm{Map}(L))$ is a quotient of $H_1(L)$; in particular it is *countable*. Consider the diagram

$$\begin{tikzcd} \bigoplus_{a\in\mathbb{R}} \mathbb{Z} \arrow[r, "(4.1)"] \arrow[d, hook] & \mathrm{Map}(L') \arrow[r] \arrow[d, two heads] & \mathrm{Map}(L) \arrow[d, two heads] \\ \bigoplus_{a\in\mathbb{R}} \mathbb{Q} \arrow[r, "(*)"] & H_1(\mathrm{Map}(L')) \arrow[r, "(**)"] & H_1(\mathrm{Map}(L)). \end{tikzcd} \tag{4.4}$$

We know from Proposition 4.2 that $(*)$ is injective and we know that $(**)$ has countable kernel by the discussion above. Thus Lemma A.2 implies that, after removing countably many terms from the direct sum on the left-hand side, the composition across the bottom of (4.4) is also injective, so we obtain the diagram

$$\begin{tikzcd} \bigoplus_{\mathfrak{c}} \mathbb{Z} \arrow[r] \arrow[d, hook] & \mathrm{Map}(L) \arrow[d, two heads] \\ \bigoplus_{\mathfrak{c}} \mathbb{Q} \arrow[r, "(*)'"] & H_1(\mathrm{Map}(L)), \end{tikzcd} \tag{4.5}$$

with $(*)'$ injective. The proposition now follows from Lemma 3.2 applied to $G = \mathrm{Map}(L)$. ☐

Remark 4.3. There are two points where this proof is not entirely constructive. The first is the choice of subsets $\Lambda_a \subseteq \mathbb{N}$ for each $a \in \mathbb{R}$ with $\Lambda_a \cap \Lambda_b$ finite if $a \neq b$. However, this may easily be made explicit by choosing an explicit bijection between $\mathbb{N}$ and $\mathbb{Q}$ and then letting $\Lambda_a \subseteq \mathbb{Q}$ be the sequence of rational numbers converging to $a \in \mathbb{R}$ given by truncating the binary expansion of a. The second point where it is non-constructive is in passing from diagram (4.4) to diagram (4.5) by throwing away countably many real numbers indexing the direct sum on the left-hand side. However, looking carefully at the proof of Lemma A.2, one may make this step constructive too.

5. Uncountability for pure mapping class groups and Torelli groups

Before completing the proof of Theorem B in §6, we first prove Theorem A in this section (Theorems 5.1 and 5.5), by adapting the methods of §4.

Theorem 5.1. *Let* S *be an infinite-type surface. Then the integral homology group*

$$H_i(\mathrm{PMap}(S))$$

is uncountable for every $i \geq 1$*. In fact it contains an isomorphic copy of* $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ *in every positive degree.*

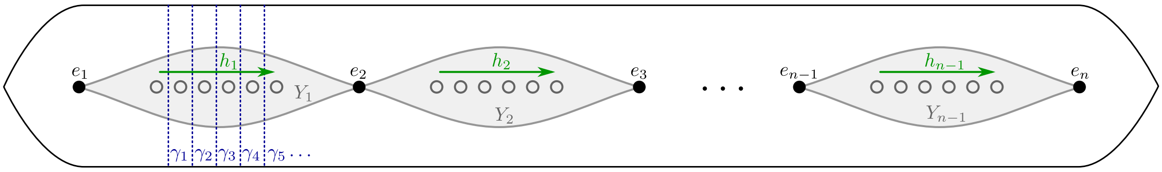

FIGURE 5.1. A surface with n non-planar ends $e_1, \ldots, e_n$ for $2 \leq n < \infty$. The top and bottom edges are identified to obtain a sphere, then the points $e_1, \ldots, e_n$ (together with a set of planar ends, which is not pictured) are removed, then we take a connected sum with a torus along each of the (infinitely many) small grey discs. The planar ends (not pictured) may have some or all of the non-planar ends $e_1, \ldots, e_n$ as limit points, but in any case lie *outside* of the subsurfaces $Y_1, \ldots, Y_{n-1}$, which support the handle shifts $h_1, \ldots, h_{n-1}$. The curves $\gamma_1, \gamma_2, \gamma_3, \ldots$ are chosen as illustrated. Thus (1) the handle shift h_1 sends γ_i to γ_{i+1} (up to isotopy) and (2) all γ_i are disjoint from $Y_2, \ldots, Y_{n-1}$, so $h_2, \ldots, h_{n-1}$ act trivially on γ_i.

We prove the theorem in three different cases depending on the number of non-planar ends of S.

5.1. **If S has at most one non-planar end.** In this case, we have $\overline{\mathrm{PMap}_c(S)} = \mathrm{PMap}(S)$ by [PV18, Theorem 4]. Let us first assume that S has at least two ends. Then, in [Dom, §8.2], Domat produces an embedding of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ into $\mathrm{PMap}(S)$ that further embeds into $H_1(\mathrm{PMap}(S))$, using countably infinite products of Dehn twists. In [Dom, §8.1], he proves that each $\mathbb{Z}$ summand of this direct sum is divisible in $H_1(\mathrm{PMap}(S))$. Hence it extends to an embedding

$$f\colon \bigoplus_{\mathfrak{c}} \mathbb{Q} \longrightarrow H_1(\mathrm{PMap}(S))$$

that makes the diagram (3.1) commute. The theorem now follows from Lemma 3.2. When S has at most one end, it necessarily must be the Loch Ness monster surface minus some open discs (since we have assumed that S has infinite type). The proof in [Dom, Appendix] (see also §4 above) then provides an embedding of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ satisfying the conditions of Lemma 3.2.

5.2. **If S has finitely many non-planar ends.** We now assume that S has n non-planar ends with $2 \leq n < \infty$. By [APV20, Corollary 6], we have

$$\mathrm{PMap}(S) \;\cong\; \overline{\mathrm{PMap}_c(S)} \rtimes \mathbb{Z}^{n-1}, \tag{5.1}$$

where $\mathbb{Z}^{n-1}$ is freely generated by $n-1$ handle shifts $h_1, \ldots, h_{n-1}$. As indicated in the proof of [APV20, Theorem 5], one may choose the handle shifts h_j to have pairwise disjoint support. Let Y_j be the support of h_j. Recall that each Y_j is a subsurface homeomorphic to the result of gluing handles onto $\mathbb{R} \times [0,1]$ periodically with respect to the transformation $(x, y) \mapsto (x+1, y)$. For convenience, we shall require that the i-th handle is attached to $[i, i+1] \times [0,1]$ and that h_j maps the i-th handle to the $(i+1)$-st handle. See Figure 5.1 for an illustration.

We now construct an embedding of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ into $\mathrm{PMap}(S)$ as in [Dom, §8.3] satisfying the conditions of Lemma 3.2. In fact, we will first construct an embedding of $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ into $H_1(\overline{\mathrm{PMap}_c(S)})$. Then, with a little more care, we further embed it into $H_1(\mathrm{PMap}(S))$.

We first choose a collection $\gamma_1, \gamma_2, \gamma_3, \ldots$ of simple closed curves, as illustrated in Figure 5.1, which satisfy the conditions in [Dom, Theorem 6.1]. Now let

$$f = \prod_{i=1}^{\infty} (T_{\gamma_i})^{i!} \in \overline{\mathrm{PMap}_c(S)},$$

where T_{γ_i} is the Dehn twist along the curve γ_i. As proved in [Dom, §8.1.1], the element f is non-trivial and divisible in $H_1(\overline{\mathrm{PMap}_c(S)})$ since our surface S has infinite genus. We now wish to choose a collection of subsets of the curves $\gamma_1, \gamma_2, \gamma_3, \ldots$ having good intersection properties even after considering the action of handle shifts.

Lemma 5.2. *For each $a \in \mathbb{R}$, one can choose an infinite subset Λ_a of $\mathbb{Z}$ such that the intersection of $\Lambda_a^k := \{i + k \mid i \in \Lambda_a\}$ with Λ_b is finite for any $a, b \in \mathbb{R}$ and $k \in \mathbb{Z}$ unless $a = b$ and $k = 0$.*

Proof. We first choose an infinite subset X of $\mathbb{Z}$ such that $X \cap X + k$ is finite for any integer $k \neq 0$. For example, we may choose $X = \{1, 2!, 3!, \ldots, n!, \ldots\}$. Then we choose a bijection ϕ between X and $\mathbb{Q}$. For $a \in \mathbb{R}$, let Λ'_a be a sequence of rational numbers approximating a. Thus $\Lambda'_a \cap \Lambda'_b$ is finite for any $a \neq b$. Let $\Lambda_a = \phi^{-1}(\Lambda'_a)$. This satisfies the conditions of the lemma. $\square$

For $a \in \mathbb{R}$ and $k \in \mathbb{Z}$, let Λ_a be chosen as in Lemma 5.2 and write $\Lambda^k_a = \{i+k \mid i \in \Lambda_a\}$. We enumerate elements of Λ^k_a as $\{a_{i,k}\}_{i\in\mathbb{N}}$ and let

$$f_{a,k} = \prod_{i=1}^{\infty} (T_{\gamma_{a_{i,k}}})^{i!} \in \overline{\mathrm{PMap}_c(S)}.$$

Just as before, $f_{a,k}$ is non-trivial and divisible in $H_1(\overline{\mathrm{PMap}_c(S)})$ for each $a \in \mathbb{R}$ and $k \in \mathbb{Z}$. Moreover, for any $a, b \in \mathbb{R}$ and $k, l \in \mathbb{Z}$, $\Lambda^k_a \cap \Lambda^l_b$ is finite unless $a = b$ and $k = l$. Thus any non-trivial finite product of elements $f_{a,k}$ again satisfies the hypotheses of [Dom, Theorem 6.1], and so any non-trivial finite product of elements $f_{a,k}$ is non-trivial and divisible in $H_1(\overline{\mathrm{PMap}_c(S)})$. This provides an an embedding of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ into $\overline{\mathrm{PMap}_c(S)}$ that extends to an embedding of $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ into $H_1(\overline{\mathrm{PMap}_c(S)})$.

The semi-direct product decomposition (5.1) implies that

$$H_1(\mathrm{PMap}(S)) \cong H_1(\overline{\mathrm{PMap}_c(S)})_{\mathbb{Z}^{n-1}} \oplus \mathbb{Z}^{n-1},$$

where $H_1(\overline{\mathrm{PMap}_c(S)})_{\mathbb{Z}^{n-1}}$ denotes the coinvariants of $H_1(\overline{\mathrm{PMap}_c(S)})$ under the action of the $n-1$ handle shifts $h_1, \ldots, h_{n-1}$. To summarise, we have so far constructed the diagram

$$\begin{tikzcd}
\bigoplus_{\mathfrak{c}} \mathbb{Z} \arrow[r, hook] \arrow[d, hook] & \overline{\mathrm{PMap}_c(S)} \arrow[r, hook] \arrow[d] & \mathrm{PMap}(S) \arrow[d] \\
\bigoplus_{\mathfrak{c}} \mathbb{Q} \arrow[r, hook] \arrow[drr, dashed] & H_1(\overline{\mathrm{PMap}_c(S)}) \arrow[r, hook] & H_1(\overline{\mathrm{PMap}_c(S)}) \rtimes \mathbb{Z}^{n-1} \arrow[d, "(*)"] \\
& & H_1(\overline{\mathrm{PMap}_c(S)})_{\mathbb{Z}^{n-1}} \oplus \mathbb{Z}^{n-1}
\end{tikzcd} \tag{5.2}$$

and by Lemma 3.2 it would be enough to show that the dotted arrow is injective. This is not true as constructed, but it will turn out to be true after restricting the direct sums on the left-hand side to an uncountable subcollection. We need to understand the images of the elements $f_{a,k}$ under the projection $(*)$ above, in other words when we quotient by the action of the handle shifts $h_1, \ldots, h_{n-1}$.

Since the support of $f_{a,k}$ is disjoint from that of h_j for all $j \geq 2$, it follows that h_j commutes with $f_{a,k}$ for any $a \in \mathbb{R}, k \in \mathbb{Z}$ and $j \geq 2$. On the other hand, for $j = 1$ we have $h_1^{-1} f_{a,k} h_1 = f_{a,k+1}$. In particular, the set of all finite products of elements in $\{f_{a,k}\}_{a\in\mathbb{R},k\in\mathbb{Z}}$ is invariant under the action of the $n-1$ handle shifts. Thus the action of $\mathbb{Z}^{n-1}$ on $H_1(\overline{\mathrm{PMap}_c(S)})$ preserves the embedded copy of $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ and on this subgroup its action is trivial for $h_2, \ldots, h_{n-1}$ and is given by $f_{a,k} \mapsto f_{a,k+1}$ for h_1. It follows that the projection $(*)$, restricted to our embedded copy of $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ indexed by $(a, k) \in \mathbb{R} \times \mathbb{Z}$, is given simply by identifying the $\mathbb{Q}$ summands indexed by (a, k) and (a, l) for all $k, l \in \mathbb{Z}$, for each fixed $a \in \mathbb{R}$.

If we now restrict the indexing set of the direct sum to $(a, 0)$ for $a \in \mathbb{R}$, the projection $(*)$ is injective on its image, so the diagonal dotted arrow in (5.2) is injective. Thus we finally have an embedding of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ in $\mathrm{PMap}(S)$ satisfying the conditions of Lemma 3.2, which completes the proof in this case.

Remark 5.3. When S has infinitely many non-planar ends, and at least one of them is isolated in the space of non-planar ends, our proof above still works. However, it does not appear to work directly for surface such as the Cantor blooming tree surface.

5.3. **If S has infinitely many non-planar ends.** By [APV20, Corollary 6], we have in this case that

$$\mathrm{PMap}(S) \cong \overline{\mathrm{PMap}_c(S)} \rtimes \mathbb{Z}^{\mathbb{N}}.$$

In particular, $\mathbb{Z}^{\mathbb{N}}$ is a retract of $\mathrm{PMap}(S)$. Thus the natural induced map $H_i(\mathbb{Z}^{\mathbb{N}}) \to H_i(\mathrm{PMap}(S))$ is split injective in every degree. Theorem 5.1 in this situation is therefore an immediate corollary of the following lemma.

Lemma 5.4. *The homology $H_i(\mathbb{Z}^{\mathbb{N}})$ contains a copy of $\mathbb{Z}^{\mathbb{N}}$ in every positive degree. Hence it contains a copy of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ in every positive degree.*

Proof. This follows from the Künneth theorem applied to the decomposition $\mathbb{Z}^{\mathbb{N}} \cong \mathbb{Z}^{\mathbb{N}} \times \mathbb{Z}^i$. $\square$

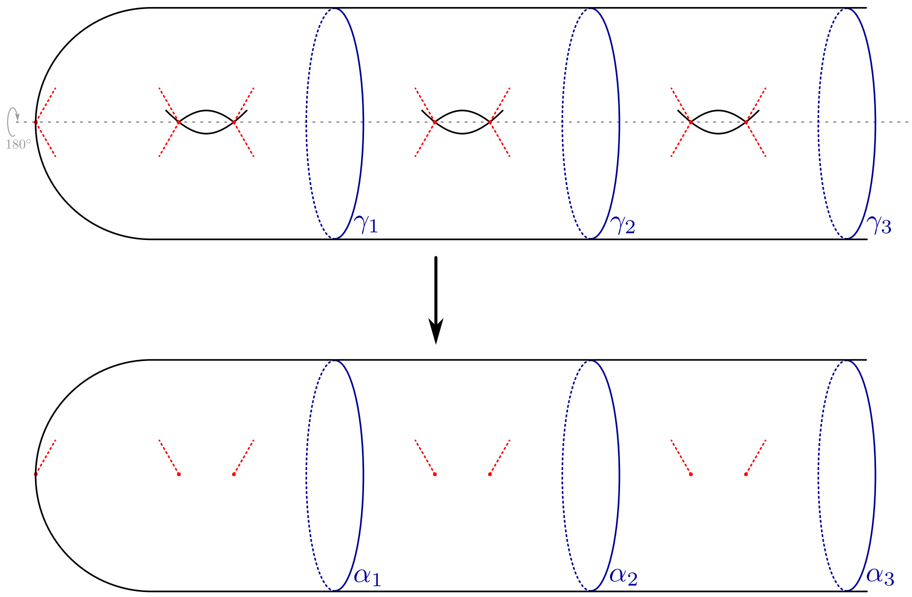

FIGURE 6.1. The branched double covering (6.1). After removing the subset marked in red (which includes the branch points), this restricts to the (genuine) double covering (6.2).

5.4. **Uncountability for Torelli groups.** Given a surface S, recall that the Torelli group $\mathcal{T}(S)$ is the kernel of the natural homomorphism $\mathrm{Map}(S) \to \mathrm{Aut}(H_1(S))$.

Theorem 5.5. *Let S be an infinite-type surface. Then the integral homology group*

$$H_i(\mathcal{T}(S))$$

is uncountable for every $i \geq 1$. In fact it contains a copy of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ in every positive degree.

Proof. This follows immediately from the proof of [Dom, Theorem 9.1] and Lemma 3.2. We sketch the proof for the reader's convenience. In fact, when the surface has at least two ends, Domat first embeds $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ into $\mathrm{PMap}(S)$ using infinite products of pseudo-Anosov elements supported on disjoint finite-type surfaces. The existence of such elements is proved in [Dom, Lemma 8.3]. By [Dom, Theorem 7.1] such elements are non-trivial in $H_1(\mathrm{PMap}(S))$ and the fact that they are divisible in $H_1(\mathcal{T}(S))$ now follows from the exact same proof as in [Dom, Theorem 7.1]. The case where S has only one end is taken care of in [Dom, Appendix]; see also §4. In particular, we always have an embedding of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ into $\mathcal{T}(S)$ satisfying the conditions of Lemma 3.2. The theorem thus follows from Lemma 3.2. □

6. Descending along double branched covers

In this section we generalise techniques of Malestein and Tao [MT] — who proved uncountability of homology in degree 1 for the mapping class group of $\mathbb{R}^2 \smallsetminus \mathcal{C}$ — to higher degrees and to the more general class of surfaces from Theorem B, completing the proof of that theorem.

To begin with, we will put stronger assumptions on the surface S: we assume that it has genus 0, empty boundary and that its space of ends is of the form $\Upsilon^+(E)$, where E has a topologically distinguished point. This means that S may be written as $\mathcal{R}(\mathbb{S}^2 \smallsetminus E)$, where $\mathcal{R}(\Sigma)$ denotes the *ray surface* associated to a surface Σ:

Definition 6.1. Let Σ be any connected surface without boundary and write Σ_1 (respectively Σ_2) for the surface obtained by removing one (respectively two disjoint) open discs from Σ. The *ray surface* $\mathcal{R}(\Sigma)$ is the surface obtained by gluing together infinitely many copies of Σ_2 and "capping off" in one direction with a single copy of Σ_1. See the top half of Figure 6.1 for an example where $\Sigma = T^2$ is the torus; thus $\mathcal{R}(T^2)$ is the Loch Ness monster surface.

Remark 6.2. This is the same as the surface denoted by $\mathfrak{L}(\Sigma)$ in [PW] with its boundary capped off by a disc.

Denote by L the Loch Ness monster surface and consider its branched double covering $L \to \mathbb{R}^2$ depicted in Figure 6.1. This may also be written as

$$L \cong \mathbb{S}^2 \sharp \mathcal{R}(T^2) \longrightarrow \mathbb{S}^2 \sharp \mathcal{R}(\mathbb{S}^2) \cong \mathbb{R}^2. \tag{6.1}$$

This decomposition corresponds to cutting along the curves depicted in the figure, together with an additional curve γ_0 or α_0, which is not depicted. Notice that there are exactly two branch points (of order 2) in each copy of $\mathbb{S}^2$ in $\mathcal{R}(\mathbb{S}^2)$ and one additional branch point in the copy of $\mathbb{S}^2$ in the extra connected summand. Let us now choose once and for all a topologically distinguished point $x \in E$ (this exists by hypothesis) and embed pairwise disjoint copies of E into $\mathbb{S}^2 \sharp \mathcal{R}(\mathbb{S}^2)$ so that:

- each copy of E lies entirely in one of the copies of $\mathbb{S}^2$,
- the point $x \in E$ is sent to a branch point of (6.1),
- each branch point of (6.1) is in the image of one of the embeddings of E.

We denote by X the complement of these embedded copies of E and we denote by $Y \subset \mathbb{S}^2 \sharp \mathcal{R}(T^2)$ the pre-image of $X \subset \mathbb{S}^2 \sharp \mathcal{R}(\mathbb{S}^2)$ under (6.1). Notice that:

$$\begin{aligned} Y &\cong (\mathbb{S}^2 \smallsetminus V) \sharp \mathcal{R}(T^2 \smallsetminus (V \sqcup V)) \\ X &\cong (\mathbb{S}^2 \smallsetminus E) \sharp \mathcal{R}(\mathbb{S}^2 \smallsetminus (E \sqcup E)) \cong \mathcal{R}(\mathbb{S}^2 \smallsetminus E) \cong S, \end{aligned}$$

where V denotes the wedge sum of two copies of E at the basepoint x. Since we have in particular removed all branch points of the branched double covering, we obtain by restriction a (genuine) double covering

$$Y \longrightarrow X \tag{6.2}$$

depicted in Figure 6.1.

We fix compatible basepoints on X and Y and denote by H the index-2 subgroup of $\pi_1(X)$ corresponding to this double covering. We also write $\mathrm{Map}_*(X)$ and $\mathrm{Map}_*(Y)$ for the *based* mapping class groups of X and Y, given by isotopy classes of self-homeomorphisms that fix the basepoint.

Lemma 6.3. *The action of* $\mathrm{Homeo}_*(X)$ *on* $\pi_1(X)$ *preserves the subgroup* H*.*

Proof. We first describe the subgroup $H \subset \pi_1(X)$ intrinsically. A based loop γ in X lies in H if and only if its lift to Y is a closed loop. This occurs if and only if the sum of its winding numbers around all branch points of the branched double covering (6.1) is even. We therefore have to show that *if* the sum of these winding numbers is even for γ, then the same is true for $\varphi \circ \gamma$, where φ is any based self-homeomorphism of X.

A subtle point here is the meaning of *winding number* (which we only need to define mod 2): a simple loop in the surface X has winding number ± 1 around an end $e \neq \infty$ if it separates X into two pieces, one containing e and the other containing the end ∞. Here ∞ denotes the end corresponding to going off to infinity to the right in Figure 6.1. More precisely, the end space of X is the one-point compactification $E\omega + 1$ of a countably infinite disjoint union of copies of E and ∞ denotes the point at infinity of the one-point compactification. By Corollary 2.7 and our assumption that E has a topologically distinguished point, the point $\infty \in E\omega + 1$ is also topologically distinguished. Thus any self-homeomorphism φ of X fixes ∞, meaning that the notion of "winding number" is preserved by φ.

Let us now show that if the sum of the winding numbers of γ around all branch points of X is even, then the same is true for $\varphi \circ \gamma$. The end space $E\omega + 1$ of X has a topologically distinguished subset $\{x\}\omega$ given by the copy of the topologically distinguished point x in each copy of E. But this is precisely the set of branch point of the branched double covering (6.1). Thus the self-homeomorphism φ must send each end of X corresponding to a branch point to another end of X corresponding to a branch point. Its effect on winding numbers around branch points is therefore simply to permute them; so in particular their *sum* is preserved. Hence if the sum of winding numbers around branch points is even for γ, then the sum of winding numbers around branch points will also be even for $\varphi \circ \gamma$. □

Remark 6.4. The proof of Lemma 6.3 is where our assumption that the space E has a topologically distinguished point is used decisively. The lemma would be false without this assumption. See also Remark 6.5.

We may now complete the proof of Theorem B under the stronger assumptions that we are currently making (we explain how to remove these assumptions at the end of this section).

Proof of Theorem B under additional assumptions. It follows from Lemma 6.3 that each based homeomorphism of X lifts uniquely to a based homeomorphism of Y, giving us a continuous map $\mathrm{Homeo}_*(X) \to \mathrm{Homeo}_*(Y)$, which on π_0 induces

$$\mathrm{Map}_*(X) \longrightarrow \mathrm{Map}_*(Y). \tag{6.3}$$

Filling in all *planar* ends of a surface is a functorial operation on the category of surfaces, so by filling in all planar ends of Y we obtain a continuous map $\mathrm{Homeo}_*(Y) \to \mathrm{Homeo}_*(L)$, which on π_0 induces

$$\mathrm{Map}_*(Y) \longrightarrow \mathrm{Map}_*(L). \tag{6.4}$$

Composing (6.3) and (6.4) with the forgetful map $\mathrm{Map}_*(L) \to \mathrm{Map}(L)$, we obtain

$$\mathrm{Map}_*(X) \longrightarrow \mathrm{Map}(L). \tag{6.5}$$

Let $\alpha_1, \alpha_2, \ldots$ be the collection of simple closed curves on X depicted in Figure 6.1. Since γ_i is a double covering of α_i, we see that

$$(T_{\alpha_i})^2 \longmapsto T_{\gamma_i}$$

under (6.5). Recalling the constructions at the beginning of §4, for each $a \in \mathbb{R}$ we let

$$g_a = \prod_{i \in \Lambda_a} (T_{\alpha_i})^{2i!} \in \mathrm{Map}_*(X).$$

Recall that the summands of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ are indexed by a (co-countable) subset of $\mathbb{R}$. If we write e_a for a generator of the summand corresponding to a, then the map

$$e_a \mapsto g_a \colon \bigoplus_{\mathfrak{c}} \mathbb{Z} \longrightarrow \mathrm{Map}_*(X)$$

makes the following triangle commute:

$$\begin{tikzcd} & \bigoplus_{\mathfrak{c}} \mathbb{Z} \arrow[dl] \arrow[dr] & \\ \mathrm{Map}_*(X) \arrow[rr, "(6.5)"] & & \mathrm{Map}(L), \end{tikzcd} \tag{6.6}$$

where the right-hand diagonal map was shown in §4 to be part of a factorisation $\bigoplus_{\mathfrak{c}} \mathbb{Z} \to \mathrm{Map}(L) \to \bigoplus_{\mathfrak{c}} \mathbb{Q}$ of the standard inclusion. We have therefore shown that the standard inclusion of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ into $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ also factors through $\mathrm{Map}_*(X)$. Now consider the diagram

$$\begin{tikzcd} \bigoplus_{\mathfrak{c}} \mathbb{Z} \arrow[r] \arrow[dr] & \mathrm{Map}_*(X) \arrow[r, "\varphi"] \arrow[d, two heads] & \bigoplus_{\mathfrak{c}} \mathbb{Q} \\ & \mathrm{Map}(X), & \end{tikzcd} \tag{6.7}$$

where the middle vertical map forgets the basepoint. This is part of the Birman exact sequence for X, and its kernel is $\pi_1(X)$, which is in particular countable. Let us denote this kernel by K and consider its image $\varphi(K) \subset \bigoplus_{\mathfrak{c}} \mathbb{Q}$. Since $\varphi(K)$ is countable and each of its elements has only finitely many non-zero coordinates in $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ (because it is a direct *sum*), it is contained in the subgroup of $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ given by the direct sum of only countably many of the copies of $\mathbb{Q}$. If we take the quotient by this subgroup, the resulting group is again isomorphic to $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ and the homomorphism φ now descends to $\mathrm{Map}(X)$. On the left-hand side of (6.7), we may compose with the inclusion of the corresponding sub-direct-summand of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ (which is again isomorphic to $\bigoplus_{\mathfrak{c}} \mathbb{Z}$); this ensures that the composition across the top row of the following diagram is still the standard inclusion of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ into $\bigoplus_{\mathfrak{c}} \mathbb{Q}$:

$$\begin{tikzcd} \bigoplus_{\mathfrak{c}} \mathbb{Z} \arrow[r] \arrow[rrrr, bend left, two heads] & \bigoplus_{\mathfrak{c}} \mathbb{Z} \arrow[r] \arrow[dr] & \mathrm{Map}_*(X) \arrow[r, "\varphi"] \arrow[d, two heads] & \bigoplus_{\mathfrak{c}} \mathbb{Q} \arrow[r, two heads] & \bigoplus_{\mathfrak{c}} \mathbb{Q} \\ & & \mathrm{Map}(X) \arrow[urr] & & \end{tikzcd} \tag{6.8}$$

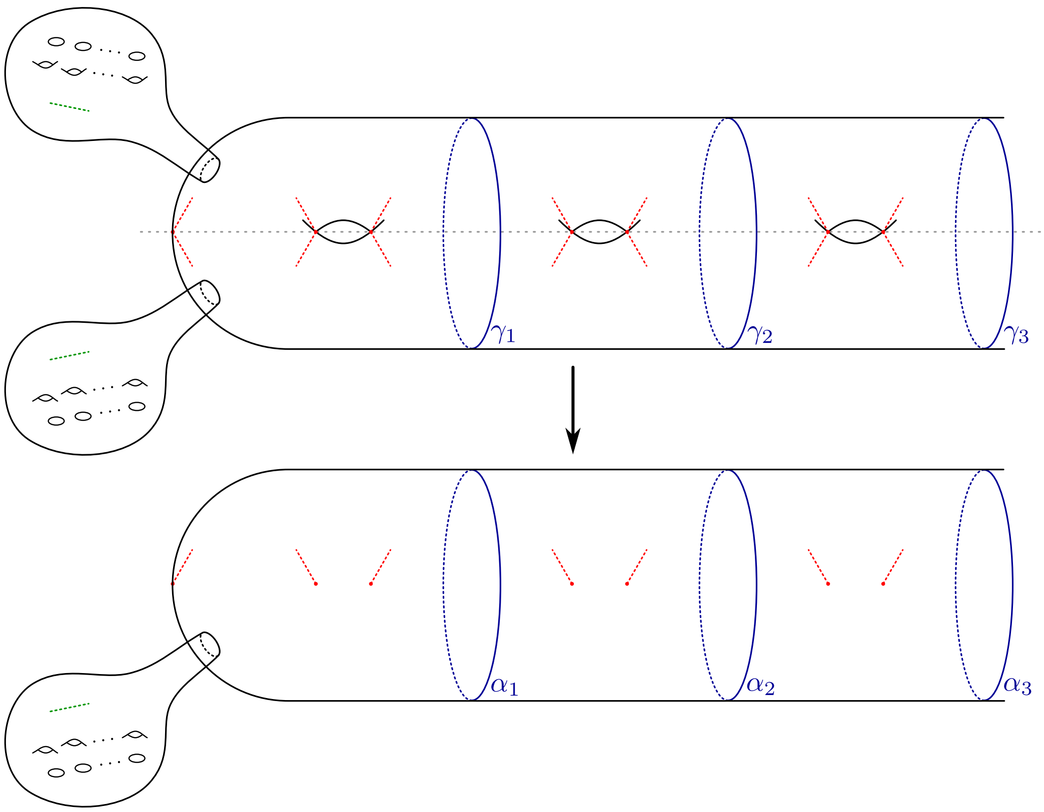

FIGURE 6.2. A modification of the branched double covering depicted in Figure 6.1.

Thus we have shown that the standard inclusion of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ into $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ factors through $\mathrm{Map}(X)$. This standard inclusion induces an injection on homology in all degrees, by Lemma A.3 and the fact that $H_*(A) = \Lambda^*(A)$ for torsion-free abelian groups A, so it follows that we have an injection

$$\Lambda^*(\bigoplus_{\mathfrak{c}} \mathbb{Z}) = H_*(\bigoplus_{\mathfrak{c}} \mathbb{Z}) \longhookrightarrow H_*(\mathrm{Map}(X)) = H_*(\mathrm{Map}(S)).$$

This completes the proof of Theorem B under the additional assumptions on the surface S. □

We next show how to modify the argument above to allow the more general surfaces S considered in the theorem.

Proof of Theorem B in general. The proof follows exactly the same strategy as the proof in the special case above, so we just explain the steps that differ slightly.

In general, the surface S is of the form pictured at the bottom of Figure 6.2, where we have taken a connected sum of the surface considered previously with another surface having finite genus, finitely many boundary components and an end space none of whose points are similar to the topologically distinguished point $x \in E$. We may correspondingly modify the total space of the double covering by taking two connected sums with this surface (no new branch points are introduced).

Lemma 6.3 generalises directly to this setting, giving us a homomorphism that lifts (based) mapping classes up the double covering. Filling in all planar ends and the finitely many boundary components upstairs, we as before obtain the Loch Ness monster surface L. The only small difference is that we have finitely many extra handles on the left of the diagram, but this just corresponds to assuming that we have chosen all of our subsets $\Lambda_a \subseteq \mathbb{N}$ to be disjoint from the initial sequence $\{1, \ldots, g\}$ for some g and then re-indexing the curves $\gamma_i \mapsto \gamma_{i-g}$. With these modifications, the rest of the proof is identical to the proof in the special case given above. □

Remark 6.5. It is essential to assume in Theorem B that E_2 has a topologically distinguished point. Indeed, if we do not assume this, then the theorem is false. For example, without this assumption, the theorem would assert that the homology of $\mathrm{Map}(\mathbb{S}^2 \smallsetminus \mathcal{C})$ is uncountable in all positive degrees, since

$\Upsilon^+(\mathcal{C}) \cong \mathcal{C}$. However, the first and second homology groups of $\mathrm{Map}(\mathbb{S}^2 \smallsetminus \mathcal{C})$ are known to be 0 and $\mathbb{Z}/2$ respectively [CC21].

7. Torsion elements

We prove in this section a splitting result which helps us to find torsion elements in the mapping class groups of genus 2 surfaces. Let $S_{g,b}$ denote the connected, compact, orientable surface of genus g with $b \geq 1$ boundary components. Let us first recall the following calculation, see for example [Pow78] and [Kor02, Theorem 5.1].

Theorem 7.1. *$H_1(\mathrm{Map}(S_{2,b})) \cong \mathbb{Z}/10$.*

We will promote this result to more complicated surfaces via the following lemma. Recall that, given any other surface S' with non-empty boundary, we can construct a new surface $S_{g,b} \natural S'$ by identifying an interval in the boundary of $S_{g,b}$ with an interval in the boundary of S'. Let us consider the map

$$H_i(\mathrm{Map}(S_{g,b})) \longrightarrow H_i(\mathrm{Map}(S_{g,b} \natural S')) \tag{7.1}$$

induced by extending any homeomorphism of $S_{g,b}$ to one of $S_{g,b} \natural S'$ by the identity on S', as well as its restriction

$$H_i(\mathrm{PMap}(S_{g,b})) \longrightarrow H_i(\mathrm{PMap}(S_{g,b} \natural S')) \tag{7.2}$$

to pure mapping class groups.

Lemma 7.2. *Let S' be a planar surface with one boundary component. Then the maps (7.1) and (7.2) split; in particular they are injective.*

Proof. This follows from the fact that the map

$$\mathrm{Map}(S_{g,b}) \longrightarrow \mathrm{Map}(S_{g,b} \natural S') \tag{7.3}$$

splits up to homotopy, as does its restriction to pure mapping class groups. To see this, first note that, from the classification of surfaces, S' must be of the form $\mathbb{D}^2 \smallsetminus E$ for some end space E. A splitting up to homotopy for (7.3) is then provided by the map

$$\mathrm{Map}(S_{g,b} \natural S') \longrightarrow \mathrm{Map}(S_{g,b} \natural \mathbb{D}^2) \cong \mathrm{Map}(S_{g,b})$$

induced by sending a homeomorphism of $S' = \mathbb{D}^2 \smallsetminus E$ to its unique extension to a homeomorphism of $\mathbb{D}^2$ (which is the Freudenthal compactification of $\mathbb{D}^2 \smallsetminus E$). □

Remark 7.3. The proof of Lemma 7.2 does not use any special property of the compact surface $S_{g,b}$. The same statement holds for any connected surface with non-empty boundary in place of $S_{g,b}$.

Proof of Theorem C. Since S has genus 2 and non-empty boundary, we may write $S = S_{2,b} \natural S'$, where $b \geq 1$ is the number of boundary components of S (recall that we always assume that the boundaries of our surfaces are compact, so they must be a finite disjoint union of circles) and S' is planar and has exactly one boundary component. The result is then an immediate consequence of Theorem 7.1 and Lemma 7.2. □

8. Some open problems

In this section we propose some open questions, in addition to Questions 0.5, 0.10 and 0.11 in the introduction. So far, our calculations suggest the answer to the following question could be positive.

Question 8.1. *Let S be an infinite-type surface. Suppose that, for some $i \geq 1$, the group $H_i(\mathrm{Map}(S))$ is countable. Is $H_i(\mathrm{Map}(S))$ finitely generated for all i?*

Question 8.2. *Let $S_{g,1}$ be the connected, compact, orientable surface of genus g and with one boundary component. Does the forgetful map $\mathrm{Map}(S_{g,1} \smallsetminus \mathcal{C}) \to \mathrm{Map}(S_{g,1})$ induce isomorphisms on homology in all degrees?*

Remark 8.3. When $g = 0$, a positive answer follows from [PW, Theorem A]. The answer in degree one (and for any g) has been proven to be positive in [CC22, Theorem 2.3]. On the other hand, the answer would be negative if we considered the sphere instead of $S_{g,1}$, since $H_2(\mathrm{Map}(\mathbb{S}^2 \smallsetminus \mathcal{C})) \cong \mathbb{Z}/2$ by [CC21, Theorem A.2]. It would also be negative if we took the plane instead of $S_{g,1}$, since $H_i(\mathrm{Map}(\mathbb{R}^2 \smallsetminus \mathcal{C})) \cong \mathbb{Z}$ for all even i by [PW, Corollary B].

Question 8.4. *Let S be an infinite-type surface with a single boundary component and suppose that its mapping class group $\mathrm{Map}(S)$ is acyclic. Is S necessarily a 1-holed binary tree surface?*

A. Abelian groups

We collect here a few facts about abelian groups that are needed in our proofs. For a comprehensive treatment of the theory of abelian groups, we refer to [Fuc70].

Recall that an abelian group A is called *divisible* if for each element $a \in A$ and positive integer n, there is another element $b \in A$ such that $a = nb$. An abelian group A is called *injective* if for every injective homomorphism of abelian groups $\iota\colon B \to C$ and homomorphism $f\colon B \to A$, there is a homomorphism $g\colon C \to A$ such that $g \circ \iota = f$. By [Fuc70, Theorems 21.1 and 24.5], an abelian group is divisible if and only if it is injective. In particular:

Lemma A.1. *Every injective homomorphism from a divisible abelian group admits a retraction.*

Proof. Let A be a divisible abelian group and let $\iota\colon A \to C$ be an injective homomorphism. Since A is injective, taking $B = A$ and $f = \mathrm{id}$ above, we obtain a retraction of ι. □

Lemma A.2. *Suppose that we have homomorphisms of abelian groups*

$$\bigoplus_{\mathfrak{c}} \mathbb{Q} \xrightarrow{f} A \xrightarrow{g} B$$

where f is injective and g has countable kernel. Then, after restricting the direct sum on the left to a subcollection of the same cardinality, the composition $g \circ f$ is also injective.

Proof. Consider the subgroup $K := \ker(g \circ f) = f^{-1}(\ker(g)) \subset \bigoplus_{\mathfrak{c}} \mathbb{Q}$. Since $\ker(g)$ is countable and f is injective, K is a countable subgroup of $\bigoplus_{\mathfrak{c}} \mathbb{Q}$. Each element of K has only finitely many non-zero coordinates in the direct sum and K has countably many elements; thus K is contained in the sub-direct-sum given by countably many $\mathbb{Q}$ summands. After removing these summands from the direct sum, the composition $g \circ f$ is injective. □

Lemma A.3. *For any set I, the canonical inclusion $\bigoplus_I \mathbb{Z} \hookrightarrow \bigoplus_I \mathbb{Q}$ induces an injective map of graded abelian groups*

$$\Lambda^*\Big(\bigoplus_I \mathbb{Z}\Big) \hookrightarrow \Lambda^*\Big(\bigoplus_I \mathbb{Q}\Big). \tag{A.1}$$

To prove this, we first recall the following basic calculation:

Lemma A.4. $\Lambda^*(\mathbb{Z}) \cong \mathbb{Z}[0] \oplus \mathbb{Z}[1]$ *and* $\Lambda^*(\mathbb{Q}) \cong \mathbb{Q}[0] \oplus \mathbb{Q}[1]$.

Proof. The only non-obvious statement is that $\Lambda^i(\mathbb{Q}) = 0$ for $i \geq 2$. To see this, first recall that

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \tag{A.2}$$

via an isomorphism that sends $a_1 \otimes a_2 \otimes \cdots \otimes a_i \mapsto a_1 a_2 \cdots a_i$. The $\mathbb{Z}$-module $\Lambda^i(\mathbb{Q})$ is the quotient of this tensor power by the sub-$\mathbb{Z}$-module generated by all elements $a_1 \otimes a_2 \otimes \cdots \otimes a_i$ with $a_j = a_k$ for some $j \neq k$. Thus to prove that $\Lambda^i(\mathbb{Q}) = 0$ we have to show that every rational number is a $\mathbb{Z}$-linear combination of rational numbers of the form $b^2 a_3 \cdots a_i$. For $i \geq 3$ this is obvious. For $i = 2$ it follows from Lagrange's four-square theorem. □

Proof of Lemma A.3. By [Bro82, §V.6.2, V.6.3], for any abelian group A we have

$$\Lambda^*\Big(\bigoplus_I A\Big) \cong \Lambda^*\Big(\operatorname*{colim}_{J \subseteq I} \bigoplus_J A\Big) \cong \operatorname*{colim}_{J \subseteq I} \Lambda^*\Big(\bigoplus_J A\Big) \cong \operatorname*{colim}_{J \subseteq I} \bigotimes_J \Lambda^*(A), \tag{A.3}$$

where the colimit is taken over finite subsets J of I. For any finite set J, the canonical map

$$\bigotimes_J \Lambda^*(\mathbb{Z}) \longrightarrow \bigotimes_J \Lambda^*(\mathbb{Q})$$

is injective by Lemma A.4 and the natural isomorphisms (A.2). Thus (A.1) is also injective since the colimit on the right-hand side of (A.3), for $A = \mathbb{Z}$ or $A = \mathbb{Q}$, is taken over a direct system in which all maps are inclusions of direct summands. □

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