Topological representations of motion groups and mapping class groups – a unified functorial construction

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Abstract

Groups of a topological origin, such as braid groups and mapping class groups, often have "wild" representation theory. One would therefore like to construct representations of these groups topologically, in order to use topological tools to study them. A key example is Bigelow and Krammer's proof of the linearity of the braid groups, which uses topologically-defined representations due to Lawrence and Bigelow.

We give a unified construction of such topological representations: a large family of representations of a category containing all *mapping class groups* and *motion groups* in a fixed dimension. This unifies many previously-known constructions, including those of Lawrence-Bigelow, together with many new families of representations. Varying one parameter of the construction also produces *pro-nilpotent* towers of representations, of which we give several non-trivial low-dimensional examples. Moreover, the rich structure of this category helps to clarify the representation theory of these families of groups.

Introduction

The representation theory of *mapping class groups* and of *motion groups* is very rich, and the subject of much active research – see Birman and Brendle's survey [BB05, §4] or Margalit's expository paper [Mar19] for instance. Their representation theory is in particular known to be *wild*, meaning (roughly) that there can be no classification system for their irreducible representations with finitely many parameters. For the braid groups on $n \geqslant 6$ strands, this follows from the work of Erdmann and Nakano [EN02] on the representation theory of Hecke algebras; for $n = 3$ (and thus $n = 4$ due to the surjection $\mathbf{B}_4 \twoheadrightarrow \mathbf{B}_3$) it follows from work of Krugljak and Samoĭlenko [KS80].

In order to understand the representation theory of these groups, we must therefore view them not just as abstract groups, but use their associated geometry and topology. Combining this philosophy (of studying these groups via their *topology*) with the philosophy of representation theory itself (of studying groups via *linear algebra*), one is led naturally to homology — more precisely, the idea of studying these groups via their actions on the homology of topological spaces naturally associated to them.

In parallel, another point of view adopted in this paper is to treat simultaneously *families of groups* that belong together geometrically. Here a family of groups means a collection of groups G_i indexed by a partially-ordered set (typically the natural numbers $\mathbb{N}$) equipped with morphisms $G_i \to G_j$ whenever $i \leqslant j$. The idea is to require representations to respect the natural coherences between the different groups in the family: this is more meaningful, since the groups naturally arise with certain coherences between them. It should also make the representation theory a little less wild, by incorporating more of the natural structure of the families of groups.

For the family of braid groups $\mathbf{B}_n$ (where the inclusions are induced by adding a strand on the left), Lawrence [Law90] and Bigelow [Big01] constructed well-known families of linear representations, called the *Lawrence-Bigelow representations*, following different methods. They may be defined via actions on twisted homology groups of configuration spaces of unordered points in a

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marked 2-disc. The most famous of these are the family of (reduced) *Burau representations* originally introduced by Burau [Bur35] and the family of *Lawrence-Krammer-Bigelow representations*, which Bigelow [Big01] and Krammer [Kra02] independently proved to be faithful.

Our goal is to develop a much more general construction, which:

- applies simultaneously to all mapping class groups and motion groups in a given dimension,
- respects the coherences between these groups, by defining the representations on a category containing all of these groups, whose morphisms encode the coherences,
- produces a much wider family of representations,
- produces interesting representations also over non-commutative rings.

We now give a general idea of the principle of our construction and refer the reader to §1 for the precise statement of our results. See in particular Theorems 1.1–1.4 for the classical braid groups, surface braid groups, loop braid groups and mapping class groups of surfaces respectively.

Inputs and outputs of the construction. For fixed $d \geqslant 2$, the construction takes as input the data (Z, ℓ, i), described in more detail below, and produces representations of all mapping class groups and motion groups in dimension d, which are "coherent" with each other in an appropriate sense.

More precisely, we consider a certain topological category $\mathfrak{U}\mathcal{D}_d$, whose automorphism groups are (roughly) diffeomorphism groups of d-manifolds fixing a given configuration of submanifolds in their interior. Its associated discrete category $\pi_0(\mathfrak{U}\mathcal{D}_d)$ therefore contains all mapping class groups of d-manifolds, as well as all motion groups in d-manifolds (as normal subgroups, see §4.3.3). Thus a coherent family of representations of mapping class groups and motion groups in dimension d means a representation of the category $\pi_0(\mathfrak{U}\mathcal{D}_d)$.

Our construction produces "twisted representations" of this category, namely functors

$$\pi_0(\mathfrak{U}\mathcal{D}_d) \longrightarrow \mathrm{Mod}_\bullet$$

to the category $\mathrm{Mod}_\bullet$ of right-modules over rings. This contains the category Mod_R for every ring R, and it is an important question when our representations take values in such a subcategory; we introduce a general criterion for this in §5.1.5. The construction depends on three parameters:

- a submanifold $Z \subseteq \mathbb{R}^d$ and a subgroup G of its mapping class group $\pi_0(\mathrm{Diff}(Z))$,
- an integer $\ell \geqslant 2$,
- an integer $i \geqslant 0$.

Varying the parameters. Each of the parameters above may be varied to obtain interesting representations. For example, the family of Lawrence-Bigelow representations of $\mathbf{B}_n$ depends on an integer parameter $k \geqslant 1$; our construction recovers this in the case when Z is a 0-dimensional manifold of size k (and $d = 2$, $\ell = 2$, $i = k$). In dimensions $d \geqslant 3$, moreover, it becomes interesting to take higher-dimensional submanifolds of $\mathbb{R}^d$ for Z, in particular in the case of the *loop braid groups* (the group of motions of an n-component unlink in $\mathbb{R}^3$).

The parameter $\ell \geqslant 2$ controls the ground ring over which the representation is defined: it is the group-ring of a group of nilpotency class at most $\ell - 1$. There are cases where the output of our construction is independent of ℓ (hence the ground ring is commutative), but there are also many interesting cases where we obtain an infinite tower of representations as $\ell \to \infty$. In particular, the family of Lawrence-Krammer-Bigelow representations is the $\ell = 2$ term of such a tower.

The parameter $i \geqslant 0$ controls the degree in which we take homology. In the case of the Lawrence-Bigelow representations, there is only one interesting degree in which we can take homology (the homology in other degrees being trivial). However, more generally there can be many interesting degrees in which to take homology. For example, if we consider the family of loop braid groups and the "naive" analogue of the Lawrence-Bigelow representations (taking Z to be a 0-dimensional manifold of size k), then there are non-trivial homology groups in all degrees $k \leqslant i \leqslant 2k$. These particular representations will be studied in more detail in the sequel paper [PS22a].

Other variations. There are also several natural variations of our construction. First, one may change the "flavour" of (twisted) homology that we use: notably *Borel-Moore* homology rather than ordinary homology (also cohomology, compactly-supported cohomology, etc.). Second, there is also a "solvable" (as opposed to "nilpotent") variant of our construction, where the parameter ℓ controls the degree of solvability of the group(-ring) over which the representations are defined; see §5.1.6. We focus in this paper on the nilpotent version since the lower central series of the relevant groups is generally better-understood than their derived series. In particular, the stopping or non-stopping of the lower central series of certain "mixed" motion groups determines whether or not we obtain an infinite tower of representations as $\ell \to \infty$ (see above). This question is answered for mixed versions of classical, loop and surface braid groups in [DPS21].

Another possible variation is to adapt this construction to as to be *iterable*, inspired by the *Long-Moody construction* [Lon94] of representations of the classical braid groups. This will be done in the sequel paper [PS22b], to obtain a general homological construction for mapping class groups and motion groups, in the philosophy of the present paper, that simultaneously generalises the Lawrence-Bigelow representations and the Long-Moody construction.

Perspectives. (*Irreducibility*) A natural question for the representations arising from the above constructions is to determine which of them are *irreducible*, which are *indecomposable* and how they decompose if they are decomposable. In particular, whether the new families of representations we construct are irreducible or not is a fundamental question insofar as these would improve the understanding of the (wild) representation theories of the families of groups we consider.

(*Polynomiality*) In addition, keeping our functorial point of view, notions of polynomiality may be introduced for functors from the discrete categories considered here to module categories: we may investigate whether the above functorial representations are *polynomial*. A key application of this property is homological stability with twisted coefficients, as there are many results in the literature establishing homological stability for families of groups with *polynomial coefficients*; see [RW17] for a general framework encompassing all of the classical examples.

These two questions are beyond the scope of the current paper, but will be addressed in the sequel paper [PS21a]: namely, using Borel-Moore homology, many of the representations constructed in the present paper have a natural free generating set; we compute explicit formulas for the group action, thus allowing us to study the irreducibility and polynomiality questions.

(*Faithfulness*) Furthermore, among the underlying prospects motivating our study stands the question of which motion groups and mapping class groups are *linear* — in the sense that they act faithfully on a finite-dimensional vector space. Indeed, this question remains wide open in the vast majority of cases. Classical braid groups provide a famous positive answer to this question by Bigelow and Krammer [Big01; Kra02], but there are also families with a negative answer, such as automorphism groups of free groups — which may be viewed as mapping class groups of certain 3-manifolds — by Formanek and Procesi [FP92]. The key representations that have provided positive answers have all been homological representations. This suggests that a systematic treatment of homological constructions that work for all motion groups and all mapping class groups is an important and natural avenue for future investigations on the linearity of these families of groups.

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Outline. First, in §1, we give a more detailed overview of our constructions and results. In §2, we introduce in detail the various tools to define general homological representations, *assuming given* a functor taking values in covering spaces. In §3–§4, we introduce the appropriate categorical framework to deal with all motion groups and mapping class groups simultaneously. In §5.1 and §5.3, we give two general topological constructions of functors taking values in covering spaces — combined with all of the previous sections, this completes our general construction of *global homological representations* of motion groups and mapping class groups. We finally apply this construction to motion groups (in §5.2) and mapping class groups (in §5.4) in dimensions 2 and 3.

Contents

1. Overview of results

In this first section, we give more precise details of our constructions, results and particular examples of homological representations that we construct.

Coherent families of representations. For a family of groups $\{G_n\}_{n\in\mathbb{N}}$ equipped with group homomorphisms $\varphi_n\colon G_n \to G_{n+1}$, there is a natural notion of *coherence* for a collection of representations $\{\varrho_n\colon G_n \to GL_R(M_n)\}_{n\in\mathbb{N}}$. Namely, the collection $\{\varrho_n\}$ is *coherent* if it comes equipped with module homomorphisms $m_n\colon M_n \to M_{n+1}$ such that m_n is *equivariant* with respect to φ_n, meaning that for all $g \in G_n$ and $x \in M_n$ we have $m_n(g.x) = \varphi_n(g).m_n(x)$.

This notion can be encoded in a functorial way. Let $\mathcal{G}$ be the groupoid with objects indexed by non-negative integers, with the groups $\{G_n\}_{n\in\mathbb{N}}$ as automorphism groups and with no morphisms between distinct objects. For instance, we consider the braid groupoid $\boldsymbol{\beta}$ to deal with braid groups and the decorated surfaces groupoid $\mathcal{M}_2$ for the mapping class groups of surfaces; see §4.4–§4.5. Suppose that we have chosen a category $\mathcal{C}_{\mathcal{G}}$ containing $\mathcal{G}$ as its underlying groupoid and with a preferred morphism $\iota_n\colon n \to n+1$ for each object n, satisfying

$$\iota_n \circ g = \varphi_n(g) \circ \iota_n$$

for each $g \in G_n$. In all of the examples addressed in this paper, such a category $\mathcal{C}_{\mathcal{G}}$ is constructed through *Quillen's bracket construction* using a monoidal structure on $\mathcal{G}$ (or $\mathcal{G}$ will be a module over another monoidal groupoid); see §3. We denote by Mod_R the category of right R-modules. A functor $\mathcal{C}_{\mathcal{G}} \to \mathrm{Mod}_R$ then gives us a coherent family of representations of $\{G_n\}_{n\in\mathbb{N}}$ in the above sense. Moreover, if $\mathcal{C}_{\mathcal{G}}$ has been constructed from $\mathcal{G}$ by "freely adjoining" the new morphisms ι_n (as will be the case in our examples), then a functor $\mathcal{C}_{\mathcal{G}} \to \mathrm{Mod}_R$ is in fact *equivalent* to a coherent family of representations of $\{G_n\}_{n\in\mathbb{N}}$.

General functorial homological constructions. Our overall procedure for constructing *homological representations* is summarised in diagram (1.1) below. A more elaborate version, allowing one to fibrewise tensor the coefficients with another functor V, is described in diagram (2.10) and Definition 2.20, but the essential ideas are the same. The desired output is the diagonal functor $\mathcal{C}_{\mathcal{G}} \to \mathrm{Mod}_{\bullet}$, a (coherent) family of (possibly twisted) representations of $\{G_n\}_{n\in\mathbb{N}}$. Here, $\mathrm{Mod}_{\bullet}$ is the category of modules, containing Mod_R for every ring R. This is constructed in three steps:

- We first construct a topologically-enriched category $\mathcal{C}^{\mathrm{t}}_{\mathcal{G}}$ whose π_0 recovers $\mathcal{C}_{\mathcal{G}}$. We will always construct the category $\mathcal{C}_{\mathcal{G}}$ using Quillen's bracket construction, so we explain in §3.1 how this construction may be lifted to topologically-enriched categories to produce a corresponding $\mathcal{C}^{\mathrm{t}}_{\mathcal{G}}$. This requires the key technical result of Lemma 3.6 about commuting Quillen's bracket construction with π_0, which depends on a fibration condition which we verify in all of our examples; see §3.3. In our examples, its morphism spaces may be identified (Proposition 4.8) with certain embedding spaces between manifolds.
  In particular, we construct new topologically-enriched categories $\mathfrak{U}\mathcal{D}_d$ that are designed to contain all diffeomorphisms of d-manifolds (equipped with configurations), together with all embeddings between such manifolds that correspond to splittings into boundary connected summands; see §3.2. Its associated discrete category $\pi_0(\mathfrak{U}\mathcal{D}_d)$ thus contains all mapping class groups and motion groups in dimension d.
- The key geometric step is to construct a (continuous) functor F from $\mathcal{C}^{\mathrm{t}}_{\mathcal{G}}$ to $\mathrm{Cov}_{\bullet}$, the category of topological spaces equipped with regular coverings (with any deck transformation group). This geometric construction is the subject of §5.
- The remaining steps simply encode the idea of taking twisted homology of covering spaces. The functor Lift takes a regular covering with deck transformation group Q to the corresponding bundle of $\mathbb{Z}[Q]$-modules, and then H_i is the twisted homology functor in degree i. See §2.2–§2.4 for more details.

$$\begin{array}{ccccccc} \mathcal{C}^{\mathrm{t}}_{\mathcal{G}} & \xrightarrow{\;F\;} & \mathrm{Cov}_{\bullet} & \xrightarrow{\;\mathrm{Lift}\;} & \mathrm{Top}_{\bullet} & \xrightarrow{\;H_i\;} & \mathrm{Mod}_{\bullet} \\ {\scriptstyle \pi_0}\downarrow & & & & & \nearrow & \\ \mathcal{C}_{\mathcal{G}} & & & \dashrightarrow{\scriptstyle L_i(F)} & & & \end{array} \tag{1.1}$$

See §2.1 for more details on the categories $\mathrm{Cov}_{\bullet}$, $\mathrm{Top}_{\bullet}$ and $\mathrm{Mod}_{\bullet}$.

There are also variants of this construction for *relative homology* where we work with categories of pairs of spaces, and for *Borel-Moore homology* where we restrict to categories of locally compact spaces and proper maps. In particular, using Borel-Moore homology is especially interesting when the image of the functor F consists of configuration spaces of points in a surface; see §5.2.

A number of representations arising in this way have previously been defined and studied — at least at the level of individual groups, i.e. when restricted to the individual automorphism groups of $\mathcal{C}_{\mathcal{G}}$ — and indeed one purpose of describing this general procedure for constructing homological representations is to give a *unified* description for various different representations appearing in the literature, as well as discovering new constructions by comparing representations coming from different settings in this unified context.

Constructions of representations for motion groups. The first type of groups to which we apply our construction are *motion groups*: given a closed submanifold A of the interior of a manifold M, the motion group $\mathrm{Mot}_A(M)$ is the fundamental group of $\mathrm{Emb}(A, \mathring{M})/\mathrm{Diff}(A)$, the space of embeddings of A into the interior of M modulo diffeomorphisms of A. If A is orientable, we may also consider $\mathrm{Mot}^{+}_{A}(M)$, which is the fundamental group of $\mathrm{Emb}(A, \mathring{M})/\mathrm{Diff}^{+}(A)$. Important examples of (M, A) are:

- $(\mathbb{D}^2, n \text{ points})$ — corresponding to the classical braid groups $\mathbf{B}_n \cong \mathrm{Mot}_n(\mathbb{D}^2)$;
- $(S, n \text{ points})$ — corresponding to the braid groups $\mathbf{B}_n(S) \cong \mathrm{Mot}_n(S)$ on a surface S;
- $(\mathbb{D}^3, U_n)$, where U_n is an n-component unlink in the interior of $\mathbb{D}^3$ — corresponding to the *loop braid groups* $\mathbf{LB}_n \cong \mathrm{Mot}^{+}_{U_n}(\mathbb{D}^3)$ and the *extended loop braid groups* $\mathbf{LB}'_n \cong \mathrm{Mot}_{U_n}(\mathbb{D}^3)$.

In §5.1, we construct a functor $\mathfrak{U}\mathcal{D}_d \to \mathrm{Cov}_{\bullet}$ depending on two parameters: a closed submanifold $Z \subseteq \mathbb{R}^d$ and an integer $\ell \geqslant 1$. (There is also a third parameter, a choice of open subgroup G of the diffeomorphism group of Z, which is assumed to be the full diffeomorphism group and thus ignored here for simplicity.) The motion group $\mathrm{Mot}_A(M)$ is a normal subgroup of the automorphism group of (M, A) in $\pi_0(\mathfrak{U}\mathcal{D}_d)$, so diagram (1.1) applied to this functor gives a (possibly twisted) representation of $\mathrm{Mot}_A(M)$.

To give an idea of how this functor $F\colon \mathfrak{U}\mathcal{D}_d \to \mathrm{Cov}_\bullet$ is constructed, we explain how to construct $\pi_0(F)\colon \pi_0(\mathfrak{U}\mathcal{D}_d) \to \pi_0(\mathrm{Cov}_\bullet)$ restricted to $\mathrm{Mot}_A(M)$. To do this, we have to give a regular covering together with an action up to homotopy of $\mathrm{Mot}_A(M)$. The key idea is to use the fact that $\mathrm{Mot}_A(M)$ acts up to homotopy on the space of embeddings (modulo diffeomorphisms) of Z into $\mathring{M} \smallsetminus A$. The fundamental group of this space is the motion group $\mathrm{Mot}_Z(M \smallsetminus A)$, and one then has to define appropriate quotients of this group so that the $\mathrm{Mot}_A(M)$-action lifts to the corresponding regular covering. To do this, we use the following diagram, which we explain below:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Mot}_Z(M \smallsetminus A) & \longrightarrow & \mathrm{Mot}_{Z\sqcup A}(M) & \longrightarrow & \mathrm{Mot}_A(M) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Q & \longrightarrow & \dfrac{\mathrm{Mot}_{Z\sqcup A}(M)}{\Gamma_\ell(\mathrm{Mot}_{Z\sqcup A}(M))} & \longrightarrow & \dfrac{\mathrm{Mot}_A(M)}{\Gamma_\ell(\mathrm{Mot}_A(M))} & \longrightarrow & 1.
\end{array}
$$

The top row consists of the split-surjection $\mathrm{Mot}_{A\sqcup Z}(M) \to \mathrm{Mot}_A(M)$ given by forgetting the manifold Z (we always require M to have non-empty boundary; the splitting uses this fact), whose kernel is $\mathrm{Mot}_Z(M \smallsetminus A)$. Write $\Gamma_\ell(G)$ for the ℓ-th term of the lower central series of a group G; this explains two of the quotients, and the diagram is completed by defining Q to be the image of $\mathrm{Mot}_Z(M \smallsetminus A)$ in the bottom-middle group. An easy diagram chase shows that the action of $\mathrm{Mot}_A(M)$ on $\mathrm{Mot}_Z(M \smallsetminus A)$ preserves $\ker(\phi)$, which is equivalent to saying that the action lifts to the corresponding regular covering. (We also identify the geometric action of $\mathrm{Mot}_A(M)$ on $\mathrm{Mot}_Z(M \smallsetminus A)$ with the action induced by the splitting on the top row; *cf.* Lemma 4.25.)

The quotient group Q is often the same for a whole family of motion groups (such as the three mentioned above); we call this phenomenon *Q-stability*. When this holds, the functor F restricted to this family takes values in a subcategory of $\mathrm{Cov}_\bullet$ denoted by $\mathrm{Cov}_Q^{\mathrm{tw}}$. In the case $\ell = 2$, we show in §5.1.5 that F takes values in $\mathrm{Cov}_Q \subset \mathrm{Cov}_Q^{\mathrm{tw}}$, and the resulting representations are *untwisted*.

If we consider all $\ell \geqslant 1$ simultaneously, these representations may be packaged together into a *"pro-nilpotent" tower of representations* of the family of motion groups; see §5.1.7. Whether or not this tower *stops* at some finite stage depends on the lower central series of the "mixed" motion group $\mathrm{Mot}_{Z\sqcup A}(M)$. This is investigated for the classical, surface and loop braid groups in [DPS21].

A natural generalisation of the above procedure consists in considering embeddings of Z modulo a fixed group $G \leqslant \mathrm{Diff}(Z)$ of diffeomorphisms, rather than modulo all diffeomorphisms. When Z is a finite set, for example, this means that we may consider *partitioned* configurations instead of unordered configurations. The corresponding lower central series are typically more subtle, and the resulting representations more sophisticated.

Classical braid groups. Many well-known representations arise as particular instances of this general homological construction. First of all, for the classical braid groups, we take Z to be a set of $k \geqslant 1$ unordered points. Quillen's bracket construction defines a category $\mathfrak{U}\boldsymbol{\beta}$ having the braid groupoid $\boldsymbol{\beta}$ as its underlying groupoid.

Theorem 1.1 (Proposition 5.28 and Corollary 5.33) *Choosing the quotients by the Γ_2-term and taking homology in degree k, the above procedure defines functors*

$$\mathfrak{LB}_1\colon \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}]} \qquad \textit{and} \qquad \mathfrak{LB}_k\colon \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}^2]} \quad (\textit{for } k \geqslant 2),$$

which encode the Lawrence-Bigelow representations [Law90; Big04]. *In particular, $\mathfrak{LB}_1$ and $\mathfrak{LB}_2$ encode the reduced Burau and the Lawrence-Krammer-Bigelow representations respectively.*

If $k = 2$, applying the procedure for the quotient by the Γ_ℓ-term for $\ell \geqslant 3$ provides twisted representations $\mathfrak{LB}_{2,\ell}$ which are not equivalent to $\mathfrak{LB}_2$. If we use Borel-Moore homology, the twisted representations of braid groups encoded by $\mathfrak{LB}_{2,\ell}^{BM}$ are ***faithful*** *for each $\ell \geqslant 3$.*

We also describe analogues of the Lawrence-Bigelow representations defined using *partitioned* configurations, whose transformation groups are more complicated; see Remark 5.29. As far as the authors know, these generalisations do not yet appear in the literature.

Surface braid groups. More generally, we consider the braid groups on a connected compact surface S with one boundary-component and again take Z to be a set of $k \geqslant 1$ unordered points. Quillen's bracket construction provides a category $\langle \boldsymbol{\beta}, \boldsymbol{\beta}^S \rangle$ having the braid groups as automorphism groups. If S is orientable of genus $g \geqslant 1$ we denote it by $\Sigma_{g,1}$, if it is non-orientable of genus $h \geqslant 1$ we denote it by $\mathcal{N}_{h,1}$.

Theorem 1.2 (Propositions 5.34 and 5.36, Corollary 5.37) *Choosing the quotients by the Γ_2-term and taking homology in degree k, the above procedure defines functors*

$$\mathfrak{L}_k(\Sigma_{g,1}, \Gamma_2) \colon \langle \boldsymbol{\beta}, \boldsymbol{\beta}^{\Sigma_{g,1}} \rangle \longrightarrow \mathrm{Mod}_{\mathbb{Z}[Q_{(k,2)}(\Sigma_{g,1})]}$$
$$\mathfrak{L}_k(\mathcal{N}_{h,1}, \Gamma_2) \colon \langle \boldsymbol{\beta}, \boldsymbol{\beta}^{\mathcal{N}_{h,1}} \rangle \longrightarrow \mathrm{Mod}_{\mathbb{Z}[Q_{(k,2)}(\mathcal{N}_{h,1})]},$$

where $Q_{(k,2)}(\Sigma_{g,1}) \cong \mathbb{Z}^{2g} \oplus (\mathbb{Z}/2)^{d_k}$ and $Q_{(k,2)}(\mathcal{N}_{h,1}) \cong \mathbb{Z}^h \oplus (\mathbb{Z}/2)^{d_k}$, with $d_1 = 0$ and $d_k = 1$ if $k \geqslant 2$. If instead we take Γ_3 quotients, we obtain

$$\mathfrak{L}_k(\Sigma_{g,1}, \Gamma_3) \colon \langle \boldsymbol{\beta}, \boldsymbol{\beta}^{\Sigma_{g,1}} \rangle \longrightarrow \mathrm{Mod}^{\mathrm{tw}}_{\mathbb{Z}[Q_{(k,3)}(\Sigma_{g,1})]}$$
$$\mathfrak{L}_k(\mathcal{N}_{h,1}, \Gamma_3) \colon \langle \boldsymbol{\beta}, \boldsymbol{\beta}^{\mathcal{N}_{h,1}} \rangle \longrightarrow \mathrm{Mod}^{\mathrm{tw}}_{\mathbb{Z}[Q_{(k,3)}(\mathcal{N}_{h,1})]},$$

where $Q_{(k,3)}(S) = \ker(\mathbf{B}_{k,n}(S)/\Gamma_3 \twoheadrightarrow \mathbf{B}_n(S)/\Gamma_3)$, which is independent of n for $n \geqslant 3$. Moreover, for $k \geqslant 3$, we obtain homological functors encoding untwisted representations

$$\mathfrak{L}^{\mathrm{u}}_k(\Sigma_{g,1}, \Gamma_3) \colon \langle \boldsymbol{\beta}, \boldsymbol{\beta}^{\Sigma_{g,1}} \rangle \longrightarrow \mathrm{Mod}_{\mathbb{Z}[Q^{\mathrm{u}}_{(k,3)}(\Sigma_{g,1})]},$$

where $Q^{\mathrm{u}}_{(k,3)}(\Sigma_{g,1}) = \mathbb{Z}^{g+1} \rtimes \mathbb{Z}^g$.

As far as the authors know, the representations encoded by the functors $\mathfrak{L}_k(\Sigma_{g,1}, \Gamma_2)$, $\mathfrak{L}_k(\mathcal{N}_{h,1}, \Gamma_2)$ and $\mathfrak{L}_k(\mathcal{N}_{h,1}, \Gamma_3)$ appear to be new. The twisted representations defined by the functor $\mathfrak{L}_k(\Sigma_{g,1}, \Gamma_3)$ are reinterpretations of those introduced by An and Ko [AK10]. However the untwisted ones arising from $\mathfrak{L}^{\mathrm{u}}_k(\Sigma_{g,1}, \Gamma_3)$ are new; see §5.2.2.1.

Loop braid groups. Finally, for loop braid groups, we have two main choices for the parameter Z: one may take either a set of $k \geqslant 1$ unordered points in $\mathbb{R}^3$, or else a k-component unlink in $\mathbb{R}^3$. In the latter case we may either consider this as an *oriented* unlink or an *unoriented* unlink. Quillen's bracket construction provides categories $\mathfrak{U}\mathcal{L}\boldsymbol{\beta}$ and $\mathfrak{U}\mathcal{L}\boldsymbol{\beta}'$ having respectively the loop braid groups and extended loop braid groups as automorphism groups.

Theorem 1.3 (Propositions 5.40, 5.42 and 5.43) *Choosing the quotients by the Γ_2-term and taking homology in degree k, the above procedure defines functors*

$$\begin{aligned}
\mathfrak{L}_1(1, \mathcal{L}\boldsymbol{\beta}) &\colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}]} & \mathfrak{L}_1(1, \mathcal{L}\boldsymbol{\beta}') &\colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta}' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}/2]} \\
\mathfrak{L}_k(k, \mathcal{L}\boldsymbol{\beta}) &\colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)]} & \mathfrak{L}_k(k, \mathcal{L}\boldsymbol{\beta}') &\colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta}' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[(\mathbb{Z}/2)^2]} \\
\mathfrak{L}_1(U_1, \mathcal{L}\boldsymbol{\beta}) &\colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)^2]} & \mathfrak{L}_1(U_1, \mathcal{L}\boldsymbol{\beta}') &\colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta}' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[(\mathbb{Z}/2)^3]} \\
\mathfrak{L}_k(U_k, \mathcal{L}\boldsymbol{\beta}) &\colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)^4]} & \mathfrak{L}_k(U_k, \mathcal{L}\boldsymbol{\beta}') &\colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta}' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[(\mathbb{Z}/2)^5]} \\
\mathfrak{L}^+_1(U_1, \mathcal{L}\boldsymbol{\beta}) &\colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}^2]} & \mathfrak{L}^+_1(U_1, \mathcal{L}\boldsymbol{\beta}') &\colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta}' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)]} \\
\mathfrak{L}^+_k(U_k, \mathcal{L}\boldsymbol{\beta}) &\colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}^3\oplus(\mathbb{Z}/2)]} & \mathfrak{L}^+_k(U_k, \mathcal{L}\boldsymbol{\beta}') &\colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta}' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}^2\oplus(\mathbb{Z}/2)^2]},
\end{aligned}$$

where $k \geqslant 2$.

In [PS21b], we compute explicitly the matrices of the representations encoded by the top two functors, $\mathfrak{L}_1(1, \mathcal{L}\boldsymbol{\beta})$ and $\mathfrak{L}_1(1, \mathcal{L}\boldsymbol{\beta}')$: these extend the *reduced Burau representations* of the classical braid groups to $\mathbf{LB}_n$ and $\mathbf{LB}'_n$ respectively. Moreover, using non-stopping results for lower central series of *partitioned tripartite welded braid groups* from [DPS21, Theorem 5.8], considering all $\ell \geqslant 2$, we obtain *infinite towers of representations* of $\mathfrak{U}\mathcal{L}\boldsymbol{\beta}$ if

$$Z \in \{2, U_1, U_2, U_3, U_2^+, U_3^+\},$$

and of $\mathfrak{U}\mathcal{L}\boldsymbol{\beta}'$ if

$$Z \in \{U_1, U_2, U_3, U_2^+, U_3^+\},$$

see §5.2.3.1–§5.2.3.2. (A superscript $^+$ indicates that we are using the oriented version of our construction.) As far as the authors are aware, other than $\mathfrak{L}_1(1, \mathcal{L}\boldsymbol{\beta})$ and $\mathfrak{L}_1(1, \mathcal{L}\boldsymbol{\beta}')$, the representations encoded by the functors of Theorem 1.3 appear to be new.

Constructions of representations for mapping class groups. The second type of groups to which we apply our construction are *mapping class groups* of manifolds. For a smooth manifold M, the mapping class group $\mathrm{MCG}(M)$ of M is the group of isotopy classes of diffeomorphisms of M fixing its boundary pointwise. More generally, if Z is a closed submanifold of M, we write $\mathrm{MCG}(M,Z)$ for the group of isotopy classes of M fixing its boundary pointwise and sending Z onto itself. We will focus on the case where M is a compact, connected surface with one boundary-component. Quillen's bracket construction provides suitable categories $\mathfrak{U}\mathcal{M}_2^+$ and $\mathfrak{U}\mathcal{M}_2^-$ having the mapping class groups of orientable surfaces $\{\Sigma_{g,1}\}_{g\geqslant 0}$ and non-orientable surfaces together with the disc $\{\mathcal{N}_{h,1}\}_{h\geqslant 0}$ respectively as automorphism groups. (These are subcategories of $\pi_0(\mathfrak{U}\mathcal{D}_2)$.)

We consider two ways to define functors $F\colon \mathfrak{U}\mathcal{D}_d \to \mathrm{Cov}_\bullet$, which will then give us homological representations of $\mathfrak{U}\mathcal{M}_2^+$ and $\mathfrak{U}\mathcal{M}_2^-$ via diagram (1.1). The first one is the construction given in §5.1, which we have already described above in the setting of motion groups. The idea for mapping class groups is similar, using the natural action (up to homotopy) of $\mathrm{MCG}(M)$ on the space of embeddings modulo diffeomorphisms of Z into the interior of M, whose fundamental group is the motion group $\mathrm{Mot}_Z(M)$. We then take a characteristic quotient of this group, typically induced by its lower central series, to lift this action to the corresponding covering space.

For example, for orientable surfaces, we construct a homological functor $\mathfrak{U}\mathcal{M}_2^+ \to \mathrm{Mod}_\bullet$ encoding representations over the group-ring of $\mathbf{B}_k(\Sigma_{g,1})/\Gamma_3$, which are *untwisted* when restricted to the Chillingworth subgroups of the mapping class groups. See §5.4.1 for the details.

However, this kind of construction is not optimal: as exemplified in the previous paragraph, it typically only gives *twisted* representations, and one has to restrict to smaller subgroups of mapping class groups so as to get untwisted representations. For this reason, we give a second construction of a functor $F\colon \mathfrak{U}\mathcal{D}_d \to \mathrm{Cov}_\bullet$ in §5.3 that is better adapted to mapping class groups in the sense that it more often gives *untwisted* representations when we plug it into diagram (1.1). (The precise meaning of this is given by Proposition 5.48.)

The idea is again to use the natural action (up to homotopy) of $\mathrm{MCG}(M)$ on the space of embeddings modulo diffeomorphisms of Z into the interior of M, whose fundamental group is $\mathrm{Mot}_Z(M)$. However, instead of directly taking lower central quotients of this group to choose coverings, we instead construct a 6-term diagram, similar to the one for motion groups, and choose the quotient ϕ indirectly via the lower central quotients of the bigger mapping class group $\mathrm{MCG}(M,Z)$:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Mot}_Z(M) & \longrightarrow & \mathrm{MCG}(M,Z) & \longrightarrow & \mathrm{MCG}(M) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow & & \Big\downarrow & & \\
1 & \longrightarrow & Q & \longrightarrow & \dfrac{\mathrm{MCG}(M,Z)}{\Gamma_\ell(\mathrm{MCG}(M,Z))} & \longrightarrow & \dfrac{\mathrm{MCG}(M)}{\Gamma_\ell(\mathrm{MCG}(M))} & \longrightarrow & 1.
\end{array}$$

Here, the top row consists of the split surjection $\mathrm{MCG}(M,Z) \twoheadrightarrow \mathrm{MCG}(M)$ given by forgetting the condition that diffeomorphisms must send Z onto itself (as before, we always require M to have non-empty boundary; the splitting uses this assumption), whose kernel may be identified with $\mathrm{Mot}_Z(M)$.

This second method produces representations of mapping class groups that appear to be new, as far as the authors are aware. For instance, when $M = S$ is a surface, we take for Z a set of $k \geqslant 1$ unordered points and obtain:

Theorem 1.4 (Propositions 5.54 and 5.57) *Choosing the quotients by the Γ_2-term and taking homology in degree k, the above procedure defines homological representation functors:*

- $\mathfrak{L}_k(\mathbf{\Gamma}_{-,1})\colon \mathfrak{U}\mathcal{M}_2^+ \longrightarrow \mathrm{Mod}_{\mathbb{Z}[Q_k]}$ *where Q_1 is trivial and $Q_k = \mathbb{Z}/2$ for $k \geqslant 2$.*
- $\mathfrak{L}_k(\boldsymbol{\mathcal{N}}_{-,1})\colon \mathfrak{U}\mathcal{M}_2^- \longrightarrow \mathrm{Mod}_{\mathbb{Z}[Q_k]}$ *where $Q_1 = \mathbb{Z}/2$ and $Q_k = (\mathbb{Z}/2)^2$ for $k \geqslant 2$.*

2. The general construction

As explained in §1, once we have constructed the topologically-enriched category $\mathfrak{U}\mathcal{D}_d$ (see §§3,4), the construction of global homological representations consists of two main parts:
(1) constructing continuous functors $\mathfrak{U}\mathcal{D}_d \to \mathrm{Cov}_\bullet$,

(2) constructing functors $\pi_0(\mathfrak{U}\mathcal{D}_d) \to \mathrm{Mod}_\bullet$ given a continuous functor $\mathfrak{U}\mathcal{D}_d \to \mathrm{Cov}_\bullet$,
where $\mathrm{Cov}_\bullet$ and $\mathrm{Mod}_\bullet$ respectively denote categories of spaces equipped with coverings and right modules over $\mathbb{A}$-algebras; *cf.* Definitions 2.2 and 2.3. Here, $\pi_0(\mathcal{C})$ is the discrete category obtained from the topologically-enriched category $\mathcal{C}$ by applying the functor π_0 to all morphism spaces.

In this section we deal with step (2) in a general setting: for any topologically-enriched category $\mathcal{C}$ and continuous functor $F\colon \mathcal{C} \to \mathrm{Cov}_\bullet$, as well as (optionally) a continuous functor $V\colon \mathcal{C} \to {}_\bullet\mathrm{Mod}_\bullet$ that is "compatible" with F, we produce a functor $L_i(F;V)\colon \pi_0(\mathcal{C}) \to \mathrm{Mod}_\bullet$ for each $i \geqslant 0$.

We first give precise definitions of the categories involved in §2.1. The three steps of the construction are then described in §2.2–§2.4, and are put together in §2.5. Throughout this section, we fix an associative, unital ring $\mathbb{A}$. The reader may prefer to assume that $\mathbb{A} = \mathbb{Z}$, since this will be the case in all of our examples later.

The reader not interested in the fine detail may wish to skip directly to §2.5 at this point.

2.1. Categories

In this subsection, we define the different categories involved in the construction.

Remark 2.1 In fact, all of the categories that we define are *topologically-enriched* categories – in other words their morphism sets are equipped with topologies such that composition is continuous – and all functors are *continuous* functors. We will therefore implicitly write *category* to mean topologically-enriched category and *functor* to mean continuous functor. The morphism spaces of the categories defined in this section are all equipped with topologies derived in an obvious way from the *compact-open topology* for sets of maps between topological spaces.

Definition 2.2 (*The category of bicoverings.*) The category ${}_\bullet\mathrm{Cov}_\bullet$ is the category of *spaces equipped with bicoverings*. An object of ${}_\bullet\mathrm{Cov}_\bullet$ is a path-connected, based space X admitting a universal covering (i.e., locally path-connected and semi-locally simply-connected), equipped with a pair of surjective homomorphisms

$$\phi_1\colon \pi_1(X) \longrightarrow Q_1 \qquad \phi_2\colon \pi_1(X) \longrightarrow Q_2$$

such that the induced homomorphism $(\phi_1 \times \phi_2) \circ \Delta\colon \pi_1(X) \to Q_1 \times Q_2$ is also surjective. Via the correspondence between path-connected, regular coverings of X and normal subgroups of $\pi_1(X)$, this is the same as a pair of normal subgroups N_1, N_2 of $\pi_1(X)$ (corresponding to regular coverings $X^{N_1} \to X$ and $X^{N_2} \to X$) such that the square

$$\begin{array}{ccc} X^{N_1 \cap N_2} & \longrightarrow & X^{N_2} \\ \big\downarrow & \lrcorner & \big\downarrow \\ X^{N_1} & \longrightarrow & X \end{array}$$

is a pullback square. A morphism in ${}_\bullet\mathrm{Cov}_\bullet$ from (X, ϕ_1, ϕ_2) to (X', ϕ_1', ϕ_2') is a based, continuous map $f\colon X \to X'$ such that the induced homomorphism $\pi_1(f)$ sends $\ker(\phi_1)$ into $\ker(\phi_1')$ and $\ker(\phi_2)$ into $\ker(\phi_2')$. This implies that there are unique homomorphisms $\alpha_1\colon Q_1 \to Q_1'$ and $\alpha_2\colon Q_2 \to Q_2'$ such that

$$\phi_1' \circ \pi_1(f) = \alpha_1 \circ \phi_1 \qquad \text{and} \qquad \phi_2' \circ \pi_1(f) = \alpha_2 \circ \phi_1.$$

If G is a group, the category ${}_G\mathrm{Cov}_\bullet$ is the subcategory of ${}_\bullet\mathrm{Cov}_\bullet$ on those objects (X, ϕ_1, ϕ_2) such that $Q_1 = G$ and those morphisms f such that the induced homomorphism α_1 mentioned above is equal to id_G. Similarly, we have a subcategory ${}_\bullet\mathrm{Cov}_G$. If G is the trivial group, we drop it from the notation, and write $\mathrm{Cov}_\bullet$ and ${}_\bullet\mathrm{Cov}$ respectively for these subcategories of ${}_\bullet\mathrm{Cov}_\bullet$. We note that ${}_\bullet\mathrm{Cov}$ and $\mathrm{Cov}_\bullet$ are abstractly isomorphic, but not equal as subcategories of ${}_\bullet\mathrm{Cov}_\bullet$. As another variant, we write $\mathrm{Cov}_G^{\mathrm{tw}} \subset \mathrm{Cov}_\bullet$ for the *full* subcategory on those objects (X, ϕ_1, ϕ_2) such that $Q_1 = G$ (this is generally larger than Cov_G).

Definition 2.3 (*The category of bimodules.*) The category ${}_\bullet\mathrm{Mod}_\bullet$ is the category of *bimodules over $\mathbb{A}$-algebras*. An object of ${}_\bullet\mathrm{Mod}_\bullet$ is a pair of associative, unital $\mathbb{A}$-algebras (R, S) together with an (R, S)-bimodule V. A morphism from (R, S, V) to (R', S', V') is a pair of $\mathbb{A}$-algebra homomorphisms $\varphi\colon R \to R'$ and $\psi\colon S \to S'$ preserving units, together with a morphism of (R, S)-bimodules $\theta\colon V \to (\varphi, \psi)^*(V')$.

For any $\mathbb{A}$-algebra R, we have a subcategory ${}_R\mathrm{Mod}_\bullet \subset {}_\bullet\mathrm{Mod}_\bullet$ on those objects (R', S', V') where $R' = R$ and those morphisms (φ, ψ, θ) where $\varphi = \mathrm{id}_R$ (note that this is generally not a full subcategory). If moreover R is the trivial $\mathbb{A}$-algebra, we drop it from the notation, and write $\mathrm{Mod}_\bullet$ for ${}_R\mathrm{Mod}_\bullet$. Similarly, we have ${}_\bullet\mathrm{Mod}_S$ and ${}_R\mathrm{Mod}_S$ for $\mathbb{A}$-algebras R and S. We note that ${}_R\mathrm{Mod}_S$ is just the category of (R, S)-bimodules, as usually defined. Furthermore, for a fixed $\mathbb{A}$-algebra S, we write $\mathrm{Mod}_S^{\mathrm{tw}} \subset \mathrm{Mod}_\bullet$ for the *full* subcategory on those objects (S', V') where $S' = S$ (this is generally larger than Mod_S).

Definition 2.4 (*Bundles of G-sets and of R-modules.*) For a path-connected space X, a bundle of left G-sets over X is a fibre bundle over X with fibre a left G-set T and structure group $\mathrm{Aut}_G(T)$, the automorphism group of T as a left G-set. See, for example, [Ste51, §2] for the definition of fibre bundle with specified fibre and structure group. In general, for a space X, a bundle of left G-sets over X is a bundle of left G-sets over each of its path-components. Bundles of left R-modules, (R, S)-bimodules and other algebraic structures are defined similarly.
We will shortly replace this definition with a slightly different one that is equivalent for our purposes.

Remark 2.5 By unique path-lifting, a bundle of left G-sets over a space X determines a functor $\pi_{\leqslant 1}(X) \to {}_G\mathrm{Set}$, where $\pi_{\leqslant 1}(X)$ is the fundamental groupoid of X. Whenever X is locally path-connected and semi-locally simply-connected, this gives an identification between isomorphism classes of left G-sets over X and isomorphism classes of functors $\pi_{\leqslant 1}(X) \to {}_G\mathrm{Set}$. In all of our examples, the base space X will have these properties, and it will often be convenient to replace Definition 2.4 with the following.

Definition 2.6 (*Bundles of G-sets and of R-modules, replacing Definition 2.4.*) For any space X, a bundle of left G-sets over X is a functor

$$\pi_{\leqslant 1}(X) \longrightarrow {}_G\mathrm{Set},$$

where $\pi_{\leqslant 1}(X)$ is the fundamental groupoid of X and ${}_G\mathrm{Set}$ is the category of left G-sets. Similarly, a bundle of left R-modules over X is a functor from $\pi_{\leqslant 1}(X)$ to the category of left R-modules and a bundle of (R, S)-bimodules over X is a functor from $\pi_{\leqslant 1}(X)$ to the category of (R, S)-bimodules.

Definition 2.7 (*The category of bundles of bimodules.*) The category ${}_\bullet\mathrm{Top}_\bullet$ is the category of *bundles of bimodules over $\mathbb{A}$-algebras*. An object of ${}_\bullet\mathrm{Top}_\bullet$ is a space X together with a pair of associative, unital $\mathbb{A}$-algebras (R, S) and a bundle of (R, S)-bimodules over X in the sense of Definition 2.6, i.e. a functor $\xi\colon \pi_{\leqslant 1}(X) \to {}_R\mathrm{Mod}_S$. A morphism from (X, R, S, ξ) to (X', R', S', ξ') is a continuous map $f\colon X \to X'$, two $\mathbb{A}$-algebra homomorphisms $\varphi\colon R \to R'$ and $\psi\colon S \to S'$ preserving units, and an endofunctor $F\colon {}_R\mathrm{Mod}_S \to {}_R\mathrm{Mod}_S$ such that $(\varphi, \psi)^* \circ \xi' \circ \pi_{\leqslant 1}(f) = F \circ \xi$, where $(\varphi, \psi)^*\colon {}_{R'}\mathrm{Mod}_{S'} \to {}_R\mathrm{Mod}_S$ is the restriction functor induced by φ and ψ.
For any $\mathbb{A}$-algebra R, we have a subcategory ${}_R\mathrm{Top}_\bullet \subset {}_\bullet\mathrm{Top}_\bullet$ on those objects (X', R', S', ξ') where $R' = R$ and those morphisms (f, φ, ψ) where $\varphi = \mathrm{id}_R$ (note that this is generally not a full subcategory). If moreover R is the trivial $\mathbb{A}$-algebra, we drop it from the notation, and write $\mathrm{Top}_\bullet$ for ${}_R\mathrm{Top}_\bullet$. Similarly, we have ${}_\bullet\mathrm{Top}_S$ and ${}_R\mathrm{Top}_S$ for $\mathbb{A}$-algebras R and S. For a fixed $\mathbb{A}$-algebra S, we write $\mathrm{Top}_S^{\mathrm{tw}} \subset \mathrm{Top}_\bullet$ for the *full* subcategory on those objects (X', S', ξ') where $S' = S$ (this is generally larger than Top_S).

Via the correspondence of Remark 2.5 between bundles of bimodules over X and functors out of $\pi_{\leqslant 1}(X)$, the endofunctor F above corresponds to a bundle map, covering f, from ξ to $(\varphi, \psi)^*(\xi')$.
Note also that ${}_\bullet\mathrm{Mod}_\bullet$ is equivalent to the full subcategory of ${}_\bullet\mathrm{Top}_\bullet$ on those objects whose underlying space is a point.

Notation 2.8 Writing $\mathbb{A}$-Alg for the category of associative, unital $\mathbb{A}$-algebras and Grp for the category of groups, there are obvious forgetful functors

$$ {}_\bullet\mathrm{Cov}_\bullet \overset{\ell}{\underset{r}{\rightrightarrows}} \mathrm{Grp} \qquad\qquad {}_\bullet\mathrm{Mod}_\bullet \subset {}_\bullet\mathrm{Top}_\bullet \overset{\ell}{\underset{r}{\rightrightarrows}} \mathbb{A}\text{-Alg} $$

that remember just the left (respectively right) underlying group or $\mathbb{A}$-algebra. For example, an object (X, R, S, ξ) of ${}_\bullet\mathrm{Top}_\bullet$ is sent under ℓ to R and under r to S.

Definition 2.9 Write ${}_\bullet\mathrm{Top}_\bullet\mathrm{Mod}_\bullet$ for the pullback of the forgetful functors $\ell\colon {}_\bullet\mathrm{Mod}_\bullet \to \mathbb{A}\text{-Alg}$ and $r\colon {}_\bullet\mathrm{Top}_\bullet \to \mathbb{A}\text{-Alg}$:

$$\begin{array}{ccc} {}_\bullet\mathrm{Top}_\bullet\mathrm{Mod}_\bullet & \longrightarrow & {}_\bullet\mathrm{Mod}_\bullet \\ \big\downarrow \quad \lrcorner & & \big\downarrow \ell \\ {}_\bullet\mathrm{Top}_\bullet & \xrightarrow[r]{} & \mathbb{A}\text{-Alg}. \end{array} \tag{2.1}$$

In other words, an object of ${}_\bullet\mathrm{Top}_\bullet\mathrm{Mod}_\bullet$ consists of a space X, a triple of $\mathbb{A}$-algebras (R, S, T), a bundle of (R, S)-bimodules over X and an (S, T)-bimodule V.

2.2. From bicoverings to bundles of bimodules

In this subsection, we introduce the key functor $\mathrm{Lift}\colon {}_\bullet\mathrm{Cov}_\bullet \longrightarrow {}_\bullet\mathrm{Top}_\bullet$.

Proposition 2.10 *There is a natural continuous functor*

$$\mathrm{Lift}\colon {}_\bullet\mathrm{Cov}_\bullet \longrightarrow {}_\bullet\mathrm{Top}_\bullet, \tag{2.2}$$

taking bicoverings to bundles of bimodules, such that the squares

$$\begin{array}{ccc} {}_\bullet\mathrm{Cov}_\bullet & \xrightarrow{\mathrm{Lift}} & {}_\bullet\mathrm{Top}_\bullet \\ \ell\big\downarrow & & \big\downarrow \ell \\ \mathrm{Grp} & \xrightarrow{\mathbb{A}[(-)^{\mathrm{op}}]} & \mathbb{A}\text{-Alg} \end{array} \qquad \begin{array}{ccc} {}_\bullet\mathrm{Cov}_\bullet & \xrightarrow{\mathrm{Lift}} & {}_\bullet\mathrm{Top}_\bullet \\ r\big\downarrow & & \big\downarrow r \\ \mathrm{Grp} & \xrightarrow{\mathbb{A}[-]} & \mathbb{A}\text{-Alg} \end{array} \tag{2.3}$$

commute, where $\mathbb{A}[-]\colon \mathrm{Grp} \to \mathbb{A}\text{-Alg}$ *takes a group to its group* $\mathbb{A}$*-algebra and* $(-)^{\mathrm{op}}\colon \mathrm{Grp} \to \mathrm{Grp}$ *takes a group to its opposite.*

Proof. On objects, this is defined as follows. Let $(X, x_0, \phi_1\colon \pi_1(X, x_0) \to Q_1, \phi_2\colon \pi_1(X, x_0) \to Q_2)$ be an object of ${}_\bullet\mathrm{Cov}_\bullet$. The normal subgroup

$$K \coloneqq \ker(\phi_1) \cap \ker(\phi_2) = \ker((\phi_1 \times \phi_2) \circ \Delta) \ \triangleleft\ \pi_1(X, x_0) \tag{2.4}$$

corresponds to a regular covering of X with deck transformation group $Q \coloneqq Q_1 \times Q_2$. In order to specify a particular regular covering of X, rather than just an *isomorphism class* of such, we will be slightly more careful. Start with the universal covering $\widetilde{X}$ of X; more specifically, the standard model for $\widetilde{X}$ whose underlying set consists of endpoint-preserving homotopy classes of paths in X starting at x_0. This is equipped with an action of $\pi_1(X, x_0)$; take the quotient of $\widetilde{X}$ by the action of the subgroup K. We denote this regular covering by $\xi_K\colon X^K = \widetilde{X}/K \to X$. Whether deck transformations act on the left or the right is an arbitrary convention. We will consider them to act on the *right*, since this agrees with the typical convention that the structure group of a principal bundle (of which regular coverings are examples) acts on the total space on the *right*. This is a bundle of right Q-sets over X (whose fibres all happen to be isomorphic to Q itself). Since Q is the product $Q_1 \times Q_2$, we may equally view ξ_K as a bundle of (Q_1^{op}, Q_2)-bisets over X, where a (G, H)-biset is a set equipped with a left G-action and a compatible right H-action.

Now replace each fibre of ξ_K with the free $\mathbb{A}$-module generated by that fibre; this forms a bundle of $(\mathbb{A}[Q_1^{\mathrm{op}}], \mathbb{A}[Q_2])$-bimodules over X. The operation of "*taking free* $\mathbb{A}$*-modules fibrewise*" is simplest to describe by viewing bundles of (G, H)-bisets over X as functors $\pi_{\leqslant 1}(X) \to {}_G\mathrm{Set}_H$ to the category of (G, H)-bisets, and similarly bundles of (R, S)-bimodules over X as functors $\pi_{\leqslant 1}(X) \to {}_R\mathrm{Mod}_S$ to the category of (R, S)-bimodules. In this viewpoint, the operation is simply post-composition with the free functor $\mathbb{A}[-]\colon {}_{Q_1^{\mathrm{op}}}\mathrm{Set}_{Q_2} \to {}_{\mathbb{A}[Q_1^{\mathrm{op}}]}\mathrm{Mod}_{\mathbb{A}[Q_2]}$. Denote the resulting bundle of bimodules by $\mathbb{A}_{\mathsf{fib}}[\xi_K]\colon \mathbb{A}_{\mathsf{fib}}[X^K] \to X$. This defines Lift on objects:

$$\mathrm{Lift}(X, x_0, \phi_1, \phi_2) \ = \ (X, \mathbb{A}[Q_1^{\mathrm{op}}], \mathbb{A}[Q_2], A_{\mathsf{fib}}[\xi_K]).$$

In order to define Lift on morphisms, we first note that, although we did not need it to define the functor on objects, the regular covering $\xi_K\colon X^K \to X$ associated to (X, x_0, ϕ_1, ϕ_2) comes equipped with a particular choice of basepoint of X^K, covering the basepoint x_0 of X. This is because the standard construction of the universal cover $\widetilde{X}$ has a canonical basepoint (namely the constant path at x_0), and therefore so does its quotient X^K. Let us denote this basepoint by $\widetilde{x}_0 \in X^K$.

Now suppose we have a morphism $(X, \phi_1, \phi_2) \to (Y, \phi'_1, \phi'_2)$ in ${}_\bullet\mathrm{Cov}_\bullet$, that is, a continuous map $f\colon X \to Y$ such that $f(x_0) = y_0$, $f_*(\ker(\phi_1)) \subseteq \ker(\phi'_1)$ and $f_*(\ker(\phi_2)) \subseteq \ker(\phi'_2)$. We recall from Definition 2.2 that this determines certain homomorphisms $\alpha_1\colon Q_1 \to Q'_1$ and $\alpha_2\colon Q_2 \to Q'_2$, which determine $\mathbb{A}$-algebra homomorphisms $\mathbb{A}[\alpha_1^{\mathrm{op}}]\colon \mathbb{A}[Q_1^{\mathrm{op}}] \to \mathbb{A}[(Q'_1)^{\mathrm{op}}]$ and $\mathbb{A}[\alpha_2]\colon \mathbb{A}[Q_2] \to \mathbb{A}[Q'_2]$. Let

$$K = \ker(\phi_1) \cap \ker(\phi_2) \qquad L = \ker(\phi'_1) \cap \ker(\phi'_2)$$

and write $\xi_K\colon X^K \to X$ and $\xi_L\colon Y^L \to Y$ for the corresponding regular covering spaces. By covering space theory, for each point $\widetilde{y} \in \xi_L^{-1}(y_0)$, there is a unique continuous map $X^K \to Y^L$ that lifts the composition $f \circ \xi_K\colon X^K \to Y$ and that takes $\widetilde{x}_0$ to $\widetilde{y}$. We therefore obtain a uniquely-determined lift

$$\widetilde{f}\colon X^K \longrightarrow Y^L$$

by requiring $\widetilde{f}(\widetilde{x}_0) = \widetilde{y}_0$. Extending this map $\mathbb{A}$-linearly in each fibre results in a map

$$\mathbb{A}_{\mathsf{fib}}[\widetilde{f}]\colon \mathbb{A}_{\mathsf{fib}}[X^K] \longrightarrow \mathbb{A}_{\mathsf{fib}}[Y^L]$$

of bundles of $\mathbb{A}$-modules. Finally, one may check that $\mathbb{A}_{\mathsf{fib}}[\widetilde{f}]$ is a map of bundles of $(\mathbb{A}[Q_1^{\mathrm{op}}], \mathbb{A}[Q_2])$-bimodules, covering f, from $\mathbb{A}_{\mathsf{fib}}[\xi_K]$ to $\mathbb{A}_{\mathsf{fib}}[\xi_L]$, where the latter is given the structure of a bundle of $(\mathbb{A}[Q_1^{\mathrm{op}}], \mathbb{A}[Q_2])$-bimodules via $\mathbb{A}[\alpha_1^{\mathrm{op}}]$ and $\mathbb{A}[\alpha_2]$. This uses the interpretation of morphisms of ${}_\bullet\mathrm{Top}_\bullet$ from Remark 2.5. Hence we may define

$$\mathrm{Lift}(f) = (f, \mathbb{A}[\alpha_1^{\mathrm{op}}], \mathbb{A}[\alpha_2], \mathbb{A}_{\mathsf{fib}}[\widetilde{f}]). \qquad \square$$

2.3. Fibrewise tensor product

In this subsection, we define the "*fibrewise tensor product*" functor

$$\otimes\colon {}_\bullet\mathrm{Top}_\bullet\mathrm{Mod}_\bullet \longrightarrow {}_\bullet\mathrm{Top}_\bullet. \tag{2.5}$$

Notation 2.11 Recall from Definition 2.7 that, for two $\mathbb{A}$-algebra homomorphisms $\alpha\colon R \to R'$ and $\beta\colon S \to S'$, we denote the corresponding restriction functor ${}_{R'}\mathrm{Mod}_{S'} \to {}_R\mathrm{Mod}_S$ by $(\alpha, \beta)^*$.

Definition 2.12 We define the functor (2.5) as follows.

- (*On objects.*) We recall that an object of ${}_\bullet\mathrm{Top}_\bullet\mathrm{Mod}_\bullet$ consists of a space X, three $\mathbb{A}$-algebras R, S, T, an (S,T)-bimodule V and a bundle $\xi\colon \pi_{\leqslant 1}(X) \to {}_R\mathrm{Mod}_S$ of (R,S)-bimodules over X. Define its image under (2.5) to be the following object of ${}_\bullet\mathrm{Top}_\bullet$:

$$(X, R, T, \pi_{\leqslant 1}(X) \xrightarrow{\xi} {}_R\mathrm{Mod}_S \xrightarrow{-\otimes_S V} {}_R\mathrm{Mod}_T).$$

- (*On morphisms.*) A morphism of ${}_\bullet\mathrm{Top}_\bullet\mathrm{Mod}_\bullet$ from (X, R, S, T, V, ξ) to $(X', R', S', T', V', \xi')$ consists of a continuous map $f\colon X \to X'$, $\mathbb{A}$-algebra homomorphisms $\alpha\colon R \to R'$, $\beta\colon S \to S'$ and $\gamma\colon T \to T'$, a homomorphism $\theta\colon V \to (\beta,\gamma)^* V'$ of (S,T)-bimodules and a natural transformation $\tau\colon \xi \Rightarrow (\alpha,\beta)^* \circ \xi'$. Define its image under (2.5) to be the morphism

$$(f, \alpha, \gamma, \hat{\tau})$$

of ${}_\bullet\mathrm{Top}_\bullet$, where $\hat{\tau}\colon (-\otimes_S V) \circ \xi \Rightarrow (\alpha,\gamma)^* \circ (-\otimes_{S'} V') \circ \xi'$ is the natural transformation defined as follows. First, note that, for any (R,S')-bimodule A and (S',T)-bimodule B, there is a canonical homomorphism of (R,T)-bimodules

$$\beta^* A \otimes_S \beta^* B \longrightarrow A \otimes_{S'} B,$$

which is an isomorphism if $\beta\colon S \to S'$ is surjective. Using this fact, we define $\hat{\tau}$ on the object x of $\pi_{\leqslant 1}(X)$ by

$$\begin{aligned} \xi(x) \otimes_S V &\longrightarrow (\alpha,\beta)^*(\xi'(x)) \otimes_S V \\ &\longrightarrow (\alpha,\beta)^*(\xi'(x)) \otimes_S (\beta,\gamma)^* V' \\ &\longrightarrow (\alpha,1)^*(\xi'(x)) \otimes_{S'} (1,\gamma)^* V' \\ &= (\alpha,\gamma)^*(\xi'(x) \otimes_{S'} V'), \end{aligned}$$

where the first arrow is induced by τ_x, the second is induced by θ and the third is the canonical homomorphism from above.

Remark 2.13 This definition is somewhat formal, but it has a very natural geometric interpretation. If we view a bundle as an actual bundle over X (*cf.* Remark 2.5) and choose an open cover $\mathcal{U}$ of X together with trivialisations of the bundle over each $U \in \mathcal{U}$, then we may take the tensor product over each U (since the bundle is now trivial over each U and it is obvious how to define this) and then glue these trivial bundles back together again using the same transition functions as for the original bundle.

Notation 2.14 There are forgetful functors ${}_\bullet\mathrm{Top}_\bullet\mathrm{Mod}_\bullet \to {}_\bullet\mathrm{Top}_\bullet$ and ${}_\bullet\mathrm{Top}_\bullet\mathrm{Mod}_\bullet \to {}_\bullet\mathrm{Mod}_\bullet$ coming from the pullback square (2.1). For a continuous functor $F\colon \mathcal{C} \to {}_\bullet\mathrm{Top}_\bullet\mathrm{Mod}_\bullet$, we denote its compositions with these two forgetful functors by

$$F_1\colon \mathcal{C} \longrightarrow {}_\bullet\mathrm{Top}_\bullet \qquad \text{and} \qquad F_2\colon \mathcal{C} \longrightarrow {}_\bullet\mathrm{Mod}_\bullet$$

respectively. With this notation, we define $F_1 \otimes F_2$ by $F_1 \otimes F_2 := \otimes \circ F$.

2.4. Twisted homology

Over a fixed $\mathbb{A}$-algebra R, one may view local coefficient systems on a space X as bundles of right R-modules over X. In this viewpoint, homology with local coefficients (in any fixed degree $i \geqslant 0$) is a functor of the form

$$H_i\colon \mathrm{Top}_R \longrightarrow \mathrm{Mod}_R. \tag{2.6}$$

See, for example [DK01, §5.4] or [Pal18, §5.1]. The following fact may be proven by directly generalising the usual construction of singular twisted homology, keeping careful track of the variable bimodule structure.

Proposition 2.15 *In any degree $i \geqslant 0$, homology with local coefficients extends to a functor*

$$H_i\colon {}_\bullet\mathrm{Top}_\bullet \longrightarrow {}_\bullet\mathrm{Mod}_\bullet \tag{2.7}$$

such that the square

$$\begin{array}{ccc} {}_\bullet\mathrm{Top}_\bullet & \xrightarrow{\;H_i\;} & {}_\bullet\mathrm{Mod}_\bullet \\ \uparrow & & \uparrow \\ \mathrm{Top}_R & \xrightarrow{\;H_i\;} & \mathrm{Mod}_R \end{array} \tag{2.8}$$

commutes for any $\mathbb{A}$-algebra R. There are exactly analogous statements for relative homology (on a category of bimodule bundles over pairs of spaces) and Borel-Moore homology (where we must restrict to locally-compact spaces and proper maps). Moreover, there are similar statements for cohomology (including compactly-supported cohomology, where we must again restrict to locally-compact spaces and proper maps), although the domain category is not the opposite of ${}_\bullet\mathrm{Top}_\bullet$, but rather another category ${}_\bullet\mathrm{Top}_\bullet^{\leftarrow}$ defined below.

Definition 2.16 The category ${}_\bullet\mathrm{Top}_\bullet^{\leftarrow}$ has the same objects as ${}_\bullet\mathrm{Top}_\bullet$, but its morphisms are slightly different: they are *contravariant* on the underlying spaces and *covariant* on bimodule bundles (local systems). More precisely, its morphisms are exactly as in Definition 2.7, except that the map f goes in the opposite direction $X' \to X$ and the compatibility condition is correspondingly changed to $(\varphi, \psi)^* \circ \xi' = F \circ \xi \circ \pi_{\leqslant 1}(f)$. See also [DK01, Theorem 5.12].

Remark 2.17 (*Interpreting twisted homology as homology of covering spaces.*) The homological representations that we construct will be a composition of functors, ending with (2.7) (or one of its variants, such as twisted Borel-Moore homology). Thus, restricted to each object of the domain, they will give a representation of its automorphism group on the twisted homology of some space X equipped with a local system $\mathcal{L}$. Typically, $\mathcal{L}$ will correspond to a regular covering $\hat{X} \to X$ (in fact, it will *always* arise from a regular covering, followed possibly by a fibrewise tensor product).

If so, the twisted homology $H_*(X;\mathcal{L})$ is canonically isomorphic to $H_*(\hat{X})$, by Shapiro's lemma for covering spaces, so we may think of the homological representation as an action on the untwisted homology of the covering space $\hat{X}$. — However, this last remark only applies to (twisted) *ordinary* homology, and not to (twisted) *Borel-Moore* homology, since Shapiro's lemma for covering spaces is generally false for Borel-Moore homology. For example if $\hat{X} \to X$ is the universal covering $\mathbb{R} \to S^1$ then $H_1^{BM}(S^1;\mathcal{L}) = 0 \ncong \mathbb{Z} \cong H_1^{BM}(\mathbb{R})$. See [AP20, §6.1–§6.2] for further details.

2.5. The general construction

Our general construction is now obtained by concatenating the lifting functor of §2.2, the fibrewise tensor product of §2.3 and twisted homology §2.4.

2.5.1. The first version.

Suppose that we are given a topologically-enriched category $\mathcal{C}$, which is assumed to be *nice*, meaning that, for each pair of objects (x, y), the connected components of the morphism space $\mathcal{C}(x, y)$ are path-connected. We note that this condition holds whenever each $\mathcal{C}(x, y)$ is locally path-connected, and it ensures that, for any discrete category $\mathcal{D}$, any continuous functor $\mathcal{C} \to \mathcal{D}$ factors (uniquely) through $\mathcal{C} \to \pi_0(\mathcal{C})$. The input for the construction consists of

- a continuous functor $F \colon \mathcal{C} \to \mathrm{Cov}_\bullet$ and
- a positive integer $i \geqslant 0$.

The induced functor $L_i(F) \colon \pi_0(\mathcal{C}) \to \mathrm{Mod}_\bullet$ is then obtained as indicated in the following diagram, where Lift is defined in §2.2 and H_i is twisted homology as described in §2.4.

$$\begin{array}{ccccccc} \mathcal{C} & \xrightarrow{\;F\;} & \mathrm{Cov}_\bullet & \xrightarrow{\;\mathrm{Lift}\;} & \mathrm{Top}_\bullet & \xrightarrow{\;H_i\;} & \mathrm{Mod}_\bullet \\ \downarrow & & & & & \nearrow_{L_i(F)} & \\ \pi_0(\mathcal{C}) & & & & & & \end{array} \tag{2.9}$$

2.5.2. The second version.

Now suppose that we are given a nice topologically-enriched category $\mathcal{C}$ as above, together with

- a continuous functor $F \colon \mathcal{C} \to \mathrm{Cov}_\bullet$,
- a continuous functor $V \colon \mathcal{C} \to {}_\bullet\mathrm{Mod}_\bullet$ and
- a positive integer $i \geqslant 0$,

where F and V satisfy Condition 2.18 below, where we recall (*cf.* Notation 2.8) that ℓ and r denote functors

$${}_\bullet\mathrm{Cov}_\bullet \underset{r}{\overset{\ell}{\rightrightarrows}} \mathrm{Grp} \qquad\qquad {}_\bullet\mathrm{Mod}_\bullet \subset {}_\bullet\mathrm{Top}_\bullet \underset{r}{\overset{\ell}{\rightrightarrows}} \mathbb{A}\text{-}\mathrm{Alg}$$

that remember just the first (respectively second) underlying group or $\mathbb{A}$-algebra.

Condition 2.18 The functors F and V are required to be compatible in the sense that

$$\mathbb{A}[r \circ F] = \ell \circ V,$$

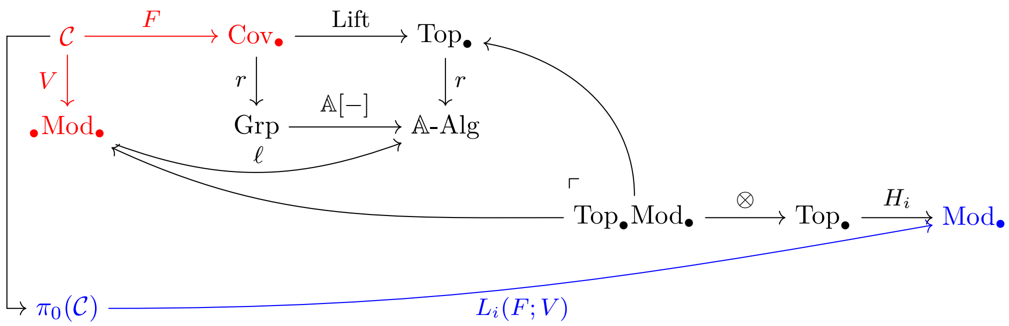
where $\mathbb{A}[-]$ denotes the free functor $\mathrm{Grp} \to \mathbb{A}\text{-}\mathrm{Alg}$.

Lemma 2.19 *Given continuous functors F and V satisfying Condition 2.18 and an integer $i \geqslant 0$, there is a naturally associated functor $\pi_0(\mathcal{C}) \to \mathrm{Mod}_\bullet$.*

Proof. The construction is as follows. Condition 2.18 implies that $\ell \circ V = r \circ \mathrm{Lift} \circ F$, and hence that the functors $\mathrm{Lift} \circ F$ and V determine a functor $\mathcal{C} \to \mathrm{Top}_\bullet\mathrm{Mod}_\bullet$ by the universal property of the pullback. This is then composed with the fibrewise tensor product (§2.3) and twisted homology (§2.4) to obtain a functor of the form $\mathcal{C} \to \mathrm{Mod}_\bullet$. Since $\mathrm{Mod}_\bullet$ is a discrete category and $\mathcal{C}$ is nice, this factors uniquely through a functor

$$\pi_0(\mathcal{C}) \longrightarrow \mathrm{Mod}_\bullet,$$

which is the output of the construction. This may be summarised diagrammatically as follows (where the input is red and the output is blue):



(2.10)

□

Definition 2.20 (*The general construction.*) Given continuous functors F and V satisfying Condition 2.18, define $L_i(F;V)$ to be the functor given by Lemma 2.19. Using the notational convention described in Notation 2.14, this may be written as

$$L_i(F;V) = H_i \circ ((\mathrm{Lift} \circ F) \otimes V).$$

Remark 2.21 (*Other variants of homology.*) All of the above constructions go through identically if we replace (twisted) ordinary homology with (twisted) Borel-Moore homology, cohomology, compactly-supported cohomology, reduced homology, etc., as long as we modify each category in the construction accordingly. For example, if we wish to use twisted Borel-Moore homology, we must restrict to locally compact spaces and proper maps, so in particular the input functor F must take values in this subcategory of ${}_\bullet\mathrm{Cov}_\bullet$. Similarly, if we wish to use twisted cohomology, we must replace ${}_\bullet\mathrm{Top}_\bullet$ with the "*ambivariant*" version ${}_\bullet\mathrm{Top}_\bullet^{\leftarrow}$, as described in Definition 2.16, and similarly for ${}_\bullet\mathrm{Cov}_\bullet$, and F must take values in ${}_\bullet\mathrm{Cov}_\bullet^{\leftarrow}$ instead of ${}_\bullet\mathrm{Cov}_\bullet$.

Remark 2.22 This construction is inspired by and recovers a method due to Bigelow [Big04, §2] of constructing $\mathbf{B}_n$-representations from $\mathbf{B}_k$-representations for any n, k; see Remark 5.30.

Remark 2.23 This construction recovers the simpler construction given in §2.5.1 if we take V to be the "trivial" coefficient system V_F, constructed as follows. We denote by $\mathrm{triv}\colon \mathbb{A}\text{-Alg} \to {}_\bullet\mathrm{Mod}_\bullet$ the functor that sends R to the bimodule (R, R, R). Then V_F is the composition $\mathrm{triv} \circ \mathbb{A}[-] \circ r \circ F$. In formulas, using the notational convention of Notation 2.14, we have $(\mathrm{Lift} \circ F) \otimes V_F \cong \mathrm{Lift} \circ F$.

Remark 2.24 A minor subtlety when we apply this construction later will be that our category $\mathcal{C} = \mathfrak{U}\mathcal{D}_d$ will actually only be a *semi-category* (a category without identities), although $\pi_0(\mathcal{C})$ will be a category (with identities). The diagrams above should therefore be interpreted as diagrams of semifunctors. However, the induced arrow from $\pi_0(\mathcal{C})$ will always turn out to be a *functor*, i.e., it will preserve identities. See §3.4 for why we must allow $\mathcal{C}$ to be a semi-category, and Lemma 5.10 for the statement that the output of the construction is nevertheless a functor.

Remark 2.25 (*Iterative constructions.*) There is an immediate generalisation of the above construction, in which F takes values in ${}_\bullet\mathrm{Cov}_\bullet$ instead of $\mathrm{Cov}_\bullet$, and the output functor $L_i(F;V)$ similarly takes values in ${}_\bullet\mathrm{Mod}_\bullet$ instead of $\mathrm{Mod}_\bullet$. One may then wonder (bearing in mind the fact that continuous functors $\mathcal{C} \to {}_\bullet\mathrm{Mod}_\bullet$ are in bijective correspondence with functors $\pi_0(\mathcal{C}) \to {}_\bullet\mathrm{Mod}_\bullet$) whether this construction, for fixed F and i, may be *iterated* by taking $V_{n+1} = L_i(F;V_n)$. However, this does not work as formulated above. In order to apply the construction above, the functor V is required to satisfy the compatibility Condition 2.18 (depending on F). That V satisfies Condition 2.18 does not imply that $L_i(F;V)$ satisfies Condition 2.18, so we cannot apply the construction $L_i(F;-)$ again to $L_i(F;V)$. To solve this issue, we introduce in [PS22b] a modified version of the construction that may be iterated indefinitely, amounting to an endofunctor

$$\mathbf{Fct}(\mathcal{C}, {}_\bullet\mathrm{Mod}_\bullet) \longrightarrow \mathbf{Fct}(\mathcal{C}, {}_\bullet\mathrm{Mod}_\bullet)$$

of the continuous functor category $\mathbf{Fct}(\mathcal{C}, {}_\bullet\mathrm{Mod}_\bullet)$. This is inspired by and recovers the *Long-Moody construction* [Lon94]. This iterative construction is not addressed in the present paper but will be the subject of the sequel paper [PS22b].

3. Topological categories of decorated manifolds

In this section, we define the categories that will serve as the domain of the "homological representations" that we will construct in §5. They are obtained from certain monoidal groupoids by the *Quillen bracket construction*, an operation that enlarges a given monoidal groupoid to a monoidal category having the original monoidal groupoid as its underlying groupoid. More precisely, we will start with certain *topologically-enriched* monoidal groupoids, so we first describe, in §3.1, a topological enrichment of the Quillen bracket construction and show that it behaves well with respect to the functor π_0 that replaces all morphism spaces with their sets of path-components, subject to a *Serre fibration condition*. In §3.2, we then define the topologically-enriched monoidal groupoids that we wish to consider, and prove that they satisfy this Serre fibration condition.

Informally, the idea is that the domain category used in §5, for a given dimension $d \geqslant 2$, will be a topologically-enriched category $\mathfrak{U}\mathcal{D}_d$ having the property that the automorphism groups of $\pi_0(\mathfrak{U}\mathcal{D}_d)$ contain all motion groups and mapping class groups in dimension d. To construct this, we define in §3.2 a topological groupoid $\mathcal{D}_d$ whose automorphism groups are the diffeomorphism groups of all d-dimensional "decorated manifolds". The topologically-enriched Quillen bracket construction of §3.1 then gives us a topologically-enriched category $\mathfrak{U}\mathcal{D}_d$ such that $\pi_0(\mathfrak{U}\mathcal{D}_d) \cong \mathfrak{U}(\pi_0(\mathcal{D}_d))$. The underlying groupoid of $\pi_0(\mathfrak{U}\mathcal{D}_d)$ is therefore $\pi_0(\mathcal{D}_d)$, consisting of all mapping class groups of d-dimensional "decorated manifolds", which contain all d-dimensional motion groups as normal subgroups.

3.1. A topological enrichment of the Quillen bracket construction

Throughout this section, we fix a *topological* monoidal groupoid $(\mathcal{G}, \natural, 0)$ and a *topological* category $(\mathcal{M}, \natural)$ with a continuous left-action of $\mathcal{G}$. We use the abbreviation ob to denote the set of objects of a category. We refer the reader to [Mac98] for a complete introduction to the notions of (strict) monoidal categories and modules over them. We recall that, by *topological category*, we mean a category enriched over the symmetric monoidal category of topological spaces with the Cartesian product.

Definition 3.1 The category $\langle \mathcal{G}, \mathcal{M} \rangle$ is defined to have the same objects as $\mathcal{M}$, and for objects X, Y of $\mathcal{M}$, we define $\mathrm{Hom}_{\langle \mathcal{G}, \mathcal{M} \rangle}(X, Y)$ to be the quotient space

$$\left[\bigsqcup_{A \in \mathrm{ob}(\mathcal{G})} \mathrm{Hom}_{\mathcal{M}}(A \natural X, Y) \right] / \sim,$$

where $\sim$ is the equivalence relation given by $(A, \varphi) \sim (A', \varphi')$ if and only if $\varphi = \varphi' \circ (\sigma \natural \mathrm{id}_X)$ for some $\sigma \in \mathrm{Hom}_{\mathcal{G}}(A, A')$. For two morphisms $[A, \varphi] : X \to Y$ and $[B, \psi] \colon Y \to Z$ in $\langle \mathcal{G}, \mathcal{M} \rangle$, the composition is defined by $[B, \psi] \circ [A, \varphi] = [B \natural A, \psi \circ (\mathrm{id}_B \natural \varphi)]$. We note that this may also be written as the colimit

$$\mathrm{Hom}_{\langle \mathcal{G}, \mathcal{M} \rangle}(X, Y) = \underset{\mathcal{G}}{\mathrm{colim}} [\mathrm{Hom}_{\mathcal{M}}(- \natural X, Y)].$$

Remark 3.2 There is a canonical faithful functor $c_{\langle \mathcal{G}, \mathcal{M} \rangle} \colon \mathcal{M} \hookrightarrow \langle \mathcal{G}, \mathcal{M} \rangle$ defined as the identity on objects and sending $\phi \in \mathrm{Hom}_{\mathcal{M}}(X, Y)$ to $[0, \phi]$.

Example 3.3 As an example, topological monoidal groupoid $\mathcal{G}$ has a continuous left-action on itself given by its monoidal structure, so we may always take $\mathcal{M} = \mathcal{G}$ and consider the category $\langle \mathcal{G}, \mathcal{G} \rangle$. As an abbreviation, we denote the category $\langle \mathcal{G}, \mathcal{G} \rangle$ by $\mathfrak{U}\mathcal{G}$.

If the categories $\mathcal{G}$ and $\mathcal{M}$ are both *discrete*, then Definition 3.1 recovers the classical *bracket construction* of Quillen. This is a particular case of a more general construction of [Gra76, p.219]. Assuming in addition that $\mathcal{M}$ is a groupoid (as for all the examples discussed in this paper, see §4), following mutatis mutandis [RW17, Proposition 1.7], if $(\mathcal{G}, \natural, 0)$ has *no zero divisors* – meaning that $A \natural B \cong 0$ if and only if $A \cong B \cong 0$ for all objects A and B of $\mathcal{G}$ – and if $\mathrm{Aut}_{\mathcal{G}}(0) = \{\mathrm{id}_0\}$, then the canonical functor of Remark 3.2 above is an isomorphism from $\mathcal{M}$ onto the maximal subgroupoid of $\langle \mathcal{G}, \mathcal{M} \rangle$ (i.e. the subcategory which has the same objects as $\langle \mathcal{G}, \mathcal{M} \rangle$ and of which the morphisms are the isomorphisms of $\langle \mathcal{G}, \mathcal{M} \rangle$). Hence, if we suppose that $\mathcal{G}$ and $\mathcal{M}$ are discrete groupoids such that $\mathcal{M}$ has non-negative integers as objects and a family of groups $\{G_n\}_{n \in \mathbb{N}}$ as isomorphisms, that

$\mathcal{G}$ has no zero divisors and that its monoidal unit has no non-trivial automorphisms, then Quillen's bracket construction consists in just "artificially" adding to $\mathcal{M}$ morphisms which go from n to $n+1$: this justifies the use of this construction as source category to encode *compatible* representations of families of groups.

Remark 3.4 A topological version of Quillen's bracket construction is mentioned briefly in Remark 2.10 of [Kra19], although there the categories are *topological* in the sense of being categories internal to the category of topological spaces, rather than topologically-enriched categories. Lemma 3.6 below is stated for topologically-enriched categories, but it is likely that it has an analogue for categories internal to the category of topological spaces, in which case Lemma 2.11 of [Kra19] would be a particular case of this analogue.

This construction is of course *functorial* in $\mathcal{M}$ and $\mathcal{G}$ in an appropriate sense. We mention some properties of this functoriality that we will need.

Lemma 3.5 *Let $\mathcal{D}$ be a topological monoidal groupoid and let $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{D}$ and $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{D}$ be subgroupoids such that, for $i = 1, 2$, $\mathcal{G}_i$ is closed under $-\natural-$ and $\mathcal{M}_i$ is closed under $g\natural-$ for each object g of $\mathcal{G}_i$. Then there is a canonical functor $\langle \mathcal{G}_1, \mathcal{M}_1 \rangle \longrightarrow \langle \mathcal{G}_2, \mathcal{M}_2 \rangle$. Moreover,*

- *if $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{G}$, the functor $\langle \mathcal{G}, \mathcal{M}_1 \rangle \longrightarrow \langle \mathcal{G}, \mathcal{M}_2 \rangle$ is an inclusion of a subcategory, which is full if the inclusion $\mathcal{M}_1 \subseteq \mathcal{M}_2$ is full;*
- *if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, the functor $\langle \mathcal{G}_1, \mathcal{M} \rangle \longrightarrow \langle \mathcal{G}_2, \mathcal{M} \rangle$ is the identity on objects, and is faithful – thus an inclusion of a subcategory – if the inclusion $\mathcal{G}_1 \subseteq \mathcal{G}_2$ is full.*

In particular, there is a canonical functor $\langle \mathcal{G}_1, \mathcal{M}_1 \rangle \longrightarrow \mathfrak{U}\mathcal{D}$, which is an inclusion of a subcategory if the inclusion $\mathcal{G}_1 \subseteq \mathcal{D}$ is full.

Proof. The objects of $\langle \mathcal{G}_i, \mathcal{M}_i \rangle$ are the objects of $\mathcal{M}_i$, so we define the functor on objects as the inclusion $\mathrm{ob}(\mathcal{M}_1) \hookrightarrow \mathrm{ob}(\mathcal{M}_2)$. A morphism in $\langle \mathcal{G}_i, \mathcal{M}_i \rangle$ from X to Y is represented by a choice of object A of $\mathcal{G}_i$ and a morphism $A\natural X \to Y$ of $\mathcal{M}_i$. We may therefore send such a morphism, for $i = 1$, to the morphism, for $i = 2$, represented by the same data, since $\mathcal{G}_1 \subseteq \mathcal{G}_2$ and $\mathcal{M}_1 \subseteq \mathcal{M}_2$. It is straightforward to check that this assignment respects the defining equivalence relation, so induces a continuous map of morphism spaces, and that it also respects composition and identities. The statements in the two bullet points may be verified by unwinding the definition of morphisms in $\langle \mathcal{G}_i, \mathcal{M}_i \rangle$. □

Lemma 3.6 *Let $\mathcal{G}$ be a topological monoidal groupoid and $\mathcal{M}$ a topological category with a continuous left-action of $\mathcal{G}$. Assume that, for each object A of $\mathcal{G}$ and each pair of objects X, Y of $\mathcal{M}$, the quotient map*

$$\mathrm{Hom}_{\mathcal{M}}(A\natural X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{M}}(A\natural X, Y)/\mathrm{Aut}_{\mathcal{G}}(A) \tag{3.1}$$

is a Serre fibration. Then there is a canonical isomorphism of categories

$$\pi_0(\langle \mathcal{G}, \mathcal{M} \rangle) \cong \langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M}) \rangle. \tag{3.2}$$

Proof. First note that $\pi_0(\langle \mathcal{G}, \mathcal{M} \rangle)$ and $\langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M}) \rangle$ have the same object set, by the definition of the discrete and topologically-enriched Quillen bracket constructions, and the functor π_0. Specifically, their common object set is $\mathrm{ob}(\mathcal{M})$. It therefore remains to show that, for objects X and Y of $\mathcal{M}$, there is a natural bijection between $\pi_0(\mathrm{Hom}_{\langle \mathcal{G}, \mathcal{M} \rangle}(X, Y))$ and $\mathrm{Hom}_{\langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M}) \rangle}(X, Y)$. Set

$$\Phi = \bigsqcup_{A \in \mathrm{ob}(\mathcal{G})} \mathrm{Hom}_{\mathcal{M}}(A\natural X, Y).$$

Unravelling the definitions, what we need to prove is that there is a natural bijection

$$\pi_0(\Phi/\sim_t) \cong \pi_0(\Phi)/\sim_h,$$

where $\sim_t$ is the equivalence relation given by $(A, \varphi) \sim_t (A', \varphi')$ if and only if there is a morphism $\sigma \in \mathrm{Hom}_{\mathcal{G}}(A, A')$ such that $\varphi = \varphi' \circ (\sigma\natural\mathrm{id}_A)$, and $\sim_h$ is the equivalence relation given by $(A, [\varphi]) \sim_h (A', [\varphi'])$ if and only if there is a morphism $\sigma \in \mathrm{Hom}_{\mathcal{G}}(A, A')$ such that $\varphi \simeq \varphi' \circ (\sigma\natural\mathrm{id}_A)$. We note that the only difference between these definitions is that the equality is replaced by a homotopy in the definition of $\sim_h$. As sets, these are both quotients of (the underlying set of) Φ, so we just need to show that, given two elements (A,φ) and (A',φ') of Φ, they have the same image in $\pi_0(\Phi/{\sim_t})$ if and only if they have the same image in $\pi_0(\Phi)/{\sim_h}$.

(a) Suppose first that (A,φ) and (A',φ') have the same image in $\pi_0(\Phi)/{\sim_h}$. This means that there is a morphism $\sigma \in \mathrm{Hom}_{\mathcal{G}}(A,A')$ and a path $\gamma\colon [0,1] \longrightarrow \mathrm{Hom}_{\mathcal{M}}(A\natural X,Y) \subseteq \Phi$ with $\gamma(0)=(A,\varphi)$ and $\gamma(1)=(A,\varphi'\circ(\sigma\natural\mathrm{id}_A))$. Composing with the projection $\Phi \to \Phi/{\sim_t}$ and writing $[-]_t$ for the equivalence classes with respect to $\sim_t$, we obtain a path in $\Phi/{\sim_t}$ from $[(A,\varphi)]_t$ to $[(A,\varphi'\circ(\sigma\natural\mathrm{id}_A))]_t = [(A',\varphi')]_t$. Hence (A,φ) and (A',φ') have the same image in $\pi_0(\Phi/{\sim_t})$.

(b) To prove the converse, we first make an assumption, which we will justify later. Namely, we assume that quotient map $q\colon \Phi \longrightarrow \Phi/{\sim_t}$ is a Serre fibration. Now assume that (A,φ) and (A',φ') have the same image in $\pi_0(\Phi/{\sim_t})$, so there is a path $\delta\colon [0,1] \to \Phi/{\sim_t}$ with $\delta(0)=[(A,\varphi)]_t$ and $\delta(1)=[(A',\varphi')]_t$. By our assumption that q is a Serre fibration, we may lift this to a path $\varepsilon\colon [0,1] \to \Phi$ with $\varepsilon(0)=(A,\varphi)$ and $\varepsilon(1) \sim_t (A',\varphi')$. Its image $\varepsilon([0,1])$ is path-connected, so it must lie in $\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y) \subseteq \Phi$. Hence we have a path $\varepsilon\colon [0,1] \longrightarrow \mathrm{Hom}_{\mathcal{M}}(A\natural X,Y)$ with $\varepsilon(0)=(A,\varphi)$ and $\varepsilon(1)=(A,\varphi'') \sim_t (A',\varphi')$, for some $\varphi'' \in \mathrm{Hom}_{\mathcal{M}}(A\natural X,Y)$. The relation $(A,\varphi'') \sim_t (A',\varphi')$ means that there is a morphism $\sigma \in \mathrm{Hom}_{\mathcal{G}}(A,A')$ such that $\varphi''=\varphi'\circ(\sigma\natural\mathrm{id}_A)$. Hence ε is a homotopy witnessing that $\varphi \simeq \varphi'\circ(\sigma\natural\mathrm{id}_A)$, so we have shown that $(A,[\varphi]) \sim_h (A',[\varphi'])$, in other words, (A,φ) and (A',φ') have the same image in $\pi_0(\Phi)/{\sim_h}$.

(c) It now just remains to prove our earlier assumption that q is a Serre fibration. Directly from the definition, one may easily verify the following two facts:

- $\bigsqcup_i f_i\colon \bigsqcup_i E_i \to B$ is a Serre fibration if and only if each $f_i\colon E_i \to B$ is a Serre fibration.
- $f\colon E \to B$ is a Serre fibration if and only if $f(E)$ is a union of path-components of B and $f\colon E \to f(E)$ is a Serre fibration.

It therefore suffices to prove that

(i) $q(\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y))$ is a union of path-components of $\Phi/{\sim_t}$ for each $A \in \mathrm{ob}(\mathcal{G})$,
(ii) $\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y) \to q(\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y))$ is a Serre fibration for each $A \in \mathrm{ob}(\mathcal{G})$.

Let us partition $\mathrm{ob}(\mathcal{G})$ into equivalence classes $\mathcal{O}_\alpha$, under the equivalence relation where two objects A,A' of $\mathcal{G}$ are equivalent if and only if there is a morphism $A \to A'$ in $\mathcal{G}$. This is an equivalence relation since $\mathcal{G}$ is a groupoid. We may then write $\Phi = \bigsqcup_\alpha \Phi_\alpha$, where

$$\Phi_\alpha = \bigsqcup_{A\in\mathcal{O}_\alpha} \mathrm{Hom}_{\mathcal{M}}(A\natural X,Y).$$

The equivalence relation $\sim_t$ on Φ clearly preserves the topological disjoint union $\bigsqcup_\alpha \Phi_\alpha$, so we have

$$\Phi/{\sim_t} = \bigsqcup_\alpha (\Phi_\alpha/{\sim_t}).$$

Also note that, for any two objects $A,A' \in \mathcal{O}_\alpha$ (for fixed α), we have $q(\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y)) = q(\mathrm{Hom}_{\mathcal{M}}(A'\natural X,Y))$. So, if we make a choice of object $A_\alpha \in \mathcal{O}_\alpha$ for each α, we have a decomposition of $\Phi/{\sim_t}$ as a topological disjoint union:

$$\Phi/{\sim_t} = \bigsqcup_\alpha q(\mathrm{Hom}_{\mathcal{M}}(A_\alpha\natural X,Y)).$$

This immediately implies point (i) above.

For point (ii), we note that two elements $\varphi,\varphi' \in \mathrm{Hom}_{\mathcal{M}}(A\natural X,Y)$ have the same image under q if and only if they are $\sim_t$-equivalent, which is equivalent to saying that they lie in the same orbit of the $\mathrm{Aut}_{\mathcal{G}}(A)$-action on $\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y)$. Hence the map

$$q_A\colon \mathrm{Hom}_{\mathcal{M}}(A\natural X,Y) \longrightarrow q(\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y)) \tag{3.3}$$

is isomorphic to (3.1), at least on underlying sets. If we can show that they are isomorphic also as continuous maps of spaces, then we will be done, since we know by hypothesis that (3.1) is a Serre fibration. Since (3.1) and (3.3) are surjective continuous maps with the same domain and the same point-fibres, and we know moreover that (3.1) is a quotient map, it suffices to prove that (3.3) is also a quotient map.

Let $U \subseteq q(\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y))$ be a subset such that $q_A^{-1}(U)$ is open in $\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y)$. We need to show that U is open in $q(\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y))$. To see this, let $A \in \mathcal{O}_\alpha$ and note that, by

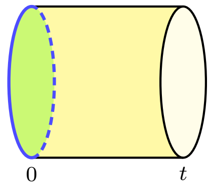


Figure 3.1 An illustration of the notation for the solid cylinder $\mathbb{B}_t^d$ from Notation 3.7 for $d = 3$. Its lower boundary $\partial_\ell \mathbb{B}_t^3$ is coloured yellow, its base $b\mathbb{B}_t^3$ is yellow-green and the codimension-2 stratum $\partial b\mathbb{B}_t^3 = \partial\mathbb{D}^2 \times \{0\}$ is blue.

the fact discussed above that the equivalence relation $\sim_t$ preserves the decomposition of Φ into a topological disjoint union, the restriction

$$q_\alpha = q|_{\Phi_\alpha} \colon \Phi_\alpha \longrightarrow q(\Phi_\alpha) = q(\mathrm{Hom}_{\mathcal{M}}(A \natural X, Y))$$

is a quotient map. So it suffices to show that $q_\alpha^{-1}(U)$ is open in Φ_α. Now, from the definitions, we observe the following description of the subset

$$q_\alpha^{-1}(U) \;\subseteq\; \Phi_\alpha \;=\; \bigsqcup_{A' \in \mathcal{O}_\alpha} \mathrm{Hom}_{\mathcal{M}}(A' \natural X, Y).$$

For each object $A' \in \mathcal{O}_\alpha$, choose an isomorphism $\sigma_{A'} \colon A' \to A$ in $\mathcal{G}$. This induces a homeomorphism

$$\Upsilon_{A'} \;=\; - \circ (\sigma_{A'} \natural \mathrm{id}) \colon \mathrm{Hom}_{\mathcal{M}}(A \natural X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{M}}(A' \natural X, Y).$$

Then we have

$$q_\alpha^{-1}(U) \;=\; \bigsqcup_{A' \in \mathcal{O}_\alpha} \Upsilon_{A'}(q_A^{-1}(U)).$$

Since $q_A^{-1}(U)$ is open in $\mathrm{Hom}_{\mathcal{M}}(A \natural X, Y)$, it follows that $\Upsilon_{A'}(q_A^{-1}(U))$ is open in $\mathrm{Hom}_{\mathcal{M}}(A' \natural X, Y)$ for each $A' \in \mathcal{O}_\alpha$. Thus $q_\alpha^{-1}(U)$ is open in Φ_α, as required. □

3.2. Topological groupoids of decorated manifolds

Fix an integer $d \geqslant 2$. First, we define the notion of *decorated manifolds* and their morphisms. The idea is that the groups $\pi_0(\mathrm{Diff}(-))$ of decorated manifolds will contain all motion groups as normal subgroups; *cf.* Remark 4.24.

Notation 3.7 (*Solid cylinders.*) We denote by $\mathbb{D}^{d-1}$ the closed unit $(d-1)$-dimensional disc in $\mathbb{R}^{d-1}$. For a real number $t > 0$, we will write $\mathbb{B}_t^d = \mathbb{D}^{d-1} \times [0,t]$ for the *solid d-dimensional cylinder of height t*. We will also write

$$\partial_\ell \mathbb{B}_t^d = (\partial\mathbb{D}^{d-1} \times [0,t]) \cup (\mathbb{D}^{d-1} \times \{0\})$$

and call this the *lower boundary* of $\mathbb{B}_t^d$, as well as $b\mathbb{B}_t^d = \mathbb{D}^{d-1} \times \{0\}$ and call this the *base* of $\mathbb{B}_t^d$. This is illustrated in Figure 3.1.

Definition 3.8 (*Boundary-cylinders.*) Let M be a smooth d-manifold. A *boundary-cylinder* for M is a topological embedding $e \colon \mathbb{B}_1^d \hookrightarrow M$ such that $e^{-1}(\partial M) = \partial_\ell \mathbb{B}_1^d$ and e is a smooth embedding except on the $(d-2)$-sphere $\partial b\mathbb{B}_1^d$. Two boundary-cylinders e, e' are *equivalent* if they are equal when restricted to $\mathbb{B}_\epsilon^d \subseteq \mathbb{B}_1^d$ for some $\epsilon > 0$. An equivalence class of boundary-cylinders is called a *boundary-cylinder germ*.

Definition 3.9 (*Decorated manifolds.*) A *decorated manifold* is a smooth d-manifold M, equipped with a closed submanifold $A \subset \mathrm{int}(M)$ and a pair (e_1, e_2) of boundary-cylinder germs for $M \smallsetminus A$ such that $e_1(b\mathbb{B}_1^d)$ and $e_2(b\mathbb{B}_1^d)$ are disjoint. See Figure 3.2 for a schematic picture.

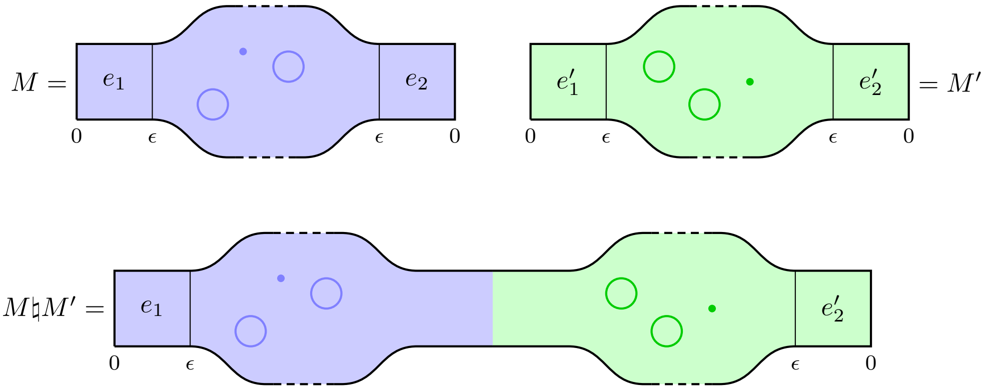


Figure 3.2 Two decorated manifolds and their boundary connected sum.

Definition 3.10 (*Morphisms of decorated manifolds*) A morphism of decorated manifolds from (M, A, e_1, e_2) to (M', A', e'_1, e'_2) is a smooth, proper (preimages of compact subspaces are compact) map $\varphi\colon M \to M'$ such that $\varphi(A) \subseteq A'$ and such that, for some $\epsilon > 0$ and for each $i \in \{1, 2\}$, we have $\varphi(e_i(\mathbb{B}^d_\epsilon)) = e'_i(\mathbb{B}^d_\epsilon)$ and the composition $(e'_i)^{-1} \circ \varphi \circ e_i \colon \mathbb{B}^d_\epsilon \to \mathbb{B}^d_\epsilon$ is the identity. Write $C^\infty_{\mathrm{dec}}(M, M')$ for the set of such morphisms, where by abuse of notation we are abbreviating (M, A, e_1, e_2) to M and (M', A', e'_1, e'_2) to M'.

Definition 3.11 (*Morphism spaces.*) The set $C^\infty_{\mathrm{dec}}(M, M')$ is topologised as a colimit of Whitney topologies, as follows. First choose representative boundary cylinders for the boundary-cylinder germs e_i and e'_i. This ensures that the condition in Definition 3.10 makes sense for a fixed $\epsilon \in (0, 1)$, not just for an unspecified $\epsilon \in (0, 1)$ that is quantified over.

Now fix $\epsilon \in (0, 1)$ and write $C^\infty_{\mathrm{dec},\epsilon}(M, M')$ for the subset of $C^\infty_{\mathrm{dec}}(M, M')$ where the condition in Definition 3.10 holds for this fixed ϵ. Equip each subset $C^\infty_{\mathrm{dec},\epsilon}(M, M')$ with the subspace topology induced from the smooth Whitney topology on the set $C^\infty(M, M')$ of *all* smooth maps from M to M'; for details of the Whitney topology, see for example [Hir76, Chapter 2]. As a set, $C^\infty_{\mathrm{dec}}(M, M')$ is the union of $C^\infty_{\mathrm{dec},\epsilon}(M, M')$ over all choices of $\epsilon \in (0, 1)$; we equip $C^\infty_{\mathrm{dec}}(M, M')$ with the colimit topology induced by the increasing filtration $\{C^\infty_{\mathrm{dec},\epsilon}(M, M')\}_{\epsilon \in (0,1)}$.

Different choices of representative boundary-cylinders for the boundary-cylinder germs e_i and e'_i will result in different filtrations. However, any two such filtrations are cofinal in each other, so the colimit topology induced on $C^\infty_{\mathrm{dec}}(M, M')$ does not depend on this choice.

Note also that this topology may differ from the subspace topology that is inherited directly from the Whitney topology on $C^\infty(M, M')$ (the colimit topology may be finer). However, these two topologies on $C^\infty_{\mathrm{dec}}(M, M')$ are weakly equivalent. In particular, they have the same π_0.

Definition 3.12 (*Boundary connected sum.*) Let (M, A, e_1, e_2) and (M', A', e'_1, e'_2) be two decorated d-manifolds. Define

$$M \natural M' = (M \sqcup M')/\!\sim,$$

where $\sim$ is the equivalence relation generated by $e_2(x, 0) \sim e'_1(x, 0)$ for all $(x, 0) \in b\mathbb{B}^d_1$. We give this a smooth structure as follows. There are obvious topological embeddings

$$M \smallsetminus e_2(b\mathbb{B}^d_1) \longhookrightarrow M \natural M' \qquad \text{and} \qquad M' \smallsetminus e'_1(b\mathbb{B}^d_1) \longhookrightarrow M \natural M',$$

and another topological embedding

$$\mathbb{D}^{d-1} \times [-1, 1] \longhookrightarrow M \natural M'$$

given by $(x, t) \mapsto e_2(x, -t)$ for $t \leqslant 0$ and $(x, t) \mapsto e'_1(x, t)$ for $t \geqslant 0$, where we have implicitly chosen representative boundary-cylinders for the boundary-cylinder germs e_2 and e'_1. We define a smooth structure on $M \natural M'$ by declaring that these are both *smooth* embeddings.

This is a well-defined smooth structure, since:

- the smooth structures induced by these embeddings are compatible on intersections, due to the fact that boundary-cylinders are *smooth* embeddings away from $\partial b\mathbb{B}^d_1$;

- the smooth structure of $M\natural M'$ is determined, except on $e_2(b\mathbb{B}^d_1) = e'_1(b\mathbb{B}^d_1)$, by the smooth structures of M and M'. The embedding of $\mathbb{D}^{d-1} \times [-1,1]$ induced by (boundary-cylinders representing the boundary-cylinder germs) e_2 and e'_1 is therefore only required to extend this smooth structure to $e_2(b\mathbb{B}^d_1) = e'_1(b\mathbb{B}^d_1)$. As a result, the smooth structure does not depend on the choice of representative boundary-cylinders, but only their germs.

Finally, we define:

$$(M, A, e_1, e_2) \natural (M', A', e'_1, e'_2) = (M\natural M', A \sqcup A', e_1, e'_2).$$

See Figure 3.2 for a schematic illustration.

Remark 3.13 The usual definition of boundary connected sum of two smooth manifolds M, M' depends on a choice of embedded disc in the boundary of each manifold, and a method of "straightening corners" after gluing these discs together. Up to diffeomorphism, the resulting smooth manifold $M\natural M'$ depends only on the choice of a boundary-component of M and of M', and orientations of these if they are orientable (this is a result of Palais' *Disc theorem* [Pal60a, Theorem B and Corollary 1] and the existence of collar neighbourhoods). However, in order for $\natural$ to induce a well-defined monoidal structure on some category of manifolds with boundary (which we will do just below), it must be well-defined on the nose, not just up to diffeomorphism (since objects are manifolds, not diffeomorphism-classes of manifolds). The definition of decorated manifolds is designed so that these additional choices are *built in*, and no additional choices are required in Definition 3.12 above.

Definition 3.14 (*Decorated manifold categories.*) Let $\mathcal{D}\mathrm{ec}_d$ be the topological category defined as follows. Its objects are all *decorated manifolds* (M, A, e_1, e_2) of dimension d, as in Definition 3.9. The space of morphisms from $M = (M, A, e_1, e_2)$ to $M' = (M', A', e'_1, e'_2)$ is the space $C^\infty_{\mathrm{dec}}(M, M')$ defined in Definition 3.10 and topologised in Definition 3.11. We note that composition is continuous in this topology, since it is a colimit of Whitney topologies and composition of smooth, proper maps is continuous in the Whitney topology.

Let $\mathcal{D}_d$ be the underlying topological groupoid of $\mathcal{D}\mathrm{ec}_d$. In other words, its objects are all decorated manifolds of dimension d and its morphisms are those morphisms $(M, A, e_1, e_2) \to (M', A', e'_1, e'_2)$ of decorated manifolds whose underlying smooth map $\varphi\colon M \to M'$ is a diffeomorphism and $\varphi(A) = A'$.

The boundary connected sum of Definition 3.12 induces a semi-monoidal structure on $\mathcal{D}\mathrm{ec}_d$, and hence on $\mathcal{D}_d$:

Definition 3.15 (*Semi-monoidal structure.*) We define the functor

$$\natural\colon \mathcal{D}\mathrm{ec}_d \times \mathcal{D}\mathrm{ec}_d \longrightarrow \mathcal{D}\mathrm{ec}_d$$

on objects via the boundary connected sum of Definition 3.12. Now suppose we are given morphisms $\varphi\colon (L, A, e_1, e_2) \to (L', A', e'_1, e'_2)$ and $\psi\colon (M, B, f_1, f_2) \to (M', B', f'_1, f'_2)$ in $\mathcal{D}\mathrm{ec}_d$. By definition, these are smooth, proper maps $L \to L'$ and $M \to M'$ that take A and B into A' and B' respectively, and are compatible with the given boundary-cylinder germs. This compatibility implies that they glue to a well-defined, smooth map $L\natural M \to L'\natural M'$, which is moreover a morphism

$$(L, A, e_1, e_2) \natural (M, B, f_1, f_2) \longrightarrow (L', A', e'_1, e'_2) \natural (M', B', f'_1, f'_2)$$

of $\mathcal{D}\mathrm{ec}_d$. It is then easily checked that this gives $\mathcal{D}\mathrm{ec}_d$ the structure of a topological semi-monoidal groupoid; see §3.4.

Remark 3.16 (*Monoidal and semi-monoidal structures.*) The semi-monoidal structure on $\mathcal{D}\mathrm{ec}_d$ defined above does not have a unit, since there is no *natural* way of identifying $M\natural\mathbb{B}^d_1$ with M for all M (although they are of course non-naturally diffeomorphic). If decorated manifolds had been defined to be equipped with boundary cylinders (not just *germs* of boundary-cylinders), then $\mathcal{D}\mathrm{ec}_d$ would have an obvious monoidal (i.e. semi-monoidal *with unit*) structure. However, the proof of Proposition 3.22 below, which tells us that the Serre fibration hypothesis of Lemma 3.6 is satisfied for subgroupoids of $\mathcal{D}_d$ (Lemma 4.4), depends crucially on the fact that morphisms of decorated manifolds are only required to preserve *germs* of boundary-cylinders, rather than entire boundary-cylinders. Thus we are forced either to formally adjoin a unit to $\mathcal{D}\mathrm{ec}_d$ (which is unnatural since $\pi_0(\mathcal{D}\mathrm{ec}_d)$ already has a unit by Lemma 3.18), or to work directly with semi-monoidal categories, which is what we shall do; see §3.4 for more details.

Finally we can analogously define a topological groupoid of *oriented* decorated manifolds:

Definition 3.17 Let $\mathcal{D}_d^+$ denote the topological groupoid whose objects are decorated d-manifolds (M, A, e_1, e_2) together with an orientation of $A \subset \mathrm{int}(M)$, and whose morphisms are diffeomorphisms φ as in Definition 3.14 such that the restriction $\varphi|_A \colon A \to A'$ is an orientation-preserving diffeomorphism. The boundary connected sum for such decorated d-manifolds is defined as in Definition 3.12, with the orientation for $A \sqcup B$ being induced from those of A and B. This then extends, just as in Definition 3.15, to a structure of a topological semi-monoidal groupoid on $\mathcal{D}_d^+$.

One may of course similarly define $\mathcal{D}_d^\theta$ for any type of tangential structure $\theta \colon X \to BO$, equipping $A \subset \mathrm{int}(M)$ with a θ-structure and requiring this to be preserved by morphisms φ. Applying π_0 to all morphism spaces, we may consider the discrete groupoids $\pi_0(\mathcal{D}_d)$ and $\pi_0(\mathcal{D}_d^+)$. The following lemma may be easily checked from the definitions.

Lemma 3.18 *The discrete groupoids $\pi_0(\mathcal{D}_d)$ and $\pi_0(\mathcal{D}_d^+)$ inherit well-defined semi-monoidal structures from those on $\mathcal{D}_d$ and $\mathcal{D}_d^+$. Moreover, these semi-monoidal structures are monoidal, with unit object given in each case by the solid cylinder $(\mathbb{B}_1^d, \emptyset, \mathrm{id}, r)$, where $r \colon \mathbb{B}_1^d \to \mathbb{B}_1^d$ is the reflection $(x, t) \mapsto (x, 1-t)$.*

More generally, if $\mathcal{G} \subseteq \mathcal{D}_d$ is any subgroupoid closed under the semi-monoidal structure and containing the solid cylinder $(\mathbb{B}_1^d, \emptyset, \mathrm{id}, r)$, then the semi-monoidal structure inherited by $\pi_0(\mathcal{G})$ is monoidal. Similarly for subgroupoids $\mathcal{G} \subseteq \mathcal{D}_d^+$.

3.3. The Serre fibration condition

We now prove a technical result that will imply that the condition (3.1) is true in a very general setting, covering all of our examples.

Definition 3.19 (*Decorated diffeomorphisms.*) A *diffeomorphism* of decorated manifolds (or a *decorated diffeomorphism*) is a morphism of decorated manifolds (*cf.* Definition 3.10) that admits an inverse. Write $\mathrm{Diff}_{\mathrm{dec}}(M, N)$ for the space of decorated diffeomorphisms $M \to N$ and abbreviate $\mathrm{Diff}_{\mathrm{dec}}(M) = \mathrm{Diff}_{\mathrm{dec}}(M, M)$. This is topologised as a subspace of $C^\infty_{\mathrm{dec}}(M, N)$, which is topologised as a colimit of Whitney topologies (*cf.* Definition 3.11).

Definition 3.20 (*Decorated embeddings.*) Let $M = (M, A)$ and $N = (N, B)$ be decorated manifolds. Define $\mathrm{Emb}_{\mathrm{dec}}(M, N)$ to be the space of smooth, proper embeddings $\varphi \colon M \hookrightarrow N$, equipped with a germ of an extension φ' to an embedding $\mathbb{B}_1^d \natural M \hookrightarrow N$, such that

- $\varphi(A) \subseteq B$;
- for some $\epsilon > 0$ we have $\varphi(e_2(\mathbb{B}_\epsilon^d)) = e_2'(\mathbb{B}_\epsilon^d)$ and $(e_2')^{-1} \circ \varphi \circ e_2$ is the identity map $\mathbb{B}_\epsilon^d \to \mathbb{B}_\epsilon^d$, where e_1, e_2 are the boundary-cylinder germs of (M, A) and e_1', e_2' are those of (N, B). This is similar to Definition 3.10, except that we only require the condition on e_2, not on e_1;
- there is a decorated manifold M' and diffeomorphism of decorated manifolds $\bar{\varphi} \colon M' \natural M \to N$ such that $\varphi = \bar{\varphi} \circ \iota_{M,M'}$, where $\iota_{M,M'}$ denotes the canonical embedding of M into $M' \natural M$. This extension of φ to $\bar{\varphi}$ should be compatible with the given germ of an extension φ' of φ.

This space is topologised as a colimit of Whitney topologies, analogously to Definition 3.11.

Definition 3.21 For decorated manifolds L, M, N, denote by $\mathrm{Emb}_{\mathrm{dec}}(M, N)_L$ the subspace of $\mathrm{Emb}_{\mathrm{dec}}(M, N)$ of those embeddings for which we may take $M' = L$ in the third point of Definition 3.20. We note that $\mathrm{Emb}_{\mathrm{dec}}(M, N)$ decomposes as a topological disjoint union:

$$\mathrm{Emb}_{\mathrm{dec}}(M, N) \;\cong\; \bigsqcup_L \mathrm{Emb}_{\mathrm{dec}}(M, N)_L, \tag{3.4}$$

where the disjoint union runs over representatives of isomorphism classes of decorated manifolds.

Let L and M be decorated manifolds. There is a continuous right action of $\mathrm{Diff}_{\mathrm{dec}}(L)$ on $\mathrm{Diff}_{\mathrm{dec}}(L \natural M)$ given by $\varphi \cdot \psi = \varphi \circ (\psi \natural \mathrm{id}_M)$, and hence a quotient map

$$\Psi \colon \mathrm{Diff}_{\mathrm{dec}}(L \natural M) \longrightarrow \mathrm{Diff}_{\mathrm{dec}}(L \natural M) / \mathrm{Diff}_{\mathrm{dec}}(L). \tag{3.5}$$

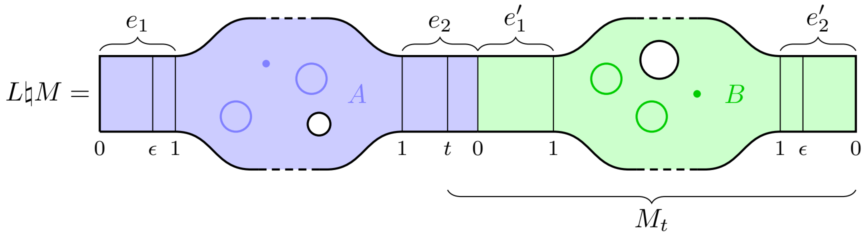


Figure 3.3 The boundary connected sum $L\natural M$ from the proof of Proposition 3.22.

Proposition 3.22 *The quotient map* (3.5) *is a Serre fibration. There is a homeomorphism between its codomain and* $\mathrm{Emb}_{\mathrm{dec}}(M, L\natural M)_L$*, induced by the restriction map.*

Remark 3.23 This is related to results of Cerf [Cer61, Corollaire 2, §II.2.2.2, page 294], Palais [Pal60b, Theorem B] and Lima [Lim63], but we were not able to find an instance of their results that covers exactly the setting that we need here. We therefore give a complete proof of Proposition 3.22 below, using as an input two key results of Cerf and Palais, namely Lemme II.2.1.2 (page 291) of [Cer61] and [Pal60b, Theorem A].

Proof of Proposition 3.22. The decorated manifolds $L = (L, A, e_1, e_2)$ and $M = (M, B, e_1', e_2')$ come equipped with germs e_1, e_2, e_1', e_2' of boundary cylinders; let us once and for all choose representative boundary cylinders for these germs, and denote them by the same symbols, by abuse of notation.

For $\epsilon \in (0,1)$, let $\mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)$ denote the group of self-diffeomorphisms of $L\natural M$ sending $A \sqcup B$ onto itself and restricting to the identity on $e_1(\mathbb{B}^d_\epsilon)$ and on $e_2'(\mathbb{B}^d_\epsilon)$. If we give this the Whitney topology, then

$$\mathrm{Diff}_{\mathrm{dec}}(L\natural M) \cong \operatorname*{colim}_{\epsilon\to 0}(\mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)), \tag{3.6}$$

by our definition of morphisms of decorated manifolds (*cf.* Definition 3.10) and the topology on morphism spaces (*cf.* Definition 3.11). Similarly, for each $\epsilon, t \in (0,1)$, let $\mathrm{Diff}_{\mathrm{dec},\epsilon,t}(L)$ denote the subgroup of $\mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)$ consisting of diffeomorphisms that restrict to the identity on the submanifold $M_t = M \cup e_2(\mathbb{B}^d_t)$ of $L\natural M$ pictured in Figure 3.3. We have a quotient map

$$\Psi_{\epsilon,t} \colon \mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M) \longrightarrow \mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)/\mathrm{Diff}_{\mathrm{dec},\epsilon,t}(L).$$

For any $\epsilon, \epsilon', t, t' \in (0,1)$ with $\epsilon' \leqslant \epsilon$ and $t' \leqslant t$ there are natural maps

$$\mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)/\mathrm{Diff}_{\mathrm{dec},\epsilon,t}(L) \longrightarrow \mathrm{Diff}_{\mathrm{dec},\epsilon'}(L\natural M)/\mathrm{Diff}_{\mathrm{dec},\epsilon',t'}(L),$$

so we may take the directed colimit of the maps $\Psi_{\epsilon,t}$ to obtain

$$\operatorname*{colim}_{\epsilon,t\to 0}(\Psi_{\epsilon,t}) \colon \mathrm{Diff}_{\mathrm{dec}}(L\natural M) \longrightarrow \operatorname*{colim}_{\epsilon,t\to 0}(\mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)/\mathrm{Diff}_{\mathrm{dec},\epsilon,t}(L)),$$

where we have used the identification (3.6) in the domain. Since each $\Psi_{\epsilon,t}$ is a quotient map, it follows from general facts about colimits in the category of topological spaces that $\operatorname*{colim}_{\epsilon,t\to 0}(\Psi_{\epsilon,t})$ is also a quotient map. The map

$$\Psi \colon \mathrm{Diff}_{\mathrm{dec}}(L\natural M) \longrightarrow \mathrm{Diff}_{\mathrm{dec}}(L\natural M)/\mathrm{Diff}_{\mathrm{dec}}(L),$$

i.e., the map (3.5) that we would like to show is a Serre fibration, is also a quotient map, with the same domain. Since M_t is a cofinal family of neighbourhoods of M in $L\natural M$, two diffeomorphisms of $\mathrm{Diff}_{\mathrm{dec}}(L\natural M)$ have the same image under Ψ if and only if they have the same image under $\operatorname*{colim}_{\epsilon,t\to 0}(\Psi_{\epsilon,t})$. As they are quotient maps of the same space, it follows that $\Psi \cong \operatorname*{colim}_{\epsilon,t\to 0}(\Psi_{\epsilon,t})$.

We will prove below that each $\Psi_{\epsilon,t}$ is a fibre bundle (and hence a Serre fibration), and then deduce that Ψ is a Serre fibration using the following general fact.

($*$) Any filtered colimit of based Serre fibrations between compactly-generated weak-Hausdorff spaces is again a Serre fibration.

For a reference for this fact, see Proposition 1.2.3.5(1) of [TV08], which states that a filtered colimit of fibrations is a fibration in any compactly generated model category. The classical model category of based compactly-generated weak-Hausdorff spaces, with its Quillen model structure in which the fibrations are the Serre fibrations, is compactly generated; see for example Proposition 6.3 of [MMSS01].

To apply $(*)$ in our situation, first note that we are taking a directed colimit, which is in particular a filtered colimit. We then need to check that the diffeomorphism groups $\mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)$ and their quotients are compactly-generated weak-Hausdorff spaces. Diffeomorphism groups of manifolds, in the Whitney topology, are always first-countable and Hausdorff, and thus compactly-generated and weak-Hausdorff. Moreover, the property of being compactly-generated is preserved when taking quotients. The property of being weak Hausdorff is *not* preserved when taking quotients; however, in the process of proving that each $\Psi_{\epsilon,t}$ is a fibre bundle below, we will also show that its target space $\mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)/\mathrm{Diff}_{\mathrm{dec},\epsilon,t}(L)$ is Hausdorff.

It therefore remains to show that each $\Psi_{\epsilon,t}$ is a fibre bundle (and its target space is Hausdorff). Write

$$\mathrm{Emb}_{\mathrm{dec},\epsilon}(M_t, L\natural M)_L$$

for the space of smooth, proper embeddings $\varphi\colon M_t \to L\natural M$ such that $\varphi(B) \subseteq A \sqcup B$, the restriction of φ to $e'_2(\mathbb{B}^d_\epsilon)$ is the identity and there exists $\bar{\varphi} \in \mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)$ such that $\varphi = \bar{\varphi} \circ \iota$, where ι is the inclusion of M_t into $L\natural M$. Then

$$\mathrm{Emb}_{\mathrm{dec}}(M, L\natural M)_L \cong \operatorname*{colim}_{\epsilon,t\to 0}(\mathrm{Emb}_{\mathrm{dec},\epsilon}(M_t, L\natural M)_L), \tag{3.7}$$

see Definitions 3.20 and 3.21. There is a restriction map

$$\Phi_{\epsilon,t}\colon \mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M) \longrightarrow \mathrm{Emb}_{\mathrm{dec},\epsilon}(M_t, L\natural M)_L,$$

which is equivariant with respect to the left-action of $\mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)$ by post-composition. This factors through the quotient map $\Psi_{\epsilon,t}$, so we have an induced map

$$\begin{array}{ccc} \mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M) & \xrightarrow{\Phi_{\epsilon,t}} & \mathrm{Emb}_{\mathrm{dec},\epsilon}(M_t, L\natural M)_L. \\ \Psi_{\epsilon,t}\big\downarrow & \nearrow \widehat{\Phi}_{\epsilon,t} & \\ \mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)/\mathrm{Diff}_{\mathrm{dec},\epsilon,t}(L) & & \end{array}$$

By definition of the right-hand embedding space, the map $\Phi_{\epsilon,t}$ is surjective, and so is the induced map $\widehat{\Phi}_{\epsilon,t}$. Moreover, if two diffeomorphisms of $\mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)$ have the same image under $\Phi_{\epsilon,t}$, their difference lies in $\mathrm{Diff}_{\mathrm{dec},\epsilon,t}(L)$, so the induced map $\widehat{\Phi}_{\epsilon,t}$ is also injective. We will prove in the next paragraphs that:

$(**)$ The map $\Phi_{\epsilon,t}$ is a fibre bundle.

In particular, it is a quotient map, since surjective fibre bundles are always quotient maps. Thus the induced map $\widehat{\Phi}_{\epsilon,t}$ must be a homeomorphism. This implies:

- The map $\Psi_{\epsilon,t}$ is also a fibre bundle, hence a Serre fibration.
- Its target space is homeomorphic to the embedding space $\mathrm{Emb}_{\mathrm{dec},\epsilon}(M_t, L\natural M)_L$, which we have given the Whitney topology, so it is Hausdorff.
- We also obtain the second statement of the proposition:

$$\begin{aligned} \mathrm{Diff}_{\mathrm{dec}}(L\natural M)/\mathrm{Diff}_{\mathrm{dec}}(L) &\cong \operatorname*{colim}_{\epsilon,t\to 0}(\mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)/\mathrm{Diff}_{\mathrm{dec},\epsilon,t}(L)) \\ &\cong \operatorname*{colim}_{\epsilon,t\to 0}(\mathrm{Emb}_{\mathrm{dec},\epsilon}(M_t, L\natural M)_L) \\ &\cong \mathrm{Emb}_{\mathrm{dec}}(M, L\natural M)_L. \end{aligned}$$

Here we combine the identification $\Psi \cong \operatorname*{colim}_{\epsilon,t\to 0}(\Psi_{\epsilon,t})$ with the colimit of $\widehat{\Phi}_{\epsilon,t}$ and (3.7).

It therefore remains just to prove statement $(**)$, that $\Phi_{\epsilon,t}$ is a fibre bundle. Since it is equivariant with respect to the left-action of $\mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)$, it suffices to prove that the action of $\mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)$

on $\mathrm{Emb}_{\mathrm{dec},\epsilon}(M_t, L\natural M)_L$ is *locally retractile* ($\equiv$ *admits local cross-sections*). This is because, by [Pal60b, Theorem A], any G-equivariant map into a G-locally retractile space is a fibre bundle.

Thus, we have to prove the following statement: given an embedding $e \in \mathrm{Emb}_{\mathrm{dec},\epsilon}(M_t, L\natural M)_L$, we may find an open neighbourhood $\mathcal{U}$ of e and a continuous map $\gamma\colon \mathcal{U} \to \mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)$ such that $\gamma(e) = \mathrm{id}$ and $\gamma(f) \circ e = f$ for any $f \in \mathcal{U}$. Note that, since $\mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)$ acts transitively on $\mathrm{Emb}_{\mathrm{dec},\epsilon}(M_t, L\natural M)_L$ (because $\Phi_{\epsilon,t}$ is both equivariant and surjective), it suffices to prove this for just one such e, which we take to be the inclusion $M_t \hookrightarrow L\natural M$.

To prove this, we apply a result of Cerf [Cer61, Lemme II.2.1.2, page 291], which we first recall. Let X be a manifold-with-corners. This means in particular that X has a stratification into *faces* (for example, if X is a connected manifold with boundary, but no higher-codimension corners, then its set of faces is $\pi_0(\partial X) \sqcup \{X\}$). Each point $x \in X$ may lie in many faces, but it has a unique *smallest* face (according to inclusion) in which it lies, which we denote by $\mathsf{F}_X(x)$. Now if Y is any submanifold-with-corners of X, we define

$$C^\infty_{\mathrm{face}}(Y, X) = \{\text{smooth maps } \varphi\colon Y \to X \text{ such that } \mathsf{F}_X(\varphi(x)) = \mathsf{F}_X(x) \text{ for each } x \in Y\},$$

equipped with the Whitney topology. The *Extension Lemma* II.2.1.2 of [Cer61] says that, if Y is closed in X and V is any neighbourhood of Y in X, then the restriction map

$$C^\infty_{\mathrm{face}}(X, X) \longrightarrow C^\infty_{\mathrm{face}}(Y, X)$$

admits a section s defined on an open neighbourhood $\mathcal{V}$ of the inclusion in $C^\infty_{\mathrm{face}}(Y, X)$, such that $s(\mathrm{incl}) = \mathrm{id}$ and $s(f)(x) = x$ for all $f \in \mathcal{V}$ and $x \in X \smallsetminus V$.

Step 1. Let us write $\partial_\bullet L$ for the union of all boundary components of L except for the one that intersects the image of e_2; see Figure 3.3. Note that $\partial_\bullet L$ may or may not intersect the image of e_1. Then there is a canonical identification:

$$\pi_0(\partial(L\natural M)) \cong \pi_0(\partial_\bullet L) \sqcup \pi_0(\partial M). \tag{3.8}$$

This is necessarily asymmetric in L and M. Each embedding $f \in \mathrm{Emb}_{\mathrm{dec},\epsilon}(M_t, L\natural M)_L$ extends to a diffeomorphism of $L\natural M$, so it induces an injection $f_\partial\colon \pi_0(\partial M) \to \pi_0(\partial(L\natural M))$. In particular, if f is the inclusion, then f_∂ is also the inclusion, under the identification (3.8). In addition, we know that f sends B into $A \sqcup B$, so it also induces a map $f_\sharp\colon \pi_0(B) \to \pi_0(A) \sqcup \pi_0(B)$, which must be an injection since A and B are closed manifolds and f is an embedding. The function $f \mapsto (f_\partial, f_\sharp)$ is locally constant, so its fibres are open. Let $\mathcal{U}'$ be the open subset of $\mathrm{Emb}_{\mathrm{dec},\epsilon}(M_t, L\natural M)_L$ consisting of all f such that f_∂ is the inclusion and $f_\sharp(\pi_0(B)) = \pi_0(B)$. Note that the second condition implies that $f(B) = B$, since f is an embedding and B is a closed manifold.

Step 2. Write $M_{\epsilon,t} = e_1(\mathbb{B}^d_\epsilon) \sqcup M_t$ (pictured in Figure 3.4). Let

$$\gamma'\colon \mathrm{Emb}_{\mathrm{dec},\epsilon}(M_t, L\natural M)_L \longrightarrow C^\infty(M_{\epsilon,t}, L\natural M)$$

be the continuous map that extends a given embedding $M_t \hookrightarrow L\natural M$ to a smooth map $M_{\epsilon,t} \to L\natural M$ by defining it to be the identity on $e_1(\mathbb{B}^d_\epsilon)$. Note that this may fail to be injective, so it is just a smooth map, not necessarily an embedding. Also observe that, if f lies in the open subset $\mathcal{U}'$ from Step 1, then $\gamma'(f)$ lies in the subspace $C^\infty_{\mathrm{face}}(M_{\epsilon,t}, L\natural M)$, since it takes points of $\mathrm{int}(L\natural M) \cap M_{\epsilon,t}$ into $\mathrm{int}(L\natural M)$ and, for any boundary-component P of $L\natural M$, it takes $P \cap M_{\epsilon,t}$ into P (this uses the fact that $f_\partial = \mathrm{id}$). Restricting γ' to $\mathcal{U}'$, we therefore have a continuous map

$$\gamma'\colon \mathcal{U}' \longrightarrow C^\infty_{\mathrm{face}}(M_{\epsilon,t}, L\natural M)$$

such that $\gamma'(\mathrm{incl}) = \mathrm{incl}$ and $\gamma'(f)|_{M_t} = f$ for all $f \in \mathcal{U}'$.

Step 3. Now set $X = L\natural M$ and $Y = M_{\epsilon,t}$ in the Extension Lemma of Cerf above, and choose V to be any open neighbourhood of $M_{\epsilon,t}$ in $L\natural M$ that is disjoint from the submanifold $A \subset \mathrm{int}(L)$. Composing the local section s obtained from the Extension Lemma with γ', we have a continuous map

$$\gamma'' = s \circ \gamma'\colon \mathcal{U}'' = (\gamma')^{-1}(\mathcal{V}) \longrightarrow C^\infty_{\mathrm{face}}(L\natural M, L\natural M)$$

such that $\gamma''(\mathrm{incl}) = \mathrm{id}$ and for any $f \in \mathcal{U}''$ we have $\gamma''(f)|_{M_t} = f$ and $\gamma''(f)(A) = A$. Moreover, by construction, we also know that $\gamma''(f)(B) = B$ and $\gamma''(f)(x) = x$ for all $x \in e_1(\mathbb{B}^d_\epsilon) \sqcup e'_2(\mathbb{B}^d_\epsilon)$.

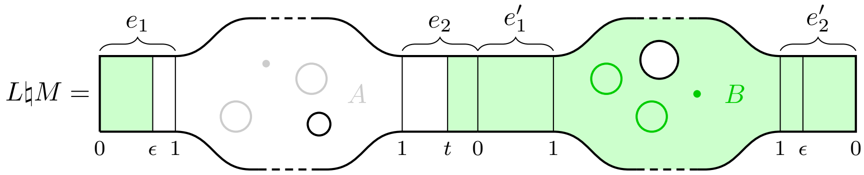


Figure 3.4 The submanifold $M_{\epsilon,t}$ (shaded in green) of $L\natural M$ from the proof of Proposition 3.22.

Step 4. Finally, note that $\mathrm{Diff}(L\natural M)$ is open in $C^\infty(L\natural M, L\natural M)$, so

$$\mathcal{U} = (\gamma'')^{-1}(C^\infty_{\mathrm{face}}(L\natural M, L\natural M) \cap \mathrm{Diff}(L\natural M))$$

is an open neighbourhood of the inclusion in $\mathrm{Emb}_{\mathrm{dec},\epsilon}(M_t, L\natural M)_L$. For each $f \in \mathcal{U}$, the diffeomorphism $\gamma''(f)$ of $L\natural M$ fixes each point of $e_1(\mathbb{B}^d_\epsilon) \sqcup e'_2(\mathbb{B}^d_\epsilon)$ and sends $A \sqcup B$ onto itself, so it is an element of $\mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)$. So we have a continuous map

$$\gamma = \gamma''|_{\mathcal{U}} \colon \mathcal{U} \longrightarrow \mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M)$$

such that $\gamma(\mathrm{incl}) = \mathrm{id}$ and, for all $f \in \mathcal{U}$, we have $\gamma(f) \circ \mathrm{incl} = \gamma(f)|_{M_t} = f$.

Summary. The 4-step construction above may be summarised in the following diagram:

$$\begin{array}{ccccc}
\mathrm{Emb}_{\mathrm{dec},\epsilon}(M_t, L\natural M) & \xrightarrow{\gamma'} & C^\infty(M_{\epsilon,t}, L\natural M) & & \\
\cup\!\!| & & \cup\!\!| & & \\
\mathcal{U}' & \longrightarrow & C^\infty_{\mathrm{face}}(M_{\epsilon,t}, L\natural M) & \longleftarrow & C^\infty_{\mathrm{face}}(L\natural M, L\natural M) \\
\cup\!\!| & & \cup\!\!| & \nearrow_{s} & \uparrow \\
\mathcal{U}'' & \longrightarrow & \mathcal{V} & & \\
\cup\!\!| & & & & \\
\mathcal{U} & \xrightarrow{\gamma} & & & C^\infty_{\mathrm{face}}(L\natural M, L\natural M) \cap \mathrm{Diff}(L\natural M) \subseteq \mathrm{Diff}(L\natural M),
\end{array}$$

where the construction of γ ensures that its image lies in $\mathrm{Diff}_{\mathrm{dec},\epsilon}(L\natural M) \subseteq \mathrm{Diff}(L\natural M)$. □

Remark 3.24 We note that all of the above may be adapted to the setting where the closed submanifolds $A \subset \mathrm{int}(L)$ and $B \subset \mathrm{int}(M)$ are equipped with orientations, and all morphisms of decorated manifolds are required to preserve these orientations. Proposition 3.22 generalises immediately to this setting.

3.4. Semi-monoidal categories and semicategories

All of the examples of categories $\mathcal{C}_\circ$ for which we would like topologically to construct representations will be of the form $\langle \mathcal{G}_\circ, \mathcal{M}_\circ \rangle$, where $\mathcal{G}_\circ$ is a braided monoidal groupoid and $\mathcal{M}_\circ$ is a groupoid with a left-action of $\mathcal{G}_\circ$. We would therefore like to find a topological monoidal groupoid $\mathcal{G}$ and a topological groupoid $\mathcal{M}$ with a left-action of $\mathcal{G}$, satisfying condition (3.1) of Lemma 3.6 and such that $\pi_0(\mathcal{G}) \cong \mathcal{G}_\circ$ and $\pi_0(\mathcal{M}) \cong \mathcal{M}_\circ$. Given this, a continuous functor $\mathcal{C} \coloneqq \langle \mathcal{G}, \mathcal{M} \rangle \to \mathrm{Cov}_\bullet$ will induce a functor

$$\mathcal{C}_\circ = \langle \mathcal{G}_\circ, \mathcal{M}_\circ \rangle \cong \pi_0(\langle \mathcal{G}, \mathcal{M} \rangle) = \pi_0(\mathcal{C}) \longrightarrow \mathrm{Mod}_\bullet,$$

via the construction summarised in the diagram (2.10). We note that there is no need for $\mathcal{G}$ to be braided, since this structure is not needed in order to form the topological Quillen bracket construction, or for Lemma 3.6. In fact, it will be convenient for our examples to drop even more structure from $\mathcal{G}$, and assume only that it is a *semi-monoidal category*. We recall that this is defined analogously to a monoidal category, but without any of the structure or conditions involving left or right units; in other words it involves just a binary operation admitting an associator that satisfies the pentagon condition. This is because we will lift the monoidal structure of $\mathcal{G}_\circ$ to a semi-monoidal structure on $\mathcal{G}$ that is not unital (see Remark 3.16 for why).

Definition 3.25 If $\mathcal{G}$ is a topological semi-monoidal groupoid and $\mathcal{M}$ is a topological category with a continuous left-action of $\mathcal{G}$, then Definition 3.1 generalises directly to this setting, and produces a semicategory $\langle \mathcal{G}, \mathcal{M} \rangle$. The associator of $\mathcal{G}$ is used to define composition in $\langle \mathcal{G}, \mathcal{M} \rangle$ and the pentagon condition for the associator implies associativity of this composition.

Lemma 3.26 *Let $\mathcal{G}$ be a topological semi-monoidal groupoid and $\mathcal{M}$ be a topological category with a continuous left-action of $\mathcal{G}$, satisfying the condition of Lemma 3.6. Then there is a canonical isomorphism of semicategories*

$$\pi_0(\langle \mathcal{G}, \mathcal{M} \rangle) \cong \langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M}) \rangle.$$

Using this definition and lemma it will therefore suffice, in our examples, to find a topological *semi*-monoidal groupoid $\mathcal{G}$, such that $\pi_0(\mathcal{G}) \cong \mathcal{G}_\circ$ as semi-monoidal groupoids and which satisfies condition (3.1) of Lemma 3.6. Then $\langle \mathcal{G}, \mathcal{M} \rangle$ is a topological *semicategory*, and we will construct, geometrically in §5.1, continuous semifunctors $\langle \mathcal{G}, \mathcal{M} \rangle \to \mathrm{Cov}_\bullet$. Via Lemma 3.26 and the construction of §2.5 summarised in diagram (2.10), we then obtain a *semifunctor*

$$\langle \mathcal{G}_\circ, \mathcal{M}_\circ \rangle \longrightarrow \mathrm{Mod}_\bullet.$$

The source and target of this semifunctor are both categories (since $\mathcal{G}_\circ$ is a monoidal groupoid, not just a semi-monoidal groupoid), and so one may ask whether this semifunctor is in fact a functor, and in all of our examples it will be trivial to verify that it does in fact preserve identities, and is therefore a functor.

Caveat 3.27 In practice, in the remainder of this paper, we will omit mention of this subtlety about a lack of units and identities at the topological level, to avoid unnecessary extra complications. Formally, however, one should modify the construction as described above.

4. Topological categories for families of groups

In this section, we discuss in more detail the topological groupoids $\mathcal{D}_d$ of "decorated manifolds" introduced in the previous section. In §4.1, we discuss their properties with respect to braidings and symmetries. In §4.2, we give an explicit description of the morphism spaces of "Quillen bracket categories of manifolds" in terms of embedding spaces. In §4.3, we establish several split homotopy fibration sequences, and hence split short exact sequences on π_1 and π_0 of embedding spaces and diffeomorphism groups, and use these to identify motion groups with "braided mapping class groups". In §§4.4–4.6, we then describe, and recall the key properties of, the relevant subgroupoids of $\mathcal{D}_2$ and $\mathcal{D}_3$ that we will be especially interested in.

4.1. Braidings and symmetries

If we restrict attention to those decorated manifolds whose two boundary-cylinder germs lie on the same boundary-component, which is a sphere, then there is a natural braiding at the level of π_0, which is symmetric if $d \geqslant 3$.

Definition 4.1 We say that a decorated manifold (M, A, e_1, e_2) has *spherical preferred boundary* if the embedded $(d-1)$-discs $e_1(b\mathbb{B}_1^d)$ and $e_2(b\mathbb{B}_1^d)$ lie on the same boundary-component $\partial_0 M$ of M, and moreover $\partial_0 M \cong S^{d-1}$. Write $\mathcal{D}_d^{\mathrm{sph}}$ for the full subgroupoid of $\mathcal{D}_d$ on decorated manifolds with spherical preferred boundary. Define $\mathcal{D}_d^{+,\mathrm{sph}} \subseteq \mathcal{D}_d^+$ similarly.

Definition 4.2 An inclusion of topologically-enriched categories $\mathcal{C} \subseteq \mathcal{D}$ is called *0-full* if, for each pair of objects c, c' of $\mathcal{C}$, the subspace $\mathcal{C}(c, c') \subseteq \mathcal{D}(c, c')$ is a union of path-components. We note that a 0-full inclusion $\mathcal{C} \subseteq \mathcal{D}$ induces an inclusion $\pi_0(\mathcal{C}) \subseteq \pi_0(\mathcal{D})$. In fact, subcategories of $\pi_0(\mathcal{D})$ correspond bijectively to 0-full subcategories of $\mathcal{D}$.

Lemma 4.3 *The subgroupoid $\pi_0(\mathcal{D}_d^{\mathrm{sph}})$ of $\pi_0(\mathcal{D}_d)$ is closed under $\natural$, so it is a monoidal groupoid. Moreover, $\pi_0(\mathcal{D}_d^{\mathrm{sph}})$ is braided if $d = 2$ and symmetric if $d \geqslant 3$.*

It follows that, if $\mathcal{G} \subseteq \mathcal{D}_d^{\mathrm{sph}}$ is a 0-full subgroupoid that is closed under $\natural$ and contains the braiding morphisms – described in the proof below – then $\pi_0(\mathcal{G})$ is braided monoidal, and symmetric monoidal if $d \geqslant 3$. The same statements hold when $\mathcal{D}_d^{\mathrm{sph}}$ is replaced by $\mathcal{D}_d^{\mathrm{sph},+}$.

Proof. The first statement is clear, since the connected sum of two spheres is again a sphere. The construction of the braiding morphisms is exactly analogous to Figure 2 on page 609 of [RW17], with the minor difference that they use two intervals, embedded in a circle boundary-component, intersecting at a point, whereas we use two disjoint discs in a spherical boundary-component. If the dimension d is at least 3, one may use the extra dimension to isotope the square of this braiding morphism to the identity, so it is a symmetry for $\pi_0(\mathcal{D}_d^{\mathrm{sph}})$. □

4.2. Quillen bracket categories of manifolds

Fix $d \geqslant 2$. Let $\mathcal{G} \subseteq \mathcal{D}_d$ be a full subgroupoid that is closed under $\natural$ and let $\mathcal{M} \subseteq \mathcal{D}_d$ be a 0-full subgroupoid that is closed under the action of $\mathcal{G}$ through $\natural$. Alternatively, we allow $\mathcal{D}_d^+$ in place of $\mathcal{D}_d$. This implies that $\mathcal{G}$ is a semi-monoidal groupoid with an action on $\mathcal{M}$, and we may form the Quillen bracket construction $\langle \mathcal{G}, \mathcal{M} \rangle$, which is a topological semicategory; see §3.1 and §3.4. It is called a *Quillen bracket category of manifolds.*

Lemma 4.4 *The Serre fibration condition* (3.1) *of Lemma 3.6 is satisfied for this $\mathcal{G}$ and $\mathcal{M}$, and hence we have*

$$\pi_0(\langle \mathcal{G}, \mathcal{M} \rangle) \cong \langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M}) \rangle.$$

Proof. If $\mathcal{M}$ is a full subgroupoid of $\mathcal{D}_d$, this follows directly from Proposition 3.22 (and Remark 3.24 for $\mathcal{D}_d^+$) together with Lemma 3.6. If $\mathcal{M}$ only satisfies the weaker property of being a 0-full subgroupoid of $\mathcal{D}_d$, it follows from these results together with Lemma 4.5 below. □

Lemma 4.5 *Let X be a space with a continuous right-action of a topological group G such that the projection $X \to X/G$ is a Serre fibration. Let $X_0 \subseteq X$ be a union of path-components such that the G-action sends X_0 into itself. Then the projection $X_0 \to X_0/G$ is also a Serre fibration.*

Proof. More generally, by considering lifting diagrams, one may prove that, in the following square:

$$\begin{array}{ccc} A & \xrightarrow{\;a\;} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[\;b\;]{} & D \end{array}$$

if f is a Serre fibration, a is an inclusion of a union of path-components, and b is injective, then g is also a Serre fibration. In our setting, a is the inclusion $X_0 \hookrightarrow X$, which is assumed to be a union of path-components, and b is the induced map $X_0/G \to X/G$, which is injective. □

Definition 4.6 (*Decorated diffeomorphisms, revisited.*) For a decorated manifold $M = (M, A)$, recall from Definition 3.19 that the topological group $\mathrm{Diff}_{\mathrm{dec}}(M)$ of decorated diffeomorphisms of M is simply the automorphism group of M in the topologically-enriched groupoid $\mathcal{D}_d$. If M lies in a subgroupoid $\mathcal{H} \subseteq \mathcal{D}_d$, write $\mathrm{Diff}_{\mathcal{H}}(M) \subseteq \mathrm{Diff}_{\mathrm{dec}}(M)$ for the subgroup of automorphisms of M in $\mathcal{H}$. We note that this is an equality if $\mathcal{H} \subseteq \mathcal{D}_d$ is full.

Definition 4.7 (*Decorated embeddings, revisited.*) For decorated manifolds $M = (M, A)$ and $N = (N, B)$, recall from Definition 3.20 the space $\mathrm{Emb}_{\mathrm{dec}}(M, N)$ of decorated embeddings. If M and N lie in $\mathcal{M} \subseteq \mathcal{D}_d$, define $\mathrm{Emb}_{\langle \mathcal{G}, \mathcal{M} \rangle}(M, N) \subseteq \mathrm{Emb}_{\mathrm{dec}}(M, N)$ to be the subspace where, in the third condition of Definition 3.20, the decorated manifold M' lies in $\mathcal{G}$ and the decorated diffeomorphism $\bar{\varphi}$ lies in $\mathcal{M}$. We note that this second condition is automatic if $\mathcal{M} \subseteq \mathcal{D}_d$ is full. For an object L of $\mathcal{G}$, we also write $\mathrm{Emb}_{\langle \mathcal{G}, \mathcal{M} \rangle}(M, N)_L$ for the subspace of $\mathrm{Emb}_{\langle \mathcal{G}, \mathcal{M} \rangle}(M, N)$ where we may take $M' = L$ in Definition 3.20 (and $\bar{\varphi}$ lies in $\mathcal{M}$). Similarly to (3.4), the space $\mathrm{Emb}_{\langle \mathcal{G}, \mathcal{M} \rangle}(M, N)$ decomposes as a topological disjoint union:

$$\mathrm{Emb}_{\langle \mathcal{G}, \mathcal{M} \rangle}(M, N) \;\cong\; \bigsqcup_L \mathrm{Emb}_{\langle \mathcal{G}, \mathcal{M} \rangle}(M, N)_L, \tag{4.1}$$

where the disjoint union runs over representatives of isomorphism classes of objects L of $\mathcal{G}$.

Proposition 4.8 *For any Quillen bracket category of manifolds $\langle \mathcal{G}, \mathcal{M} \rangle$, its morphism spaces may be identified as follows:*

$$\langle \mathcal{G}, \mathcal{M} \rangle(M, N) \;\cong\; \mathrm{Emb}_{\langle \mathcal{G}, \mathcal{M} \rangle}(M, N).$$

Remark 4.9 When $\mathcal{M} \subseteq \mathcal{D}_d$ is full, this says that morphisms in $\langle \mathcal{G}, \mathcal{M} \rangle$ are embeddings of decorated manifolds such that the complement of the image of the embedding is an object of $\mathcal{G}$. In particular the space of morphisms is non-empty if and only if there exists an object M' of $\mathcal{G}$ such that $M' \natural M$ is diffeomorphic to N as decorated manifolds.

Proof of Proposition 4.8. The space $\langle \mathcal{G}, \mathcal{M} \rangle(M, N)$ is described in Definition 3.1. In the notation of the proof of Lemma 3.6 (setting $X = M$ and $Y = N$), this is the quotient space $\Phi/{\sim_t}$. In that proof, it is shown that this splits as the topological disjoint union of certain spaces denoted $q(\mathrm{Hom}_{\mathcal{M}}(Z \natural X, Y))$, as Z runs over representatives of isomorphism classes of objects of $\mathcal{G}$. It is also proved that this space is homeomorphic to the quotient space $\mathrm{Hom}_{\mathcal{M}}(Z \natural X, Y)/\mathrm{Aut}_{\mathcal{G}}(Z)$. We therefore have the following homeomorphism, where the disjoint union runs over representatives Z of isomorphism classes of objects of $\mathcal{G}$:

$$\langle \mathcal{G}, \mathcal{M} \rangle(M, N) \;\cong\; \bigsqcup_{Z} \mathrm{Hom}_{\mathcal{M}}(Z \natural X, Y)/\mathrm{Aut}_{\mathcal{G}}(Z).$$

Since $\mathcal{M}$ is a groupoid, the space $\mathrm{Hom}_{\mathcal{M}}(Z \natural X, Y)/\mathrm{Aut}_{\mathcal{G}}(Z)$ is empty unless $Z \natural X$ is isomorphic to Y in $\mathcal{M}$, in which case we may rewrite it as $\mathrm{Aut}_{\mathcal{M}}(Z \natural X)/\mathrm{Aut}_{\mathcal{G}}(Z) = \mathrm{Diff}_{\mathcal{M}}(L \natural M)/\mathrm{Diff}_{\mathrm{dec}}(L)$, using the notation of Definition 4.7 and setting $L = Z$ (recall that $\mathcal{G} \subseteq \mathcal{D}_d$ is full). The second part of Proposition 3.22 says that the restriction map induces a homeomorphism

$$\mathrm{Diff}_{\mathrm{dec}}(L \natural M)/\mathrm{Diff}_{\mathrm{dec}}(L) \;\cong\; \mathrm{Emb}_{\mathrm{dec}}(M, L \natural M)_L,$$

and one may easily see that this sends the subspace $\mathrm{Diff}_{\mathcal{M}}(L \natural M)/\mathrm{Diff}_{\mathrm{dec}}(L)$ homeomorphically onto the subspace $\mathrm{Emb}_{\langle \mathcal{G}, \mathcal{M} \rangle}(M, L \natural M)_L$. Putting this all together, we have a homeomorphism

$$\langle \mathcal{G}, \mathcal{M} \rangle(M, N) \;\cong\; \bigsqcup_{L} \mathrm{Emb}_{\langle \mathcal{G}, \mathcal{M} \rangle}(M, L \natural M)_L, \tag{4.2}$$

where the disjoint union runs over representatives L of isomorphism classes of objects of $\mathcal{G}$ such that $L \natural M$ is isomorphic to N in $\mathcal{M}$.

Finally, consider the topological decomposition (4.1), where the disjoint union is indexed by representatives L of isomorphism classes of all objects of $\mathcal{G}$. If $L \natural M \ncong N$ in $\mathcal{M}$, the corresponding term is empty, whereas if $L \natural M \cong N$ in $\mathcal{M}$, we may rewrite the corresponding term by replacing N with $L \natural M$, to obtain:

$$\mathrm{Emb}_{\langle \mathcal{G}, \mathcal{M} \rangle}(M, N) \;\cong\; \bigsqcup_{L} \mathrm{Emb}_{\langle \mathcal{G}, \mathcal{M} \rangle}(M, L \natural M)_L, \tag{4.3}$$

where the disjoint union now runs over representatives L of isomorphism classes of objects of $\mathcal{G}$ such that $L \natural M$ is isomorphic to N in $\mathcal{M}$. Combining (4.2) and (4.3), we obtain the desired result. □

Remark 4.10 Suppose that *cancellation* holds for our chosen subgroupoids $\mathcal{G}$ and $\mathcal{M}$ — meaning that $L \natural M \cong L' \natural M$ in $\mathcal{M}$ implies $L \cong L'$ in $\mathcal{G}$ for objects L, L' of $\mathcal{G}$ and M of $\mathcal{M}$. Then the disjoint union (4.2) is taken either over the empty set or a set of size one, so we have:

$$\langle \mathcal{G}, \mathcal{M} \rangle(M, N) \;\cong\; \begin{cases} \mathrm{Emb}_{\langle \mathcal{G}, \mathcal{M} \rangle}(M, L \natural M)_L & \text{if } \exists L \in \mathcal{G} \text{ such that } L \natural M \cong N \text{ in } \mathcal{M}; \\ \emptyset & \text{otherwise.} \end{cases}$$

All of our examples of $\mathcal{G}$ and $\mathcal{M}$ in §4.4–§4.6 satisfy cancellation. This follows from the classification of compact surfaces in all of our examples in dimension 2. In the examples of §4.6, it follows from the fact that the groupoid of finite sets and bijections under disjoint union satisfies cancellation, since the objects of the groupoids in the examples of §4.6 are determined up to isomorphism by the number of copies of $\mathbb{S}^1$ in the submanifold $A \subset M$.

4.3. Split short exact sequences

We now establish several split homotopy fibration sequences whose associated split short exact sequences will be used in our construction. In particular, the split short exact sequences (4.6) and (4.9) below will be used in the two versions of our general construction of global homological representations in §5.1 and §5.3 respectively. In addition, the split short exact sequence (4.9) implies that any motion group is a *braided mapping class group* (*cf.* Proposition 4.23) in particular, a normal subgroup of a mapping class group.

Fix a closed submanifold $Z \subset \mathbb{R}^d$ and an open subgroup $G \leqslant \mathrm{Diff}(Z)$. We note that, since $\mathrm{Diff}(Z)$ is locally path-connected, this corresponds to a choice of subgroup of $\pi_0(\mathrm{Diff}(Z))$.

4.3.1. The first short exact sequence

Definition 4.11 For smooth manifolds X and Y, let us write $\mathcal{E}(X,Y) = \mathrm{Emb}(X,Y)/\mathrm{Diff}(X)$. For a subgroup $G \leqslant \mathrm{Diff}(X)$, we also write $\mathcal{E}_G(X,Y) = \mathrm{Emb}(X,Y)/G$.

Lemma 4.12 *For any decorated manifold $(M,A) \in \mathcal{D}_d$ there is a homotopy fibration sequence*

$$\mathcal{E}_G(Z, \mathring{M} \smallsetminus A) \longrightarrow \mathcal{E}_{\mathrm{Diff}(A)\times G}(A \sqcup Z, \mathring{M}) \overset{\dashleftarrow}{\longrightarrow} \mathcal{E}(A, \mathring{M}), \tag{4.4}$$

in which the second map admits a section up to homotopy, as pictured.

Proof. The map $\mathcal{E}_{\mathrm{Diff}(A)\times G}(A \sqcup Z, \mathring{M}) \to \mathcal{E}(A, \mathring{M})$ that forgets the embedding (modulo G) of Z is equivariant with respect to the left action of the topological group $\mathrm{Diff}_c(\mathring{M})$ of compactly-supported diffeomorphisms of $\mathring{M}$. By [Pal21, Proposition 4.15], the action of $\mathrm{Diff}_c(\mathring{M})$ on $\mathcal{E}(A, \mathring{M})$ is locally retractile, i.e., it admits local sections. Thus, by [Pal60b, Theorem A], the map

$$\mathcal{E}_{\mathrm{Diff}(A)\times G}(A \sqcup Z, \mathring{M}) \longrightarrow \mathcal{E}(A, \mathring{M}) \tag{4.5}$$

is a fibre bundle, in particular a Serre fibration. Write incl. for the inclusion of A into $\mathring{M}$ and [incl.] for its $\mathrm{Diff}(A)$-orbit; this is a natural basepoint for $\mathcal{E}(A, \mathring{M})$. The point-set fibre of (4.5) over $[\mathrm{incl.}] \in \mathcal{E}(A, \mathring{M})$ is clearly equal to $\mathcal{E}_G(Z, \mathring{M} \smallsetminus A)$, so (4.4) is a fibration sequence.

We construct a section-up-to-homotopy as follows. Let $[\varphi] \in \mathcal{E}(A, \mathring{M})$. This determines an embedding $A \sqcup Z \hookrightarrow \mathring{M} \sqcup \mathbb{R}^d$ modulo the action of $\mathrm{Diff}(A)$, since Z is given as a submanifold of $\mathbb{R}^d$. We recall from Definition 3.9 that a decorated manifold comes equipped with two *germs* of boundary-cylinders; we choose one of these and then choose a boundary-cylinder representing the given germ. Using this boundary-cylinders of M, we construct an embedding $\mathring{M} \sqcup \mathbb{R}^d \hookrightarrow \mathring{M}$ whose restriction to $\mathring{M}$ is isotopic to the identity. Composing this with the embedding above gives us an embedding $\varphi' \colon A \sqcup Z \hookrightarrow \mathring{M}$ modulo the action of $\mathrm{Diff}(A)$. The desired splitting is then given by $[\varphi] \mapsto [\varphi']$. This is a section up to homotopy because, forgetting Z, φ' is isotopic to φ. □

Corollary 4.13 *For any decorated manifold $(M,A) \in \mathcal{D}_d$ there is a split short exact sequence*

$$1 \longrightarrow \pi_1(\mathcal{E}_G(Z, \mathring{M} \smallsetminus A)) \longrightarrow \pi_1(\mathcal{E}_{\mathrm{Diff}(A)\times G}(A \sqcup Z, \mathring{M})) \overset{\dashleftarrow}{\longrightarrow} \pi_1(\mathcal{E}(A, \mathring{M})) \longrightarrow 1, \tag{4.6}$$

where the basepoint of each space of embeddings modulo diffeomorphisms is given by the inclusion.

We may, alternatively, consider spaces of embeddings whose image is allowed to lie either in the interior of M or in its boundary (but it must lie wholly in one or the other).

Definition 4.14 For $(M,A) \in \mathcal{D}_d$ and $Z \subset \mathbb{R}^d$ and $G \leqslant \mathrm{Diff}(Z)$ as above, we write $\mathrm{Emb}'(Z, M \smallsetminus A)$ for the subspace of smooth maps $Z \to M \smallsetminus A$ that are either smooth embeddings of Z into $\mathring{M} \smallsetminus A$ or smooth embeddings of Z into ∂M. Write $\mathcal{E}'_G(Z, M \smallsetminus A) = \mathrm{Emb}'(Z, M \smallsetminus A)/G$. Similarly, write $\mathrm{Emb}'(A \sqcup Z, M)$ for the subspace of smooth maps $A \sqcup Z \to M$ that are either embeddings of $A \sqcup Z$ into $\mathring{M}$ or disjoint unions of an embedding of A into $\mathring{M}$ together with an embedding of Z into ∂M. Write $\mathcal{E}'_{\mathrm{Diff}(A)\times G}(A \sqcup Z, M) = \mathrm{Emb}'(A \sqcup Z, M)/(\mathrm{Diff}(A) \times G)$.

Lemma 4.15 *The natural inclusions*

$$\mathcal{E}_G(Z, \mathring{M} \smallsetminus A) \lhook\joinrel\longrightarrow \mathcal{E}'_G(Z, M \smallsetminus A) \quad \text{and} \quad \mathcal{E}_{\mathrm{Diff}(A)\times G}(A \sqcup Z, \mathring{M}) \lhook\joinrel\longrightarrow \mathcal{E}'_{\mathrm{Diff}(A)\times G}(A \sqcup Z, M)$$

are homotopy equivalences. Hence we may rewrite the split short exact sequence (4.6) above as

$$1 \longrightarrow \pi_1(\mathcal{E}'_G(Z, M \smallsetminus A)) \longrightarrow \pi_1(\mathcal{E}'_{\mathrm{Diff}(A)\times G}(A \sqcup Z, M)) \overset{\dashleftarrow}{\longrightarrow} \pi_1(\mathcal{E}(A, \mathring{M})) \longrightarrow 1. \tag{4.7}$$

Proof. In each case, a deformation retraction may be defined by post-composing embeddings with an isotopy of self-embeddings of M starting from the identity and "shrinking" a collar neighbourhood of its boundary. □

4.3.2. The second short exact sequence

We now construct the second key split short exact sequence (sequence (4.9) below). We first need some notation.

Definition 4.16 For a decorated manifold $(M, A) = (M, A, e_1, e_2) \in \mathcal{D}_d$, recall that we write $\mathrm{Diff}_{\mathrm{dec}}(M, A)$ for its automorphism group in $\mathcal{D}_d$, in other words, the self-diffeomorphisms of M that send A onto itself and that are compatible with the boundary-cylinder germs e_1 and e_2. If we have a decomposition $A = A_1 \sqcup A_2$, we now also write $\mathrm{Diff}_{\mathrm{dec}}(M, A_1, A_2)$ for the subgroup of $\mathrm{Diff}_{\mathrm{dec}}(M, A)$ of those diffeomorphisms that preserve this decomposition, in other words, that send A_1 onto itself and A_2 onto itself. We note that this is a finite covering of $\mathrm{Diff}_{\mathrm{dec}}(M, A)$.

Notation 4.17 We identify $\mathbb{R}^d$ with the interior of the solid cylinder $\mathbb{B}^d_1$ in a standard way, so the choice of closed submanifold $Z \subset \mathbb{R}^d$ determines a decorated manifold $(\mathbb{B}^d_1, Z) = (\mathbb{B}^d_1, Z, \mathrm{id}, r)$, where r is the reflection of the solid cylinder $\mathbb{B}^d_1 = \mathbb{D}^{d-1} \times [0, 1]$ in its second coordinate. For any other decorated manifold $(M, A) \in \mathcal{D}_d$, we may therefore consider the boundary connected sum $(M, A)\natural(\mathbb{B}^d_1, Z)$, which we denote by an abuse of notation by $(M, A \sqcup Z)$.

Lemma 4.18 *For any decorated manifold $(M, A) \in \mathcal{D}_d$ there is a homotopy fibration sequence*

$$\mathrm{Diff}_{\mathrm{dec}}(M, A, Z) \xrightarrow{} \mathrm{Diff}_{\mathrm{dec}}(M, A) \longrightarrow \mathcal{E}(Z, \mathring{M} \smallsetminus A), \tag{4.8}$$

in which the first map admits a section up to homotopy, as pictured.

Proof. The right-hand map above is equivariant with respect to the left action of $\mathrm{Diff}_c(\mathring{M} \smallsetminus A)$. By [Pal21, Proposition 4.15], its action on $\mathcal{E}(Z, \mathring{M} \smallsetminus A)$ is locally retractile, so by [Pal60b, Theorem A], the right-hand map above is a fibre bundle. Its point-set fibre over the basepoint [incl.] is clearly equal to $\mathrm{Diff}_{\mathrm{dec}}(M, A, Z)$. The section up to homotopy is constructed similarly to the proof of Lemma 4.12, using a self-embedding $M \hookrightarrow M$ that restricts to the identity on $A \subset M$, is isotopic to the identity and whose image is disjoint from Z (when considered as a submanifold of M via a chosen boundary-cylinder). □

Corollary 4.19 *For any decorated manifold $(M, A) \in \mathcal{D}_d$ there is a split short exact sequence*

$$1 \longrightarrow \pi_1(\mathcal{E}(Z, \mathring{M} \smallsetminus A)) \longrightarrow \pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, A, Z)) \longrightarrow \pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, A)) \longrightarrow 1, \tag{4.9}$$

where the basepoint of the space $\mathcal{E}(Z, \mathring{M} \smallsetminus A)$ is given by the inclusion.

The split homotopy fibration sequence (4.8) may be generalised as follows.

Lemma 4.20 *For any decorated manifold $(M, A) \in \mathcal{D}_d$ and any open subgroup $H \leqslant \mathrm{Diff}_{\mathrm{dec}}(M, A)$, the split homotopy fibration sequence (4.8) restricts to a split homotopy fibration sequence*

$$\mathrm{Diff}_{\mathrm{dec}}(M, A, Z) \cap H \longrightarrow H \longrightarrow \mathcal{E}(Z, \mathring{M} \smallsetminus A), \tag{4.10}$$

and hence a corresponding split short exact sequence.

Proof. Since H is an open subgroup of a topological group, it is also closed and thus a union of path-components. The restriction map $\mathrm{Diff}_{\mathrm{dec}}(M, A) \to \mathcal{E}(Z, \mathring{M} \smallsetminus A)$ is a fibre bundle, hence a Serre fibration, by the proof of Lemma 4.18. In general, if $E \to B$ is a Serre fibration and $E_0 \subseteq E$ is a union of path-components, then the restriction $E_0 \to B$ is also a Serre fibration. This establishes the homotopy fibration sequence. To see that the section up to homotopy of (4.8) restricts to give a section up to homotopy of (4.10), one again just has to use the fact that H is a union of path-components of $\mathrm{Diff}_{\mathrm{dec}}(M, A)$. □

4.3.3. Motion groups are braided mapping class groups

Definition 4.21 The *braided diffeomorphism group* $\mathrm{Diff}^{\mathrm{br}}_{\mathrm{dec}}(M, A)$ of a decorated manifold (M, A) is the kernel of the natural homomorphism

$$\mathrm{Diff}_{\mathrm{dec}}(M, A) \longrightarrow \pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, \emptyset)),$$

in other words, the subgroup of diffeomorphisms of (M, A) that become isotopic to the identity after forgetting A.

Definition 4.22 Given a closed manifold Z and an embedding $Z \hookrightarrow \mathring{M}$, the corresponding *motion group* $\mathrm{Mot}_Z(M)$ is the fundamental group $\pi_1(\mathcal{E}(Z, \mathring{M}))$.

Corollary 4.19 implies the following identification of motion groups with π_0 of braided diffeomorphism groups, in other words, *motion groups are braided mapping class groups*.

Proposition 4.23 *For any closed submanifold $Z \subset \mathbb{R}^d$, we have a canonical isomorphism*

$$\mathrm{Mot}_Z(M) = \pi_1(\mathcal{E}(Z, \mathring{M})) \cong \pi_0(\mathrm{Diff}^{\mathrm{br}}_{\mathrm{dec}}(M, Z)).$$

Proof. This follows immediately from Corollary 4.19 in the special case where $A = \emptyset$. □

Remark 4.24 (*Globality*) In particular, Proposition 4.23 tells us that all motion groups are normal subgroups of the appropriate $\pi_0(\mathrm{Diff}(-))$'s of decorated manifolds. Also, more obviously, mapping class groups of d-manifolds are clearly examples of $\pi_0(\mathrm{Diff}(-))$'s of decorated manifolds, where we take A to be the empty submanifold. Thus *the discrete groupoid $\pi_0(\mathcal{D}_d)$ contains all motion groups and all mapping class groups of d-dimensional manifolds.*

The homological representations that we shall construct in §5.1 and §5.3 below are functors of the form $\mathfrak{U}\mathcal{D}_d \to \mathrm{Mod}_\bullet$. Since the target category is discrete, they factor through $\pi_0(\mathfrak{U}\mathcal{D}_d) \cong \mathfrak{U}(\pi_0(\mathcal{D}_d))$, whose underlying groupoid is $\pi_0(\mathcal{D}_d)$. Each such homological representation therefore restricts to homological representations of all motion groups and mapping class groups in dimension d. This is why we refer to these as *global* homological representations.

4.3.4. Actions of (braided) mapping class groups

The following lemma will be important for the proof of the "*Q-stability lemma*" in §5.1.

Lemma 4.25 *There is a natural action of* $\mathrm{Diff}_{\mathrm{dec}}(M, A)$ *on* $\mathcal{E}(Z, \mathring{M} \smallsetminus A)$ *given by post-composition, which induces an action of* $\pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, A))$ *on* $\pi_1(\mathcal{E}(Z, \mathring{M} \smallsetminus A))$. *This action agrees with the one coming from the splitting in the short exact sequence* (4.9).

Similarly, there is a natural action of $\mathrm{Diff}_{\mathrm{dec}}(M, A)$ *on* $\mathcal{E}_G(Z, \mathring{M} \smallsetminus A)$ *given by post-composition, which induces an action of*

$$\pi_1(\mathcal{E}(A, \mathring{M})) = \pi_0(\mathrm{Diff}^{\mathrm{br}}_{\mathrm{dec}}(M, A)) \leqslant \pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, A)) \tag{4.11}$$

on $\pi_1(\mathcal{E}_G(Z, \mathring{M} \smallsetminus A))$. *This action agrees with the one coming from the splitting in the short exact sequence* (4.6).

Proof. We first of all note that there is a slight generalisation of the split short exact sequence (4.9), as follows. Lemma 4.18 generalises (with the same proof) to the setting where, for an open subgroup $G \leqslant \mathrm{Diff}(Z)$, the base space of the fibration is replaced with $\mathcal{E}_G(Z, \mathring{M} \smallsetminus A)$ and the fibre is replaced with $\mathrm{Diff}_{\mathrm{dec}}(M, A, Z|G)$, the subgroup of $\varphi \in \mathrm{Diff}_{\mathrm{dec}}(M, A, Z)$ such that $\varphi|_Z \in G$. The corresponding split short exact sequence is the middle row of the following diagram.

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(\mathcal{E}_G(Z, \mathring{M} \smallsetminus A)) & \longrightarrow & \pi_1(\mathcal{E}_{\mathrm{Diff}(A)\times G}(A \sqcup Z, \mathring{M})) & \longrightarrow & \pi_1(\mathcal{E}(A, \mathring{M})) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(\mathcal{E}_G(Z, \mathring{M} \smallsetminus A)) & \longrightarrow & \pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, A, Z|G)) & \longrightarrow & \pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, A)) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & 1 & \longrightarrow & \pi_0(\mathrm{Diff}_{\mathrm{dec}}(M)) & \xrightarrow{\mathrm{id}} & \pi_0(\mathrm{Diff}_{\mathrm{dec}}(M)) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
\tag{4.12}
$$

Moreover, in this diagram:

- the top row is the split short exact sequence (4.6),
- the right-hand column is an instance of (4.9) (with $A = \emptyset$ and replacing $Z \mapsto A$),
- the middle column is an instance of the middle row (with $A = \emptyset$ and replacing $Z \mapsto A \sqcup Z$ and $G \mapsto \mathrm{Diff}(A) \times G$)

In particular, the top-right vertical map is the inclusion (4.11). The top-right square of (4.12) also commutes when the horizontal arrows are replaced by the given splittings. Thus both parts of the lemma (for (4.6) and for (4.9)) will follow once we prove the analogue for the middle row of (4.12).

To see this, let us first describe geometrically the action of an element $\varphi \in \pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, A))$ on $\pi_1(\mathcal{E}_G(Z, \mathring{M} \smallsetminus A))$ that arises from the split short exact sequence forming the middle row of (4.12). By definition, φ is an isotopy class of diffeomorphisms of M fixing (pointwise) a neighbourhood of its two boundary-cylinders and fixing A setwise. First choose an identification of M with $M \natural \mathbb{B}_1^d$ so that $A \subseteq M$ and $Z \subseteq \mathbb{B}_1^d$. The section sends φ to the diffeomorphism φ' of $M \natural \mathbb{B}_1^d$ that acts by φ on M and by the identity on $\mathbb{B}_1^d$. The diffeomorphism φ' then acts on $\pi_1(\mathcal{E}_G(Z, \mathring{M} \smallsetminus A))$ by conjugation, where the latter group is viewed as a subgroup of $\pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, A, Z|G))$ via the connecting homomorphism of the long exact sequence, which may be viewed geometrically as sending a loop of embeddings of Z to the corresponding *submanifold-pushing* diffeomorphism of M (named by analogy with the well-known *point-pushing* diffeomorphism in the case when Z is zero-dimensional). But using the submanifold-pushing construction to view $\pi_1(\mathcal{E}_G(Z, \mathring{M} \smallsetminus A))$ as a subgroup of $\pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, A, Z|G))$ and then acting by conjugation is equivalent to acting directly on $\pi_1(\mathcal{E}_G(Z, \mathring{M} \smallsetminus A)$ through the obvious action of $\mathrm{Diff}_{\mathrm{dec}}(M, A, Z|G)$ on $\mathcal{E}_G(Z, \mathring{M} \smallsetminus A)$. $\square$

4.4. Mapping class groups of surfaces

We define $\mathcal{M}_2^{\mathrm{t}}$ to be the full subgroupoid of $\mathcal{D}_2^{\mathrm{sph}}$ on decorated surfaces $(S, \emptyset, e_1, e_2)$ where S is a compact, connected, smooth surface with boundary. This collection of objects is clearly closed under the operation $\natural$, so it is a semi-monoidal subgroupoid, so the category $\mathcal{M}_2^{\mathrm{t}}$ inherits a topological semi-monoidal structure from $\mathcal{D}_2$ by Lemma 3.18. In the general notation of this section, this first example is $\mathcal{M} = \mathcal{G} = \mathcal{M}_2^{\mathrm{t}}$.

We denote by $\mathcal{M}_2$ the groupoid $\pi_0(\mathcal{M}_2^{\mathrm{t}})$. We recall that for diffeomorphisms of surfaces, the condition of fixing (a neighbourhood of) an interval in a boundary-component is equivalent to fixing two (neighbourhoods of) intervals in that boundary-component. Therefore $\mathcal{M}_2$ can be described as the groupoid of decorated surfaces (S, I) where S is a surface as above, equipped with a parametrised interval $I\colon [-1, 1] \hookrightarrow \partial_0 S$ in the boundary *decorating* one boundary component denoted by $\partial_0 S$. This *decorated surfaces groupoid* $\mathcal{M}_2$ was introduced already in [RW17, §5.6]. When there is no ambiguity, we omit the parametrised interval I from the notation.

The morphisms in $\mathcal{M}_2$ are the isotopy classes of diffeomorphisms of surfaces which restrict to the identity on a neighbourhood of their parametrised intervals I (or equivalently diffeomorphisms which fix pointwise the boundary component $\partial_0 S$). The non-distinguished boundary components may be freely moved by the mapping classes. The automorphism group of S forms the *mapping class group* of S, which we denote by $\pi_0 \mathrm{Diff}_I(S)$. When the surface S is orientable, the diffeomorphisms then automatically preserve orientations, as they restrict to the identity on a neighbourhood of I. By Lemma 4.3, the groupoid $\mathcal{M}_2$ has a braided monoidal structure induced by gluing, which is already considered in [RW17, §5.6.1]. Hence we may apply Quillen's bracket construction, and we have $\pi_0(\mathfrak{U}\mathcal{M}_2^{\mathrm{t}}) \cong \mathfrak{U}\mathcal{M}_2$ as semicategories by Lemma 4.4.

Notation 4.26 We denote by $\mathbb{D}^2$ the closed unit 2-disc. Let $\Sigma_{0,1}^1$ denote the cylinder $\mathbb{S}^1 \times [0,1]$ (which may be thought of as the disc $\mathbb{D}^2$ with a smaller open disc removed from its interior), $\Sigma_{1,1}$ denote the torus with one boundary component ($\mathbb{S}^1 \times \mathbb{S}^1 \smallsetminus \mathrm{Int}(\mathbb{D}^2)$) and $\mathcal{N}_{1,1}$ denote a Möbius band. For S an object of the groupoid $\mathcal{M}_2$, by the classification of surfaces, there exist $g, s, h \in \mathbb{N}$ such that there is a diffeomorphism $S \cong (\natural_s \Sigma_{0,1}^1)\natural(\natural_g \Sigma_{1,1})\natural(\natural_h \mathcal{N}_{1,1})$.

If $h = 0$, we denote by $\Sigma_{g,1}^s$ the boundary connected sum $(\natural_s \Sigma_{0,1}^1)\natural(\natural_g \Sigma_{1,1})$ and by $\mathbf{\Gamma}_{g,1}^s$ the mapping class group $\pi_0\mathrm{Diff}_I(\Sigma_{g,1}^s)$. If $g = 0$, we denote by $\mathcal{N}_{h,1}^s$ the boundary connected sum $(\natural_s \Sigma_{0,1}^1)\natural(\natural_h \mathcal{N}_{1,1})$ and by $\boldsymbol{\mathcal{N}}_{h,1}^s$ the mapping class group $\pi_0\mathrm{Diff}_I(\mathcal{N}_{h,1}^s)$. When $s = 0$, we often omit it from the notation.

4.5. Surface braid groups

Let S be a compact, connected, smooth surface with a chosen boundary-component $\partial_0 S$. For each non-negative integer k, we denote by $\underline{k}$ a closed submanifold of S consisting in k distinct points of the interior of S. Let $\mathcal{B}r^S$ be the subgroupoid of $\mathcal{D}_2^{\mathrm{sph}}$ with objects all decorated surfaces $(S', \underline{n}, e_1, e_2)$ such that n is any non-negative integer and there is a diffeomorphism $S \cong S'$ taking $\partial_0 S$ onto $\partial_0 S'$; the morphisms of $\mathcal{B}r^S$ are given by the subgroups of the braided diffeomorphisms of Definition 4.21 and denoted by $\mathrm{Diff}_{\mathrm{dec}}^{\mathrm{br}}(S, \underline{n})$. If $S = \mathbb{D}^2$, this collection of objects is closed under the operation $\natural$, so $\mathcal{B}r^{\mathbb{D}^2}$ is a monoidal subgroupoid of $\mathcal{D}_2$ by Lemma 3.18. Also, the groupoid $\mathcal{B}r^S$ is closed under the left-action of $\mathcal{B}r^{\mathbb{D}^2}$ via $\natural$. Hence, in the general notation of this section, we take $\mathcal{G} = \mathcal{B}r^{\mathbb{D}^2}$ and $\mathcal{M} = \mathcal{B}r^S$.

In general, $\mathcal{B}r^S$ is not a full subgroupoid of $\mathcal{D}_2^{\mathrm{sph}}$, since there may be diffeomorphisms of S that restrict to the identity on the set $\underline{n}$ that are not isotopic to the identity. However, the special case of $\mathcal{B}r^{\mathbb{D}^2}$ *is* a full subgroupoid, since *all* diffeomorphisms of $\mathbb{D}^2$ fixing a pair of disjoint intervals in the boundary are isotopic to the identity. In other words, the diffeomorphism group $\mathrm{Diff}_{I \sqcup I}(\mathbb{D}^2)$ is path-connected, as we show in the following lemma.

Lemma 4.27 *Let $I \sqcup I$ be a pair of disjoint closed intervals in the boundary of the disc $\mathbb{D}^2$ and write $\mathrm{Diff}_{I \sqcup I}(\mathbb{D}^2)$ for the group of diffeomorphisms of $\mathbb{D}^2$ that fix $I \sqcup I$ pointwise. Then $\mathrm{Diff}_{I \sqcup I}(\mathbb{D}^2)$ is weakly contractible, in particular path-connected.*

Proof. By [Cer61, §II.2.2.2, Corollaire 2] (see also [Pal60b; Lim63]), the map

$$\mathrm{Diff}_{I \sqcup I}(\mathbb{D}^2) \longrightarrow \mathrm{Diff}_\partial(I \sqcup I) \cong (\mathrm{Diff}_\partial(I))^2$$

that remembers just the action of a diffeomorphism restricted to the *complementary* pair of intervals $\partial\mathbb{D}^2 \smallsetminus (I \sqcup I) \cong I \sqcup I$ is a fibre bundle. Its fibre over the identity is $\mathrm{Diff}_\partial(\mathbb{D}^2)$, the group of diffeomorphisms of $\mathbb{D}^2$ fixing all of $\partial\mathbb{D}^2$ pointwise. The diffeomorphism group $\mathrm{Diff}_\partial(I)$ is easily seen to be contractible, and the diffeomorphism group $\mathrm{Diff}_\partial(\mathbb{D}^2)$ was shown to be contractible by Smale [Sma59]. The long exact sequence of the fibre bundle above then implies that $\mathrm{Diff}_{I \sqcup I}(\mathbb{D}^2)$ is weakly contractible. □

Notation 4.28 For simplicity, we denote the decorated surface $(\mathbb{D}^2, \underline{n})$ for each non-negative integer n by $\mathbb{D}_n$ and call it the *n-th marked 2-disc*, and we denote the decorated surface $(S, \underline{n})$ by $S^{(n)}$. If $n = 0$, we simply denote $(S, \underline{0}) = S^{(0)}$ by S.

The groupoid $\pi_0(\mathcal{B}r^{\mathbb{D}^2})$ is clearly isomorphic to the *braid groupoid* $\boldsymbol{\beta}$ whose objects are non-negative integers and whose automorphism groups are the *classical braid groups* $\{\mathbf{B}_n\}_{n \in \mathbb{N}}$; see [Mac98, Chapter XI, §4] for instance. Let $\boldsymbol{\beta}^S$ be the groupoid $\pi_0(\mathcal{B}r^S)$. For each non-negative integer n, its automorphism group is the surface braid group $\pi_0(\mathrm{Diff}_{\mathrm{dec}}^{\mathrm{br}}(S^{(n)}))$ of S, which we denote by $\mathbf{B}_n(S)$.

By Lemma 4.3, the groupoid $\boldsymbol{\beta}$ inherits the braided monoidal structure of $\mathcal{M}_2$ which also induces a left $\boldsymbol{\beta}$-module structure on $\pi_0(\mathcal{B}r^S)$. Hence Quillen's bracket construction defines categories $\mathfrak{U}\mathcal{B}r^{\mathbb{D}^2}$, $\langle \mathcal{B}r^{\mathbb{D}^2}, \mathcal{B}r^S \rangle$, $\mathfrak{U}\boldsymbol{\beta}$ and $\langle \boldsymbol{\beta}, \boldsymbol{\beta}^S \rangle$. By Lemma 4.4, we deduce that, as semicategories, we have $\pi_0(\mathfrak{U}\mathcal{B}r^{\mathbb{D}^2}) \cong \mathfrak{U}\boldsymbol{\beta}$ and $\pi_0(\langle \mathcal{B}r^{\mathbb{D}^2}, \mathcal{B}r^S \rangle) \cong \langle \boldsymbol{\beta}, \boldsymbol{\beta}^S \rangle$.

Alternatively, braid groups on surfaces may be defined as fundamental groups of configuration spaces. We fix non-negative integers n and k, and a surface $S^{(n)}$ as above. The embedding space $\mathrm{Emb}(\underline{k}, \mathring{S} \setminus \underline{n})$ is the *ordered* configuration space of k points in S, denoted by $F_k(S^{(n)})$ (or $F_k(\mathbb{D}_n)$ if $S = \mathbb{D}^2$). The quotient space $F_k(S^{(n)})/\mathfrak{S}_k$, induced by the natural action of the symmetric group on the coordinates, is the *unordered* configuration space of k points in S, and denoted by $C_k(S^{(n)})$. Corollary 4.19 with $M = S$, $A = \emptyset$ and $Z = \underline{n}$ implies that $\mathbf{B}_n(S)$ is isomorphic to the fundamental group of the unordered configuration space $C_n(S)$. The group $\pi_0(\mathrm{Diff}^{\mathrm{br}}_{\mathrm{dec}}(S, \underline{k}, \underline{n}))$ is called the *partitioned* (k,n)*-braid group* $\mathbf{B}_{k,n}(S)$.

4.6. Loop braid groups

We now focus on the families of extended and non-extended loop braid groups. Their definitions are recalled here and we refer to [Dam17] for a complete and unified introduction to these groups.

Notation 4.29 We denote by $\mathbb{D}^3$ the unit 3-disc. For each non-negative integer n, we denote by $\underline{n}\mathbb{S}^1$ a closed submanifold of $\mathbb{D}^3$ consisting of a collection of n disjoint, unknotted circles forming a trivial link of n components in the interior of $\mathbb{D}^3$. The notation $\underline{n}\mathbb{S}^1_+$ indicates that we require any action on this unlink to be orientation-preserving; otherwise orientation-reversing actions are permitted. For simplicity, we sometimes denote the decorated manifold $(\mathbb{D}^3, \underline{n}\mathbb{S}^1)$ by $\mathbb{D}^3_n$.

Let $\mathrm{Diff}_\partial(\mathbb{D}^3, \underline{n}\mathbb{S}^1)$ be the group of self-diffeomorphisms of $\mathbb{D}^3$ that fix $\partial\mathbb{D}^3$ pointwise and fix $\underline{n}\mathbb{S}^1$ as a subset. The *extended loop braid group* $\mathbf{LB}'_n$ is the group of isotopy classes $\pi_0(\mathrm{Diff}_\partial(\mathbb{D}^3, \underline{n}\mathbb{S}^1))$. Let $\mathrm{Diff}_\partial\big(\mathbb{D}^3, \underline{n}\mathbb{S}^1_+\big)$ be the subgroup of diffeomorphisms that also preserve the orientation of $\underline{n}\mathbb{S}^1$. The (non-extended) *loop braid group* $\mathbf{LB}_n$ is the group of isotopy classes $\pi_0(\mathrm{Diff}_\partial\big(\mathbb{D}^3, \underline{n}\mathbb{S}^1_+\big))$.

We now set a categorical framework for handling these families of groups:

- Let $\mathcal{LB}'$ be the full subgroupoid of $\mathcal{D}_3$ on those decorated manifolds (M, A, e_1, e_2) such that the pair (M, A) is diffeomorphic to the 3-disc relative to an embedded n-component unlink. This is closed under the operation $\natural$, so it inherits a topological semi-monoidal structure from $\mathcal{D}_3$ by Lemma 3.18. We then consider $\mathcal{G} = \mathcal{M} = \mathcal{LB}'$.
- Let $\mathcal{LB}$ be the full subgroupoid of $\mathcal{D}_3^+$ on those decorated manifolds (M, A, e_1, e_2) such that the pair (M, A) is diffeomorphic to the 3-disc relative to an embedded n-component *oriented* unlink. Again, $\mathcal{LB}$ is closed under the operation $\natural$, so it inherits a topological semi-monoidal structure from $\mathcal{D}_3^+$ by Lemma 3.18. We then consider $\mathcal{G} = \mathcal{M} = \mathcal{LB}$.

We note that $\mathcal{LB}$ and $\mathcal{LB}'$ are subcategories of $\mathcal{D}_3^{\mathrm{sph},+}$ and of $\mathcal{D}_3^{\mathrm{sph}}$ respectively. We denote their discrete versions by $\pi_0(\mathcal{LB}') = \mathcal{L\beta}'$ and $\pi_0(\mathcal{LB}) = \mathcal{L\beta}$.

Let us show that there are isomorphisms $\mathrm{Aut}_{\mathcal{L\beta}'}(\mathbb{D}^3_n) \cong \mathbf{LB}'_n$ and $\mathrm{Aut}_{\mathcal{L\beta}}(\mathbb{D}^3_n) \cong \mathbf{LB}_n$. We show this for the first case, the other one following mutatis mutandis. The automorphism group of $\mathbb{D}^3_n$ in $\mathcal{L\beta}'$ is $\pi_0(\mathrm{Diff}_{\mathrm{dec}}(\mathbb{D}^3_n))$, where $\mathrm{Diff}_{\mathrm{dec}}(\mathbb{D}^3_n)$ is the topological group of diffeomorphisms of $\mathbb{D}^3$ that send the embedded the n-component unlink onto itself and that restrict to the identity on a neighbourhood of two disjoint 2-discs in $\partial\mathbb{D}^3$. The condition of fixing *a neighbourhood of* two discs in the boundary is equivalent (on π_0) to fixing just the two discs themselves. On the other hand, $\mathbf{LB}'_n$ has the same description except that diffeomorphisms must fix the whole boundary $\partial\mathbb{D}^3$. It therefore suffices to show that:

Lemma 4.30 *Let M be a 3-manifold with a spherical boundary-component $\partial_0 M$. Then, for isotopy classes of diffeomorphisms, fixing two disjoint 2-discs in $\partial_0 M$ is equivalent to fixing all of $\partial_0 M$.*

Proof. Let $\mathrm{Diff}(M, \partial_0 M)$ be the group of diffeomorphisms of M that send $\partial_0 M$ onto itself. The restriction map $\mathrm{Diff}(M, \partial_0 M) \to \mathrm{Diff}(\partial_0 M) = \mathrm{Diff}(\mathbb{S}^2)$ is a fibre bundle, by [Cer61, Corollaire 2, §II.2.2.2, page 294]. Hence its restriction $\mathrm{Diff}_{\mathbb{D}^2\sqcup\mathbb{D}^2}(M) \longrightarrow \mathrm{Diff}_{\mathbb{D}^2\sqcup\mathbb{D}^2}(\mathbb{S}^2) \cong \mathrm{Diff}_{\partial C}(C)$ is also a fibre bundle, where the subscript ${}_{\mathbb{D}^2\sqcup\mathbb{D}^2}$ means that diffeomorphisms must restrict to the identity on a given pair of disjoint discs in $\partial_0 M = \mathbb{S}^2$, and C is the 2-dimensional cylinder $\mathbb{S}^1 \times [0,1]$. The fibre is $\mathrm{Diff}_{\partial_0 M}(M)$ and we obtain an exact sequence

$$\cdots \to \pi_1(\mathrm{Diff}_{\partial C}(C)) \longrightarrow \pi_0(\mathrm{Diff}_{\partial_0 M}(M)) \xrightarrow{(*)} \pi_0(\mathrm{Diff}_{\mathbb{D}^2\sqcup\mathbb{D}^2}(M)) \longrightarrow \pi_0(\mathrm{Diff}_{\partial C}(C)).$$

By [Gra73, Théorème 1], $\mathrm{Diff}_{\partial C}(C)$ is contractible, and hence $(*)$ is a bijection. □

Since their automorphism groups are the (extended) loop braid groups, we therefore call $\mathcal{L}\beta'$ and $\mathcal{L}\beta$ respectively the *extended loop braid groupoid* and the (non-extended) *loop braid groupoid*. By Lemma 4.3, the groupoids $\mathcal{L}\beta'$ and $\mathcal{L}\beta$ inherit a symmetric monoidal structure from the semi-monoidal structure of $\mathcal{D}_3$. Hence Quillen's bracket construction defines categories $\mathfrak{U}\mathcal{L}\beta'$ and $\mathfrak{U}\mathcal{L}\beta$. By Lemma 4.4, we see that $\pi_0(\mathfrak{U}\mathcal{L}\mathcal{B}') \cong \mathfrak{U}\mathcal{L}\beta'$ and $\pi_0(\mathfrak{U}\mathcal{L}\mathcal{B}) \cong \mathfrak{U}\mathcal{L}\beta$.

Partitioned versions and an alternative interpretation. For non-negative integers n and k, the group $\pi_0(\mathrm{Diff}_{\mathrm{dec}}(\mathbb{D}^3, \underline{k}\mathbb{S}^1, \underline{n}\mathbb{S}^1))$ is called the *partitioned* (k,n)*-extended loop braid group* and denoted by $\mathbf{LB}'_{k,n}$. Similarly, the group $\pi_0(\mathrm{Diff}_{\mathrm{dec}}(\mathbb{D}^3, \underline{k}\mathbb{S}^1_+, \underline{n}\mathbb{S}^1_+))$ is called the *partitioned* (k,n)*-loop braid group* and denoted by $\mathbf{LB}_{k,n}$.

We recall here for further use an alternative way to define loop braid groups. Consider the embedding spaces $F_k(\mathbb{D}^3_n) = \mathrm{Emb}(\underline{k}, \mathring{\mathbb{D}}^3 \smallsetminus \underline{n}\mathbb{S}^1)$ and $\overline{U}_k(\mathbb{D}^3_n) = \mathrm{Emb}^{\mathrm{unl}}(\underline{k}\mathbb{S}^1, \mathring{\mathbb{D}}^3 \smallsetminus \underline{n}\mathbb{S}^1)$, where the superscript $^{\mathrm{unl}}$ means the path-component of the embedding space corresponding to $(n+k)$-component unlinks, and let

$$C_k(\mathbb{D}^3_n) = F_k(\mathbb{D}^3_n)/\mathfrak{S}_k \qquad U^+_k(\mathbb{D}^3_n) = \overline{U}_k(\mathbb{D}^3_n)/\mathrm{Diff}^+(\underline{k}\mathbb{S}^1) \qquad U_k(\mathbb{D}^3_n) = \overline{U}_k(\mathbb{D}^3_n)/\mathrm{Diff}(\underline{k}\mathbb{S}^1).$$

The first is usual unordered configuration space of k points in the unlink-complement $\mathring{\mathbb{D}}^3 \smallsetminus \underline{n}\mathbb{S}^1$. The middle space is the space of unordered, oriented k-component unlinks in $\mathring{\mathbb{D}}^3 \smallsetminus \underline{n}\mathbb{S}^1$ such that the resulting $(n+k)$-component link is again trivial. The right-hand space is similar, except that the k-component unlinks are unoriented. We then have isomorphisms $\pi_1(U^+_n(\mathbb{D}^3)) \cong \mathbf{LB}_n$ and $\pi_1(U_n(\mathbb{D}^3)) \cong \mathbf{LB}'_n$, and embeddings $\pi_1(U^+_k(\mathbb{D}^3_n)) \hookrightarrow \mathbf{LB}_{k,n}$ and $\pi_1(U_k(\mathbb{D}^3_n)) \hookrightarrow \mathbf{LB}'_{k,n}$.

5. Global homological representations

In this section, we apply the general construction of §2 to the categories of decorated manifolds $\mathfrak{U}\mathcal{D}_d$ introduced and studied in §3.2 and §4. This is the more geometric or topological part of our construction of representations, whereas §2 is the more algebraic and formal part.

In more detail, in §5.1, we construct continuous functors $\mathfrak{U}\mathcal{D}_d \to \mathrm{Cov}_\bullet$, which lead, via the general construction of §2, to "global homological representations" $\pi_0(\mathfrak{U}\mathcal{D}_d) \to \mathrm{Mod}_\bullet$. In §5.2.2–§5.2.3, we study in more detail certain particular cases of these global homological representations, restricted to the subcategories of $\mathfrak{U}\mathcal{D}_2$ and $\mathfrak{U}\mathcal{D}_3$ that are relevant for *surface braid groups* and *loop braid groups* (see §4.5–4.6). The key point is that, in many cases, these restrictions of global homological representations have image contained in a subcategory $\mathrm{Mod}_R \subset \mathrm{Mod}_\bullet$, for some ring R. In §5.3, we vary this construction in such a way that it is better adapted to mapping class groups of manifolds (rather than motion groups), and we study particular cases of this construction for *mapping class groups of surfaces* in §5.4.

Although we do not discuss it in more detail in this section, one may equally well consider examples of these global homological representations restricted to subcategories of $\mathfrak{U}\mathcal{D}_2$ and $\mathfrak{U}\mathcal{D}_3$ (or $\mathfrak{U}\mathcal{D}_d$ for higher d) relevant for automorphism groups of free groups, Torelli groups, handlebody mapping class groups, pure braid groups and mapping class groups of non-orientable surfaces, as well as higher-dimensional motion groups and mapping class groups.

Recall (Remark 4.24) that the automorphism groups of the category $\mathfrak{U}\mathcal{D}_d$ contain all mapping class groups and all motion groups of d-dimensional (decorated) manifolds M with *non-empty* boundary. Representations of $\mathfrak{U}\mathcal{D}_d$ therefore give a "global" way to describe representations of all of these groups simultaneously. Our construction below does not, on the other hand, apply directly to give representations of mapping class groups or motion groups of *closed* manifolds M. However, the idea of the construction may also be carried over to this setting, to produce representations of these groups too.

5.1. Global functors for motion groups

Fix an integer $d \geqslant 2$, a closed submanifold $Z \subset \mathbb{R}^d$ and an open subgroup $G \leqslant \mathrm{Diff}(Z)$. Also fix another integer $\ell \geqslant 0$. From this data, we construct two functors

$$\mathring{F}_{(Z,G,\ell)} \text{ and } F_{(Z,G,\ell)} \colon \mathfrak{U}\mathcal{D}_d \longrightarrow \mathrm{Cov}_\bullet. \tag{5.1}$$

The construction of these functors occupies §5.1.1–§5.1.3, and their key basic properties are established in §5.1.4–§5.1.5. In §5.1.6, we discuss possible variations and in §5.1.7 we explain how, fixing (Z,G) and allowing ℓ to vary, these functors fit together into a tower that may be thought of as a single "pro-nilpotent" representation of the category $\mathfrak{U}\mathcal{D}_d$; see diagram (5.15).

Before we begin the construction of these functors, we recall their place in our general construction, which consists in concatenating:

1. one of the functors (5.1),
2. a functor that passes from covering spaces to bundles of modules,
3. an optional fibrewise tensor product with a functor $V\colon \mathfrak{U}\mathcal{D}_d \to {}_\bullet\mathrm{Mod}_\bullet$,
4. twisted homology.

See §2.2–§2.4 for more details of steps (2)–(4). The resulting functor takes values in the category $\mathrm{Mod}_\bullet$, which is a *discrete* category, and so it factors through $\pi_0(\mathfrak{U}\mathcal{D}_d)$, which is identified canonically with $\mathfrak{U}\pi_0(\mathcal{D}_d)$, by Lemma 3.6 and Proposition 3.22, yielding a functor:

$$L_i(F_{(Z,G,\ell)};V)\colon \mathfrak{U}\pi_0(\mathcal{D}_d) \longrightarrow \mathrm{Mod}_\bullet. \tag{5.2}$$

See §2.5, and in particular Lemma 2.19, for the precise construction.

Semi-functors and functors. As discussed in §3.4, the left-hand side of (5.1) is only a (topological) *semi*-category, since it is constructed from the (topological) *semi*-monoidal groupoid $\mathcal{D}_d$. So (5.1) is, necessarily, just a (continuous) semi-functor. However, it induces a functor on π_0; see Lemma 5.10 below. All of the other steps (2)–(4) in our general construction are functors, so the final output (5.2) of the construction is a functor (not just a semi-functor). We will not emphasise this small subtlety in the rest of this section, however, and in particular we will henceforth write "functor" instead of "semi-functor".

Borel-Moore homology and functoriality. We will show that the functor $F_{(Z,G,\ell)}$ takes values in the subcategory $\mathrm{Cov}_\bullet^{\mathrm{pr}}$ of $\mathrm{Cov}_\bullet$ with the same objects and whose morphisms are those which are *proper* as maps between spaces; *cf.* Lemma 5.11. This functor therefore works equally well as an input for our general construction when step (4) is twisted *Borel-Moore* homology, which is functorial only with respect to proper maps of spaces.

On the other hand, the functor $\mathring{F}_{(Z,G,\ell)}$ does not take values in $\mathrm{Cov}_\bullet^{\mathrm{pr}}$, although its restriction to the underlying groupoid $\mathcal{D}_d$ of $\mathfrak{U}\mathcal{D}_d$ does. Thus, if we wish to use Borel-Moore homology together with the functor $\mathring{F}_{(Z,G,\ell)}$ in our general construction, it will not be fully functorial, but only functorial for automorphisms: $\pi_0(\mathcal{D}_d) \to \mathrm{Mod}_\bullet$. In other words, we just obtain representations of the individual groups in this case.

A natural homotopy equivalence. There is a natural homotopy equivalence $\mathring{F}_{(Z,G,\ell)} \Rightarrow F_{(Z,G,\ell)}$; *cf.* Lemma 5.13. Thus, in the case where step (4) consists of *ordinary* twisted homology (or any other homotopy invariant flavour of twisted homology), it will not matter which of these two functors ($F_{(Z,G,\ell)}$ or $\mathring{F}_{(Z,G,\ell)}$) we use in step (1).

On the other hand, the natural homotopy equivalence $\mathring{F}_{(Z,G,\ell)} \Rightarrow F_{(Z,G,\ell)}$ is not a *proper* natural homotopy equivalence. Hence, in the case where step (4) consists of *Borel-Moore* twisted homology – which is invariant only under proper homotopy equivalences – the two choices $F_{(Z,G,\ell)}$ and $\mathring{F}_{(Z,G,\ell)}$ in step (1) will lead to *a priori* different homological representations. As noted above, the homological representations obtained using Borel-Moore homology in step (4) are defined on all of $\mathfrak{U}\mathcal{D}_d$ when using $F_{(Z,G,\ell)}$ in step (1), but only on $\mathcal{D}_d$ when using $\mathring{F}_{(Z,G,\ell)}$ in step (1). Hence we can only compare them on $\mathcal{D}_d \subset \mathfrak{U}\mathcal{D}_d$.

In certain specific cases, such as the classical *Lawrence-Bigelow representations* of the braid groups in Borel-Moore homology, if the coefficients are "generic" – a certain condition on the fibrewise tensor product taken in step (3) – the two choices of $F_{(Z,G,\ell)}$ and $\mathring{F}_{(Z,G,\ell)}$ in step (1) do in fact lead to the same homological representations, due to the fact that they actually agree with the corresponding representations using ordinary homology instead of Borel-Moore homology. This is due to [Koh17, Theorem 3.1] when the ground ring is $\mathbb{C}$; see also [AP20, Proposition D] for more general ground rings.

Cohomology. One can of course replace step (4) of the general construction with twisted cohomology or twisted compactly-supported cohomology, to obtain functors of the form $\mathfrak{U}\pi_0(\mathcal{D}_d)^{\mathrm{op}} \to$

Mod$_\bullet$. The same considerations apply to compactly-supported cohomology as for Borel-Moore homology, so choosing the version $\mathring{F}_{(Z,G,\ell)}$ in step (1) together with twisted compactly-supported cohomology in step (4) leads only to functors $\pi_0(\mathcal{D}_d)^{\mathrm{op}} \to \mathrm{Mod}_\bullet$.

5.1.1. The construction of the functor on objects

In this section and the next, we describe the construction of the functor (5.1) (on objects and on morphisms respectively) in its $\mathring{F}_{(Z,G,\ell)}$ ("open") variant. The modifications involved in defining the $F_{(Z,G,\ell)}$ ("closed") variant of (5.1) are summarised in §5.1.3.

Let $(M, A) \in \mathcal{D}_d$ be a decorated d-dimensional manifold. We show how to associate to this:

(i) a based, path-connected space $X_{(Z,G,\ell)}(M, A)$ that admits a universal cover,

(ii) a surjective homomorphism $\phi_{(Z,G,\ell)}(M, A)\colon \pi_1(X_{(Z,G,\ell)}(M, A)) \to Q_{(Z,G,\ell)}(M, A)$.

Together, these data determine an object of Cov$_\bullet$. To simplify the notation, since the choice of (Z, G, ℓ) is fixed throughout this construction, we will drop the subscripts, denoting the space by $X(M, A)$ and the surjective homomorphism by $\phi(M, A)\colon \pi_1(X(M, A)) \to Q(M, A)$.

The space. We denote by $\breve{M}$ the interior of $\mathbb{B}^d_1 \natural M$, where $\natural$ denotes the boundary connected sum along boundary-cylinder-germs (the semi-monoidal structure of $\mathcal{D}_d$). Note that the interior of $\mathbb{B}^d_1$ may be identified canonically with $\mathbb{R}^d$, so there is a canonical embedding $\mathbb{R}^d \hookrightarrow \breve{M}$. Its image is disjoint from M, hence in particular disjoint from $A \subset M$. Thus, restricting this to $Z \subset \mathbb{R}^d$, we obtain a canonical embedding $Z \hookrightarrow \breve{M} \smallsetminus A$, which determines a basepoint of the relative embedding space

$$\mathcal{E}_G(Z, \breve{M} \smallsetminus A) = \mathrm{Emb}(Z, \breve{M} \smallsetminus A)/G.$$

Definition 5.1 We define $X(M, A)$ to be the path-component of this space containing the basepoint.

To complete step (i) of the construction, we will show that $\mathcal{E}_G(Z, \breve{M} \smallsetminus A)$ is locally path-connected and semi-locally simply-connected (it will then follow that $X(M, A)$ also has these properties, and hence admits a universal cover). There is a quotient map

$$\mathrm{Emb}(Z, \breve{M} \smallsetminus A) \longrightarrow \mathcal{E}_G(Z, \breve{M} \smallsetminus A). \tag{5.3}$$

Embedding spaces, equipped with the Whitney topology, are locally path-connected. Hence it follows that $\mathcal{E}_G(Z, \breve{M} \smallsetminus A)$ is also locally path-connected (since this property is preserved under taking quotients).

For semi-local simply-connectedness we need to use a stronger property of (5.3) than just the fact that it is a quotient map. The space $\mathcal{E}_G(Z, \breve{M} \smallsetminus A)$ is locally retractile with respect to the action of $\mathrm{Diff}_c(\breve{M} \smallsetminus A)$, by [Pal21, Proposition 4.15] (here we use the assumption that G is an *open* subgroup of $\mathrm{Diff}(Z)$). Now, the map (5.3) is equivariant with respect to the action of $\mathrm{Diff}_c(\breve{M} \smallsetminus A)$, so by [Pal60b, Theorem A], the map (5.3) is a fibre bundle.

In general, we have the following point-set topological lemma:

Lemma 5.2 *Let $f\colon X \to Y$ be a surjective fibre bundle, and suppose that X is semi-locally simply-connected. Then Y is also semi-locally simply-connected.*

Proof. Let $y \in Y$ and let U be an open neighbourhood of y in Y. We need to find a smaller open neighbourhood $V \subseteq U$ of y such that any loop in V based at y is nullhomotopic in U. First, choose a smaller open neighbourhood $U' \subseteq U$ such that f is trivialisable over U', and choose a trivialisation $\varphi\colon f^{-1}(U') \cong U' \times F$. Also choose a point $z \in F$ (this is possible since we have assumed that f is surjective). Since X is semi-locally simply-connected, we may find an open neighbourhood $W \subseteq f^{-1}(U')$ of $\tilde{y} = \varphi^{-1}(y, z)$ such that any loop in W based at $\tilde{y}$ is nullhomotopic in $f^{-1}(U')$. By the definition of the product topology, we may then find open subsets $V \subseteq U'$ and $F' \subseteq F$ such that $y \in V$, $z \in F'$ and $\varphi^{-1}(V \times F') \subseteq W$. Now let γ be any loop in V based at y. Then $\tilde{\gamma} = \varphi^{-1} \circ (\gamma \times \{z\})$ is a loop in W based at $\tilde{y}$. By above, we may find a nullhomotopy of $\tilde{\gamma}$ in $f^{-1}(U')$. Composing this nullhomotopy with f, it becomes a nullhomotopy of γ in $U' \subseteq U$. □

Corollary 5.3 *The space $X(M, A)$ admits a universal cover.*

Proof. As remarked above, it will suffice to show that the space $\mathcal{E}_G(Z,\breve{M}\smallsetminus A)$ is locally path-connected and semi-locally simply-connected, since $X(M,A)$ is one path-component of this space. We have already explained why $\mathcal{E}_G(Z,\breve{M}\smallsetminus A)$ is locally path-connected. For the second property, note that the embedding space $\mathrm{Emb}(Z,\breve{M}\smallsetminus A)$ is locally contractible, thus in particular semi-locally simply-connected. The quotient map (5.3) is a fibre bundle, so Lemma 5.2 implies that its target $\mathcal{E}_G(Z,\breve{M}\smallsetminus A)$ is also semi-locally simply-connected. □

The surjective homomorphism. To complete the definition of the functor (5.1) on objects, we need to choose a quotient $\phi(M,A)$ of $\pi_1(X(M,A))=\pi_1(\mathcal{E}_G(Z,\breve{M}\smallsetminus A))$.

First note that there is a canonical isomorphism $\pi_1(\mathcal{E}_G(Z,\breve{M}\smallsetminus A))\cong\pi_1(\mathcal{E}_G(Z,\mathring{M}\smallsetminus A))$, where $\mathring{M}$ denotes the interior of M, given by identifying $\mathbb{B}^d_1\natural M$ with M using one of the boundary-cylinders of the decorated manifold (M,A);[1] see the proof of Lemma 4.12. We therefore need to choose a quotient of $\pi_1(\mathcal{E}_G(Z,\mathring{M}\smallsetminus A))$. To do this, we use the split homotopy fibration sequence (4.4) of Lemma 4.12, which induces the split short exact sequence (4.6); this is the top row of diagram (5.4) below. We then apply the functor $G\mapsto G/\Gamma_\ell(G)$ that quotients a group by the ℓ-th term in its lower central series to the middle and right-hand terms. Finally, we factor the composition of the top-left horizontal map and the middle vertical map in the unique possible way as a surjection followed by an injection, and use these to define the left-hand vertical map $\phi(M,A)$ and the bottom-left horizontal map in the 6-term diagram below.

$$
\begin{array}{ccccccccc}
 & & \pi_1(X(M,A)) & & & & & & \\
 & & \wr\Vert & & & & & & \\
1 & \longrightarrow & \pi_1(\mathcal{E}_G(Z,\mathring{M}\smallsetminus A)) & \longrightarrow & \pi_1(\mathcal{E}_{\mathrm{Diff}(A)\times G}(A\sqcup Z,\mathring{M})) & \rightleftarrows & \pi_1(\mathcal{E}(A,\mathring{M})) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \phi(M,A)} & & \Big\downarrow{\scriptstyle \gamma(M,A)} & & \Big\downarrow{\scriptstyle \bar{\gamma}(M,A)} & & \\
1 & \longrightarrow & Q(M,A) & \longrightarrow & \dfrac{\pi_1(\mathcal{E}_{\mathrm{Diff}(A)\times G}(A\sqcup Z,\mathring{M}))}{\Gamma_\ell(\pi_1(\mathcal{E}_{\mathrm{Diff}(A)\times G}(A\sqcup Z,\mathring{M})))} & \rightleftarrows & \dfrac{\pi_1(\mathcal{E}(A,\mathring{M}))}{\Gamma_\ell(\pi_1(\mathcal{E}(A,\mathring{M})))} & \longrightarrow & 1
\end{array}
\tag{5.4}
$$

This completes the construction of $X(M,A)$ and $\phi(M,A)$, and hence of the functor (5.1) on objects.

An immediate key observation about this diagram is that the bottom row is again a split short exact sequence; the only property of the quotient $G/\Gamma_\ell(G)$ that this uses is that it is *functorial*. We recall that Grp denotes the category of groups.

Lemma 5.4 *Let $Q\colon \mathrm{Grp}\to\mathrm{Grp}$ be a functorial quotient of groups, i.e., it is equipped with a natural transformation $q\colon \mathrm{id}\Rightarrow Q$ such that $q(G)\colon G\to Q(G)$ is a quotient map for each group G. Then for any split short exact sequence*

$$1\longrightarrow A\xrightarrow{\;f\;}B\underset{g}{\rightleftarrows}C\longrightarrow 1$$

we have $\mathrm{im}(q(B)\circ f\colon A\to B\twoheadrightarrow Q(B))=\ker(Q(g)\colon Q(B)\to Q(C))$. Denoting this group by $Q^*(A)$, this means that we have an induced 6-term diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & A & \xrightarrow{\;f\;} & B & \underset{g}{\rightleftarrows} & C & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Q^*(A) & \longrightarrow & Q(B) & \rightleftarrows & Q(C) & \longrightarrow & 1
\end{array}
$$

in which both rows are split short exact sequences and the middle and right-hand vertical maps are given by the natural transformation q.

[1] We recall that (M,A) comes equipped with an ordered pair of *germs* of boundary-cylinders. We choose the first of this ordered pair, and then choose a boundary cylinder representing this germ. The identification of $\mathbb{B}^d_1\natural M$ with M depends on this choice of representative, but the induced isomorphism $\pi_1(\mathcal{E}_G(Z,\mathring{M}\smallsetminus A))\cong\pi_1(\mathcal{E}_G(Z,\breve{M}\smallsetminus A))$ does not.

Proof. The second statement is clear, given the first one: we define the right-hand square by applying the functor Q to the map g and its given section and by applying the natural transformation q to the groups B and C. We then fill in the left-hand square by factoring $q(B) \circ f$ uniquely as a surjection followed by an injection. The final thing to check is exactness in the middle of the bottom row, which is precisely the first statement of the lemma.

To prove the first statement, first note that the inclusion $\mathrm{im}(q(B) \circ f) \subseteq \ker(Q(g))$ follows immediately from exactness of the top row. To prove the opposite inclusion, let $x \in Q(B)$ with $Q(g)(x) = 1$; we need to find a lift $y \in B$ of x such that $y \in f(A)$. To do this, first pick any lift $y' \in B$ of x and set $z = g(y') \in C$. Denoting the given section of g by s, note that $s(z)$ projects to $1 \in Q(B)$, since z projects to $1 \in Q(C)$. Thus $y = y' \cdot s(z)^{-1} \in B$ is another lift of x, and moreover $g(y) = g(y') \cdot z^{-1} = z \cdot z^{-1} = 1$, so $y \in f(A)$ by exactness of the top row. □

Notation 5.5 Two notational points concerning the above diagram:

- To be fully rigorous, the vertical morphisms in (5.4) should have subscripts with the data (Z, G, ℓ), which have been elided above to avoid cluttering the diagram. For example, the middle vertical morphism is called $\gamma_{(Z,G,\ell)}(M, A)$ in general.
- As convention, we take $\Gamma_0(G)$ to be the trivial subgroup of G (of course, when $\ell \geqslant 1$, $\Gamma_\ell(G)$ is the ℓ-th term in the lower central series of G).

Remark 5.6 We note that when $\ell = 1$ we have $Q(M, A) = \{\mathrm{id}\}$, corresponding to the trivial cover of $X(M, A)$, and when $\ell = 0$ we have $\phi(M, A) = \mathrm{id}$, corresponding to the universal cover of $X(M, A)$.

5.1.2. The construction of the functor on morphisms

Given a morphism $\varphi\colon (M, A) \to (N, B)$ in $\mathfrak{UD}_d$, we use the identification of Proposition 4.8, which describes it as an embedding of manifolds satisfying the three properties of Definition 3.20. From this description, we see that it induces a map of split homotopy fibration sequences of the form $(4.4)_{(M,A)} \to (4.4)_{(N,B)}$, in particular a map

$$f_\varphi\colon \mathcal{E}_G(Z, \mathring{M} \smallsetminus A) \longrightarrow \mathcal{E}_G(Z, \mathring{N} \smallsetminus B).$$

Notation of the form $(4.4)_{(N,B)}$ means the diagram (4.4) with each instance of (M, A) replaced by (N, B). If we first use the semi-monoidal structure of $\mathfrak{UD}_d$ to multiply φ with the identity map of $(\mathbb{B}^d_1, \emptyset)$, then we also obtain a map

$$f_\varphi^{\smile}\colon \mathcal{E}_G(Z, \breve{M} \smallsetminus A) \longrightarrow \mathcal{E}_G(Z, \breve{N} \smallsetminus B),$$

which preserves basepoints and therefore restricts to a based map

$$f_\varphi^X\colon X(M, A) \longrightarrow X(N, B).$$

As explained in §5.1.1 above, $\pi_1(X(M, A))$ is naturally identified with $\pi_1(\mathcal{E}_G(Z, \mathring{M} \smallsetminus A))$ (and similarly for (N, B)); under these identifications the homomorphisms $\pi_1(f_\varphi)$ and $\pi_1(f_\varphi^X)$ agree.

The map of split homotopy fibration sequences $(4.4)_{(M,A)} \to (4.4)_{(N,B)}$ induces a map of split short exact sequences $(4.6)_{(M,A)} \to (4.6)_{(N,B)}$, and hence (since the construction of (5.4) from (4.6) is functorial) a map of diagrams of the form $(5.4)_{(M,A)} \to (5.4)_{(N,B)}$, in particular a homomorphism

$$\theta_\varphi\colon Q(M, A) \longrightarrow Q(N, B).$$

That this is a map of diagrams says, in particular, that

$$\theta_\varphi \circ \phi(M, A) = \phi(N, B) \circ \pi_1(f_\varphi) = \phi(N, B) \circ \pi_1(f_\varphi^X),$$

and hence $\pi_1(f_\varphi^X)$ sends $\ker(\phi(M, A))$ into $\ker(\phi(N, B))$. Thus $(f_\varphi^X, \theta_\varphi)$ is a morphism in $\mathrm{Cov}_\bullet$ from $(X(M, A), \phi(M, A))$ to $(X(N, B), \phi(N, B))$; *cf.* Definition 2.2. This completes the definition of the functor (5.1) on morphisms.

Two actions agree. The entire diagram (5.4) is functorial in the input (M, A) as an object of $\mathfrak{U}\mathcal{D}_d$. If we restrict to the automorphism group $\mathrm{Diff}_{\mathrm{dec}}(M, A)$ of a single object (M, A) and look just at the bottom-left group $Q(M, A)$ of (5.4), we obtain an action of $\mathrm{Diff}_{\mathrm{dec}}(M, A)$ on $Q(M, A)$. Since $Q(M, A)$ is discrete, this factors through an action of $\pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, A))$ on $Q(M, A)$. If we restrict this further to the subgroup $\pi_0(\mathrm{Diff}^{\mathrm{br}}_{\mathrm{dec}}(M, A)) \subseteq \pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, A))$, we obtain an action of $\pi_0(\mathrm{Diff}^{\mathrm{br}}_{\mathrm{dec}}(M, A))$ on $Q(M, A)$.

We also have an action of the top-right group of (5.4) on the bottom-left group $Q(M, A)$ of (5.4), given either by lifting elements of $Q(M, A)$ along the vertical map $\phi(M, A)$ and using the conjugation action of the semi-direct product on the top row, or equivalently by projecting along the vertical map $\bar{\gamma}(M, A)$ to the bottom-right group of (5.4) and then using the conjugation action of the semi-direct product on the bottom row. By Proposition 4.23, the top-right group of (5.4) is naturally identified with $\pi_0(\mathrm{Diff}^{\mathrm{br}}_{\mathrm{dec}}(M, A))$, so this gives us another action of $\pi_0(\mathrm{Diff}^{\mathrm{br}}_{\mathrm{dec}}(M, A))$ on $Q(M, A)$.

Proposition 5.7 *The two actions of* $\pi_0(\mathrm{Diff}^{\mathrm{br}}_{\mathrm{dec}}(M, A))$ *on* $Q(M, A)$ *described above are equal.*

Proof. This follows immediately from the second part of Lemma 4.25. □

5.1.3. A closed variant

The construction of the "closed" variant $F_{(Z,G,\ell)}$ of the functor (5.1) is a very slight modification of the constructions in §5.1.1–§5.1.2, in which we constructed the "open" variant $\mathring{F}_{(Z,G,\ell)}$.

To do this, we simply replace $\mathring{M}$ with M (and similarly $\breve{M}$ with $\mathbb{B}^d_1 \natural M$) everywhere, and we consider the embedding space $\mathcal{E}'_G(Z, M \smallsetminus A)$ of Definition 4.14 instead of $\mathcal{E}_G(Z, \mathring{M} \smallsetminus A)$. In other words, we consider embeddings of Z whose image is either contained entirely in the interior or entirely in the boundary of M. This has the effect of replacing $X(M, A)$ with another space that is homotopy equivalent (cf. Lemma 4.15),[2] so it does not change $\pi_1(X(M, A))$.[3] The construction of the quotient $\phi(M, A)\colon \pi_1(X(M, A)) \twoheadrightarrow Q(M, A)$ is identical. On morphisms, the construction is exactly analogous, viewing the split short exact sequence (4.6) as (4.7).

5.1.4. Elementary properties

By Lemma 3.5, if we have subgroupoids $\mathcal{M}$ and $\mathcal{G}$ of $\mathcal{D}_d$ such that $\mathcal{G}$ is closed under $\natural$ and $\mathcal{M}$ is closed under the action of $\mathcal{G}$ via $\natural$, and if $\mathcal{G}$ is moreover a *full* subgroupoid of $\mathcal{D}_d$, then there is a natural inclusion of categories

$$\langle \mathcal{G}, \mathcal{M} \rangle \longhookrightarrow \mathfrak{U}\mathcal{D}_d. \tag{5.5}$$

On objects, this is just the inclusion of the objects of $\mathcal{M}$ into the objects of $\mathcal{D}_d$. On morphisms, under the identification of morphism spaces of Proposition 4.8, this is an inclusion of embedding spaces. Roughly, morphisms in $\mathfrak{U}\mathcal{D}_d$ are given by embeddings whose complement is another decorated manifold, whereas morphisms in $\langle \mathcal{G}, \mathcal{M} \rangle$ are given by embeddings whose complement is a decorated manifold in the set $\mathrm{ob}(\mathcal{G})$; *cf.* Definition 4.7 for precise details.

Summarising the constructions in §5.1.1–§5.1.3, we have:

Proposition 5.8 *For any integers $d \geqslant 2$ and $\ell \geqslant 0$, closed submanifold $Z \subset \mathbb{R}^d$ and open subgroup $G \leqslant \mathrm{Diff}(Z)$, the recipe described above gives well-defined functors*

$$\mathring{F}_{(Z,G,\ell)} \text{ and } F_{(Z,G,\ell)} \colon \mathfrak{U}\mathcal{D}_d \longrightarrow \mathrm{Cov}_\bullet, \tag{5.6}$$

and hence, by restriction, well-defined functors $\mathring{F}_{(Z,G,\ell)}$ *and* $F_{(Z,G,\ell)} \colon \langle \mathcal{G}, \mathcal{M} \rangle \to \mathrm{Cov}_\bullet$*, for any subcategory* $\langle \mathcal{G}, \mathcal{M} \rangle \subseteq \mathfrak{U}\mathcal{D}_d$ *as in* (5.5).

[2] Although not *proper* homotopy equivalent, which is why the open and closed variants of (5.1) give rise to different homological representations when using Borel-Moore homology.

[3] The only small technicality comes when checking that the closed version of $X(M, A)$ has good local properties, so it admits a universal cover (Corollary 5.3): the argument is exactly analogous, except that one has to use a variant of [Pal21, Proposition 4.15] where the target manifold is allowed to have non-empty boundary. This may be proved by a small adaptation of the proof of Proposition 4.15 of [Pal21], using ideas of [Cer61] (where all manifolds are allowed to have corners of any codimension).

Remark 5.9 We note that the construction, up to homotopy, depends only on the isotopy class (mod G) of the embedded submanifold $Z \subseteq \mathbb{R}^d$. More precisely, if we choose two embeddings of the closed manifold Z in $\mathbb{R}^d$ such that the corresponding points of $\mathrm{Emb}(Z, \mathbb{R}^d)/G$ lie in the same path-component, then the two resulting continuous functors $\mathfrak{U}\mathcal{D}_d \to \mathrm{Cov}_\bullet$ will be homotopic.

We now record a few immediate observations about the the topological functors (5.6). First, recall that these are in fact only semifunctors, since $\mathfrak{U}\mathcal{D}_d$ is only a semi-category; see §3.4. The first observation says that this technical issue does not matter once we pass to π_0.

Lemma 5.10 *The topological semifunctors* (5.6) *induce functors on* π_0*.*

Proof. Let (M, A) be an object of $\pi_0(\mathfrak{U}\mathcal{D}_d)$ (i.e., an object of $\mathcal{D}_d$, a decorated manifold). We have to show that π_0(5.6) sends $\mathrm{id}_{(M,A)}$ to an identity morphism of $\pi_0(\mathrm{Cov}_\bullet)$. To do this, we first have to *find* the identity morphism of (M, A) in $\pi_0(\mathfrak{U}\mathcal{D}_d)$.

Let $e_1 \colon \mathbb{D}^{d-1} \times [0, 1] = \mathbb{B}^d_1 \hookrightarrow M$ denote one of the boundary cylinders that (M, A) is equipped with (more precisely, a representative of one of the *germs of* boundary cylinders that (M, A) is equipped with). The boundary connected sum $(\mathbb{B}^d_1, \emptyset)\natural(M, A)$ may be viewed as the union of $\mathbb{D}^{d-1} \times [-1, 1]$ with M along $\mathbb{D}^{d-1} \times [0, 1]$ via the embedding e_1. Choose a diffeomorphism $[-1, 1] \to [0, 1]$ that is given by $t \mapsto t + 1$ on $[-1, -1 + \epsilon]$ and by $t \mapsto t$ on $[1 - \epsilon, 1]$ for some $\epsilon > 0$. Multiplying this by $\mathbb{D}^{d-1}$ and extending by the identity over $M \smallsetminus \mathrm{im}(e_1)$, this determines an isomorphism of decorated manifolds

$$\upsilon_{(M,A)} \colon (\mathbb{B}^d_1, \emptyset)\natural(M, A) \longrightarrow (M, A). \tag{5.7}$$

Consider the endomorphism of (M, A) in $\mathfrak{U}\mathcal{D}_d$ given by

$$\Upsilon_{(M,A)} = ((\mathbb{B}^d_1, \emptyset), \upsilon_{(M,A)}). \tag{5.8}$$

One may check that, for any endomorphism φ of (M, A) in $\mathfrak{U}\mathcal{D}_d$, the compositions $\Upsilon_{(M,A)} \circ \varphi$ and $\varphi \circ \Upsilon_{(M,A)}$ are both isotopic to φ. Hence $\Upsilon_{(M,A)}$ is the identity of (M, A) in the category $\pi_0(\mathfrak{U}\mathcal{D}_d)$. Under the identification of Proposition 4.8, this corresponds to the self-embedding of (M, A) given by restricting the diffeomorphism (5.7) to the submanifold $(M, A) \subset (\mathbb{B}^d_1, \emptyset)\natural(M, A)$. Since this self-embedding is isotopic to the identity, the induced self-map of embedding spaces $X(M, A) \to X(M, A)$ is homotopic to the identity, and hence is an identity morphism in $\pi_0(\mathrm{Cov}_\bullet)$. □

Lemma 5.11 *The topological semifunctor* $F_{(Z,G,\ell)}$ *of* (5.6) *takes values in the subcategory* $\mathrm{Cov}^{\mathrm{pr}}_\bullet$ *of* $\mathrm{Cov}_\bullet$*.*

Proof. We have to show that, for any morphism $\varphi \colon (M, A) \to (N, B)$ of decorated manifolds, the induced map of spaces $X(M, A) \to X(N, B)$ (in the *closed* variant of the construction in §5.1.1) is a proper map (preimages of compact subspaces are compact). We recall that this is (a restriction to particular path-components of) an inclusion of embedding spaces

$$\mathcal{E}'_G(Z, \mathbb{B}^d_1\natural M \smallsetminus A) \longrightarrow \mathcal{E}'_G(Z, \mathbb{B}^d_1\natural N \smallsetminus B)$$

induced by an embedding of pairs of manifolds $(M, A) \hookrightarrow (N, B)$ satisfying the three properties of Definition 3.20 (cf. Proposition 4.8). In particular, the third property implies that the inclusion $M \smallsetminus A \hookrightarrow N \smallsetminus B$ has closed image, so the inclusion of embedding spaces above also has closed image; any closed inclusion is a proper map. □

Remark 5.12 On the other hand, the topological semifunctor $\mathring{F}_{(Z,G,\ell)}$ of (5.6) does *not* take values in the subcategory $\mathrm{Cov}^{\mathrm{pr}}_\bullet$ of $\mathrm{Cov}_\bullet$. The above proof breaks down in this setting because the inclusion of the *interior* of $M \smallsetminus A$ into the *interior* of $N \smallsetminus B$ does not have closed image.

Lemma 5.13 *There is a natural homotopy equivalence* $\mathring{F} \Rightarrow F$ *between the two functors* (5.6)*.*

Proof. The natural transformation may easily be constructed from the inclusions of spaces of embeddings into the interior of $M \smallsetminus A$ into spaces of embeddings with image contained either in the interior or the boundary of $M \smallsetminus A$. The fact that this is a natural *homotopy equivalence* (although not a natural *proper* homotopy equivalence) follows from Lemma 4.15. □

5.1.5. The image of the functor

Under certain conditions, the functor (5.6), restricted to a subcategory of the form $\langle \mathcal{G}, \mathcal{M} \rangle$ from (5.5), takes values in a subcategory of $\mathrm{Cov}_\bullet$ of the form Cov_Q or $\mathrm{Cov}_Q^{\mathrm{tw}}$ for a fixed group Q.

Definition 5.14 (Q*-stability.*) Let $\mathcal{M}$ and $\mathcal{G}$ be subgroupoids as in (5.5). Suppose that there is a group Q and there are identifications $Q(M,A) \cong Q$ for each object (M,A) of $\langle \mathcal{G}, \mathcal{M} \rangle$ (recall that these are exactly the objects of $\mathcal{M}$), such that, for each object (M',A') of $\mathcal{G}$, the homomorphism of groups

$$Q(M,A) \longrightarrow Q((M',A') \natural (M,A)),$$

induced by the canonical morphism $((M',A'),\mathrm{id})$ of $\langle \mathcal{G}, \mathcal{M} \rangle$, is equal to the identity under these identifications. In this case, we say that *the functor* $F_{(Z,G,\ell)}$ *is* Q*-stable on* $\langle \mathcal{G}, \mathcal{M} \rangle$.

We recall that $\mathrm{Cov}_Q^{\mathrm{tw}} \subset \mathrm{Cov}_\bullet$ is the full subcategory on objects $(X, \phi \colon \pi_1(X) \twoheadrightarrow Q')$ where $Q' = Q$. As an immediate observation, we note:

Lemma 5.15 *Suppose that* $F_{(Z,G,\ell)}$ *is* Q*-stable on* $\langle \mathcal{G}, \mathcal{M} \rangle$*. Then* $F_{(Z,G,\ell)}$ *is equivalent to a functor with image contained in* $\mathrm{Cov}_Q^{\mathrm{tw}} \subset \mathrm{Cov}_\bullet$.

Proof. The hypothesis implies that the image of $F_{(Z,G,\ell)}$ is contained in the slightly larger full subcategory $\mathrm{Cov}_{\cong Q}^{\mathrm{tw}} \subset \mathrm{Cov}_\bullet$ on objects $(X, \phi \colon \pi_1(X) \twoheadrightarrow Q')$ where Q' is *isomorphic* to Q. Then we observe that the inclusion $\mathrm{Cov}_Q^{\mathrm{tw}} \hookrightarrow \mathrm{Cov}_{\cong Q}^{\mathrm{tw}}$ is an equivalence of categories. □

Under certain additional conditions, when $\ell = 2$, we may restrict the image further to Cov_Q, after passing to π_0. Write $|\mathcal{G}|$ for the set of isomorphism classes of objects of $\pi_0(\mathcal{G})$, which is naturally a monoid, and similarly write $|\mathcal{M}|$ for the set of isomorphism classes of objects of $\pi_0(\mathcal{M})$, which is naturally a $|\mathcal{G}|$-set.

Proposition 5.16 *Let* $\mathcal{M}$ *and* $\mathcal{G}$ *be subgroupoids of* $\mathcal{D}_d$ *such that* $\mathcal{G}$ *is closed under* $\natural$ *and* $\mathcal{M}$ *is closed under the action of* $\mathcal{G}$ *via* $\natural$*. Set* $\ell = 2$*, suppose that* $F_{(Z,G,2)}$ *is* Q*-stable on* $\langle \mathcal{G}, \mathcal{M} \rangle$*, and*

- $\mathcal{G}$ *is full in* $\mathcal{D}_d$ *and* $\mathcal{M}$ *is 0-full in* $\mathcal{D}_d$*,*
- *for each object* (M,A) *of* $\mathcal{M}$*, the subgroup* $\mathrm{Aut}_{\mathcal{M}}(M,A) \subseteq \mathrm{Diff}_{\mathrm{dec}}(M,A)$ *lies in* $\mathrm{Diff}_{\mathrm{dec}}^{\mathrm{br}}(M,A)$*,*
- $|\mathcal{G}|$ *is free as a monoid and* $|\mathcal{M}|$ *is free as a* $|\mathcal{G}|$*-set.*

Under these assumptions, the functor

$$\pi_0(F_{(Z,G,2)}|_{\langle \mathcal{G}, \mathcal{M} \rangle}) \colon \pi_0(\langle \mathcal{G}, \mathcal{M} \rangle) \longrightarrow \pi_0(\mathrm{Cov}_\bullet)$$

is equivalent to a functor with image contained in $\pi_0(\mathrm{Cov}_Q) \subset \pi_0(\mathrm{Cov}_\bullet)$.

Proof. To prove this, we will go step by step through the following diagram, where $F = F_{(Z,G,2)}|_{\langle \mathcal{G}, \mathcal{M} \rangle}$.

$$\begin{array}{ccccc} \langle \mathcal{G}, \mathcal{M} \rangle & & \xrightarrow{\quad F \cong \hat{F} \quad} & & \mathrm{Cov}_\bullet \\ \downarrow & & & & \downarrow \\ \pi_0(\langle \mathcal{G}, \mathcal{M} \rangle) & \cong & \langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M}) \rangle & \xrightarrow{\pi_0(\hat{F})} & \pi_0(\mathrm{Cov}_\bullet) \\ & & \uparrow & & \uparrow \\ & & \langle \mathcal{G}_0, \mathcal{M}_0 \rangle & \xrightarrow{\pi_0(\hat{F})} & \pi_0(\mathrm{Cov}_Q) \end{array} \tag{5.9}$$

The functor $\hat{F}$ is defined just like F, except that, on objects, the surjection $\phi(M,A) \colon \pi_1(X(M,A)) \twoheadrightarrow Q(M,A)$ is composed with the given identification $Q(M,A) \cong Q$ from the definition of Q-stability. It is easy to see that this is again a well-defined functor, and the isomorphisms $Q(M,A) \cong Q$ induce an isomorphism of functors $F \cong \hat{F}$. Clearly, $\hat{F}$ has image contained in $\mathrm{Cov}_Q^{\mathrm{tw}}$: this is a slightly different alternative proof of Lemma 5.15, with the slightly stronger conclusion that $F_{(Z,G,\ell)}$ is *isomorphic* to a functor with image contained in $\mathrm{Cov}_Q^{\mathrm{tw}}$. Moreover, the definition of Q-stability immediately tells us that each canonical morphism $(M',\mathrm{id}) \colon M \to M' \natural M$ in $\langle \mathcal{G}, \mathcal{M} \rangle$ is sent by $\hat{F}$ to the subcategory Cov_Q.

We now pass to the middle row of diagram (5.9). Using the first assumption of the proposition, Lemma 4.4 gives us a canonical isomorphism of categories $\pi_0(\langle \mathcal{G}, \mathcal{M}\rangle) \cong \langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M})\rangle$, so we may view $\pi_0(\hat{F})$ as defined on the category $\langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M})\rangle$. Since $\pi_0(\mathcal{G})$ has a monoidal unit $\mathbf{1}$, we may consider, for each automorphism φ of (M, A) in $\pi_0(\mathcal{M})$, the automorphism $(\mathbf{1}, \varphi)$ of (M, A) in $\langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M})\rangle$. More precisely, since the unit $\mathbf{1} = (\mathbb{B}^d_1, \emptyset)$ of $\langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M})\rangle$ is not *strict*, the corresponding automorphism of (M, A) in $\langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M})\rangle$ is given by $(\mathbf{1}, \varphi \circ \lambda_{(M,A)})$, where λ is the left unitor for the monoidal structure. Under the functor $\pi_0(\hat{F})$, this is sent to an automorphism of the object $(X(M, A), \pi_1(X(M, A)) \twoheadrightarrow Q(M, A) \cong Q)$, so in particular it induces an automorphism of Q. We now show that this is the *identity* automorphism; in other words, this says that $\pi_0(\hat{F})$ sends $(\mathbf{1}, \varphi)$ into the subcategory Cov_Q. Since the induced automorphism of Q is pulled back along the given identification $Q(M, A) \cong Q$ from the induced automorphism of $Q(M, A)$, it is equivalent to show that the induced automorphism of $Q(M, A)$ is the identity. To see this, note that, by the second assumption of the proposition and by Proposition 5.7, the action of $(\mathbf{1}, \varphi)$ on $Q(M, A)$ is given by viewing φ as an element of the top-right group of diagram (5.4) (via the identification of Proposition 4.23), projecting it to the bottom-right group and then letting it act on $Q(M, A)$ via the semi-direct product structure of the bottom row of (5.4). But we have assumed that $\ell = 2$, so all groups on the bottom row of (5.4) are abelian, so the semi-direct product is a *direct* product, and the action on $Q(M, A)$ is trivial.

Putting together the two paragraphs above, we see that the functor $\pi_0(\hat{F})$ has image contained in $\pi_0(\mathrm{Cov}^{\mathrm{tw}}_Q)$, and moreover it takes all canonical morphisms (M', id) and all automorphisms of the form $(\mathbf{1}, \varphi)$ (for φ an automorphism of $\pi_0(\mathcal{M})$) into the subcategory $\pi_0(\mathrm{Cov}_Q)$. However, we cannot yet conclude: although each morphism of $\langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M})\rangle$ decomposes into a composition of a canonical morphism (M', id) followed by $(\mathbf{1}, \varphi)$ for an isomorphism φ of $\pi_0(\mathcal{M})$, it is not necessarily the case that φ is an *automorphism* of $\pi_0(\mathcal{M})$.

To deal with this, we now choose skeleta $\mathcal{G}_0 \subseteq \pi_0(\mathcal{G})$ and $\mathcal{M}_0 \subseteq \pi_0(\mathcal{M})$ such that:

- $\mathcal{G}_0$ is a *monoidal* subcategory of $\pi_0(\mathcal{G})$, and
- $\mathcal{M}_0$ is a $\mathcal{G}_0$-*module* subcategory of $\pi_0(\mathcal{M})$.

It is possible to construct skeleta with these properties due to the third assumption of the proposition. Since the set $|\mathcal{G}|$ of isomorphism classes of objects of $\pi_0(\mathcal{G})$ is a *free* monoid, we may choose a free generating set for $|\mathcal{G}|$, then choose arbitrarily one object of $\pi_0(\mathcal{G})$ in each isomorphism class corresponding to a generator, and then use the monoidal structure of $\pi_0(\mathcal{G})$ to choose all other representatives of isomorphism classes of objects. This procedure is well-defined since we used a *free* generating set for the monoid $|\mathcal{G}|$. We then define $\mathcal{G}_0$ to be the full subcategory of $\pi_0(\mathcal{G})$ on the objects that we have chosen in this way. Exactly the same idea allows us to construct a skeleton $\mathcal{M}_0$ of $\pi_0(\mathcal{M})$ that is closed under the action of $\mathcal{G}_0 \subseteq \pi_0(\mathcal{G})$, using a free generating set for $|\mathcal{M}|$ as a $|\mathcal{G}|$-set.

There is a canonical functor $\langle \mathcal{G}_0, \mathcal{M}_0\rangle \to \langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M})\rangle$ that is full, faithful and essentially surjective on objects. In other words, it is an inclusion of a subcategory that is also an equivalence of categories. It will therefore suffice to show that $\pi_0(\hat{F})$, restricted to $\langle \mathcal{G}_0, \mathcal{M}_0\rangle$, takes values in $\pi_0(\mathrm{Cov}_Q)$. This now follows from what we have already shown above: We have shown that $\pi_0(\hat{F})$ sends any morphism of the form (M', id) or $(\mathbf{1}, \varphi)$ for φ an automorphism of $\pi_0(\mathcal{M})$ into $\pi_0(\mathrm{Cov}_Q)$. Now, any morphism in $\langle \mathcal{G}_0, \mathcal{M}_0\rangle$ decomposes as $(\mathbf{1}, \varphi) \circ (M', \mathrm{id})$ for an isomorphism φ of $\mathcal{M}_0$; but $\mathcal{M}_0$ is a skeletal category, so φ must be an *automorphism*. □

In several of our examples, Q-stability does not hold on $\langle \mathcal{G}, \mathcal{M}\rangle \subseteq \mathfrak{UD}_d$ (for certain $\mathcal{G}$ and $\mathcal{M}$), but *does* hold on the full subcategory $\langle \mathcal{G}, \mathcal{M}'\rangle \subseteq \langle \mathcal{G}, \mathcal{M}\rangle$ for a certain full sub-$\mathcal{G}$-module $\mathcal{M}' \subset \mathcal{M}$. Typically, the isomorphism classes of objects of $\mathcal{G}$ will form the monoid $\mathbb{N}$, the isomorphism classes of objects of $\mathcal{M}$ will form the $\mathbb{N}$-module $\mathbb{N}$ (acting on itself by addition), and $\mathcal{M}'$ will correspond to the sub-$\mathbb{N}$-module $\{n, n+1, n+2, \ldots\}$ for some $n \geqslant 1$.

The functor $F_{(Z,G,2)}$ restricted to $\pi_0(\langle \mathcal{G}, \mathcal{M}'\rangle)$ will therefore factor (up to equivalence) through $\pi_0(\mathrm{Cov}_Q)$, and hence the general construction will result in a functor of the form $\pi_0(\langle \mathcal{G}, \mathcal{M}'\rangle) \to \mathrm{Mod}_R$ for some ring R. It will sometimes be convenient to extend this functor "trivially" to the larger category $\pi_0(\langle \mathcal{G}, \mathcal{M}\rangle) \supseteq \pi_0(\langle \mathcal{G}, \mathcal{M}'\rangle)$ with the help of the following observation.

Lemma 5.17 *Let $\mathcal{D}$ be a category and $\mathcal{C} \subseteq \mathcal{D}$ the full subcategory on a subset* $\mathrm{ob}(\mathcal{C}) \subseteq \mathrm{ob}(\mathcal{D})$. *Assume that there are no morphisms in $\mathcal{D}$ from* $\mathrm{ob}(\mathcal{C})$ *to* $\mathrm{ob}(\mathcal{D}) \smallsetminus \mathrm{ob}(\mathcal{C})$. *Then any functor $\mathcal{C} \to \mathcal{A}$ to a category $\mathcal{A}$ with initial object I may be extended to $\mathcal{D}$ by sending each object of* $\mathrm{ob}(\mathcal{D}) \smallsetminus \mathrm{ob}(\mathcal{C})$

to I.

5.1.6. Choosing quotients

The construction of quotients of $\pi_1(X(M,A))$ in diagram (5.4) admits several natural generalisations.

Functorial descending normal series. We recall that a *functorial descending normal series* is a sequence of endofunctors $s_i \colon \mathrm{Grp} \to \mathrm{Grp}$ with $s_1 = \mathrm{id}$, such that $s_{i+1}(G) \subseteq s_i(G)$ for each i and these inclusions assemble to a natural transformation $s_{i+1} \Rightarrow s_i$, and each subgroup $s_i(G) \subseteq G$ is normal in G. Examples are the lower central series and derived series. In the construction of (5.1), we have used the lower central series to define certain quotients of fundamental groups. *However, exactly the same construction goes through if one replaces the lower central series by any other functorial descending normal series. More generally, one could alternatively index series by all ordinals, i.e., consider transfinite series.* We have chosen to focus on constructions based on the lower central series due to the fact that the lower central series of (partitioned) surface braid groups and loop braid groups are well-understood (see for example [DPS21]), which helps us to understand the specific examples of homological representations fitting into this framework that we discuss in §5.2.

Functorial quotients of groups. More generally, it would suffice to fix any functorial choice of normal subgroup (rather than an entire functorial descending normal series), in other words, a functor $s \colon \mathrm{Grp} \to \mathrm{Grp}$ with the property that $s(G) \lhd G$ for all groups G and $s(\phi) = \phi|_G$ for any homomorphism $\phi \colon G \to H$ (so that the inclusions assemble into a natural transformation $s \Rightarrow \mathrm{id}$). We may then define $\gamma(M,A)$ and $\bar{\gamma}(M,A)$ in (5.4) by the natural transformation $G \mapsto G/s(G)$.

Invariant non-functorial quotients. The diagram (5.4) is functorial in $\mathfrak{U}\mathcal{D}_d$, so in particular there is an action of $\mathrm{Aut}_{\mathcal{D}_d}(M,A)$ on $Q(M,A)$. If we focus on the single object (M,A) and its automorphisms, we may wish to modify the construction so that the quotient of $\pi_1(X(M,A))$ has the *trivial* action of $\mathrm{Aut}_{\mathcal{D}_d}(M,A)$. There are two options:

- Restrict the automorphism group $\mathrm{Aut}_{\mathcal{D}_d}(M,A)$ to the kernel of its action on $Q(M,A)$; see §5.4.1.1 for a specific example of this.
- Replace $Q(M,A)$ by a further quotient that is invariant under the action of $\mathrm{Aut}_{\mathcal{D}_d}(M,A)$; see Corollary 5.37 for a specific example of this.

5.1.7. Pro-nilpotent representations

If we fix the inputs $Z \subseteq \mathbb{R}^d$ and $G \leqslant \mathrm{Diff}(Z)$ in our construction and allow the integer ℓ to vary, one may package together the homological representations arising from the general construction for each value of ℓ into a kind of "pro-nilpotent" homological representation, which may be truncated to recover the representation corresponding to each level ℓ. Namely, the resulting homological representations fit together into a tower, which one may think of as a single *pro-nilpotent representation*. In this section we explain how this may be done.

Remark 5.18 More generally, one may do this for any functorial descending normal series in place of the lower central series; in particular one may easily adapt this section to describe "pro-solvable" homological representations.

Definition 5.19 For an integer $\ell \geqslant 0$, let $\mathrm{Cov}_\bullet^{(\ell)}$ be the full subcategory of $\mathrm{Cov}_\bullet$ on those objects $(X, \pi_1(X) \twoheadrightarrow Q)$ where the group Q has nilpotency class at most ℓ. If $\ell = 0$ this means Q must be the trivial group, corresponding to the trivial covering of X. If $\ell = 1$, this means that Q must be abelian, so $\mathrm{Cov}_\bullet^{(1)}$ is the category of spaces equipped with abelian coverings.

Definition 5.20 Fix a unital ring $\mathbb{A}$ and an integer $\ell \geqslant 0$, and let $\mathrm{Mod}_\bullet^{(\ell)}$ be the full subcategory of $\mathrm{Mod}_\bullet$ on those objects (R,V) where $R = \mathbb{A}[Q]$ for a group Q of nilpotency class at most ℓ. Similarly, $\mathrm{Top}_\bullet^{(\ell)}$ is the full subcategory of $\mathrm{Top}_\bullet$ on those objects (X,R,ξ) where $R = \mathbb{A}[Q]$ for a group Q of nilpotency class at most ℓ (and ξ is a bundle of right R-modules over the space X). We also write $\mathrm{Mod}_\bullet^{(\infty)} \subset \mathrm{Mod}_\bullet$ for the full subcategory on objects (R,V) where $R = \mathbb{A}[Q]$ for any group Q, and similarly $\mathrm{Top}_\bullet^{(\infty)} \subset \mathrm{Top}_\bullet$.

Construction 5.21 For $k \geqslant 0$, there is a functor

$$\varpi_k \colon \mathrm{Cov}_\bullet \longrightarrow \mathrm{Cov}_\bullet^{(k)},$$

which is a section for the inclusion $\mathrm{Cov}_\bullet^{(k)} \subseteq \mathrm{Cov}_\bullet$, defined as follows. On objects, it is given by

$$(X, \pi_1(X) \twoheadrightarrow Q) \longmapsto (X, \pi_1(X) \twoheadrightarrow Q \twoheadrightarrow Q/\Gamma_k(Q)).$$

We recall that a morphism $(X, \pi_1(X) \twoheadrightarrow Q) \to (X', \pi_1(X') \twoheadrightarrow Q')$ is a based map $f \colon X \to X'$ with the property that $\pi_1(f)$ descends to a map $Q \to Q'$. Clearly this implies that it also descends to a map $Q/\Gamma_k(Q) \to Q'/\Gamma_k(Q')$. Hence we may define the functor on morphisms simply by $f \mapsto f$.

Construction 5.22 Similarly, for $k \geqslant 0$ there are functors

$$\varpi_k \colon \mathrm{Mod}_\bullet^{(\infty)} \longrightarrow \mathrm{Mod}_\bullet^{(k)} \qquad \text{and} \qquad \varpi_k \colon \mathrm{Top}_\bullet^{(\infty)} \longrightarrow \mathrm{Top}_\bullet^{(k)}$$

which are sections for the inclusions $\mathrm{Mod}_\bullet^{(k)} \subseteq \mathrm{Mod}_\bullet^{(\infty)}$ and $\mathrm{Top}_\bullet^{(k)} \subseteq \mathrm{Top}_\bullet^{(\infty)}$, defined as follows. The first is defined on objects by

$$(\mathbb{A}[Q], V) \longmapsto (\mathbb{A}[Q/\Gamma_k(Q)], V \otimes_{\mathbb{A}[Q]} \mathbb{A}[Q/\Gamma_k(Q)])$$

and the second is defined on objects by

$$(X, \mathbb{A}[Q], \xi \colon \pi_{\leqslant 1}(X) \to \mathrm{Mod}_{\mathbb{A}[Q]}) \longmapsto (X, \mathbb{A}[Q/\Gamma_k(Q)], \pi_{\leqslant 1}(X) \to \mathrm{Mod}_{\mathbb{A}[Q]} \to \mathrm{Mod}_{\mathbb{A}[Q/\Gamma_k(Q)]}),$$

where the functor $\mathrm{Mod}_{\mathbb{A}[Q]} \to \mathrm{Mod}_{\mathbb{A}[Q/\Gamma_k(Q)]}$ is given by $-\otimes_{\mathbb{A}[Q]}\mathbb{A}[Q/\Gamma_k(Q)]$. As in the construction above, it is easy to see how to extend these constructions to morphisms.

The functors $\mathrm{Lift} \colon \mathrm{Cov}_\bullet \to \mathrm{Top}_\bullet$ and $H_i \colon \mathrm{Top}_\bullet \to \mathrm{Mod}_\bullet$ respect the filtrations of these categories given by nilpotency class. With respect to the functors ϖ_k, they fit together as follows:

$$\begin{array}{ccccc} \mathrm{Cov}_\bullet & \xrightarrow{\ \mathrm{Lift}\ } & \mathrm{Top}_\bullet^{(\infty)} & \xrightarrow{\ H_i\ } & \mathrm{Mod}_\bullet^{(\infty)} \\ {\scriptstyle \varpi_k}\downarrow & /\!/ & {\scriptstyle \varpi_k}\downarrow & \Swarrow & \downarrow{\scriptstyle \varpi_k} \\ \mathrm{Cov}_\bullet^{(k)} & \xrightarrow[\ \mathrm{Lift}\ ]{} & \mathrm{Top}_\bullet^{(k)} & \xrightarrow[\ H_i\ ]{} & \mathrm{Mod}_\bullet^{(k)} \end{array} \tag{5.10}$$

The left-hand square commutes on the nose, but the second one does not. However, for any space X, local system $\mathcal{L}$ on X defined over a ring R and ring homomorphism $\theta \colon R \to S$, there is a natural homomorphism of S-modules

$$H_i(X; \mathcal{L}) \otimes_R S \longrightarrow H_i(X; \mathcal{L} \otimes_R S),$$

where S is viewed as an (R, S)-bimodule via θ. This homomorphism is the subject of the universal coefficient theorem (although we will not need more than its existence). This natural homomorphism provides the natural transformation filling the right-hand square as indicated.

Lemma 5.23 *The homological functor* $F_{(Z,G,\ell)} \colon \mathfrak{U}\mathcal{D}_d \to \mathrm{Cov}_\bullet$ *has image contained in* $\mathrm{Cov}_\bullet^{(\ell)}$.

Proof. This follows from the fact that, in diagram (5.4), the middle group on the bottom row has nilpotency class at most ℓ by construction. The property of having nilpotency class at most ℓ passes to subgroups, so $Q(M, A)$ also has nilpotency class at most ℓ. □

This means that we may consider the following triangle, where $\ell \geqslant k$.

$$\begin{array}{ccc} & \overset{F_{(Z,G,\ell)}}{\nearrow} & \mathrm{Cov}_\bullet^{(\ell)} \\ \mathfrak{U}\mathcal{D}_d & & \downarrow{\scriptstyle \varpi_k} \\ & \underset{F_{(Z,G,k)}}{\searrow} & \mathrm{Cov}_\bullet^{(k)} \end{array} \tag{5.11}$$

Lemma 5.24 *There is a natural transformation* $\tau\colon \varpi_k \circ F_{(Z,G,\ell)} \Rightarrow F_{(Z,G,k)}$ *filling the triangle* (5.11).

Proof. The functor going clockwise around (5.11) sends the object (M, A) of $\mathfrak{U}\mathcal{D}_d$ to the space $X(M, A)$ together with the quotient

$$\varpi_k \circ \phi_\ell(M,A)\colon \pi_1(X(M,A)) \longrightarrow Q_\ell(M,A) \longrightarrow Q_\ell(M,A)/\Gamma_k(Q_\ell(M,A)), \tag{5.12}$$

where $\phi_\ell(M, A)$ is defined in diagram (5.4). The functor going anticlockwise around (5.11) sends the object (M, A) to the same space $X(M, A)$ together with the quotient

$$\phi_k(M,A)\colon \pi_1(X(M,A)) \longrightarrow Q_k(M,A). \tag{5.13}$$

We recall that a morphism in $\mathrm{Cov}_\bullet^{(k)}$ is a based map of spaces f having the property that $\pi_1(f)$ descends to a homomorphism between the respective quotients. We define the natural transformation $\tau_{(M,A)}$ to be the identity map $X(M, A) \to X(M, A)$. To see that this really is a morphism, we have to check that the quotient (5.13) factors through the quotient (5.12). Naturality will then be clear. First, note that the quotient of $\pi_1(X(M, A))$ onto $Q_k(M, A)$ factors through its quotient onto $Q_\ell(M, A)$, by the construction of these quotients in the diagram (5.4), so we have a quotient

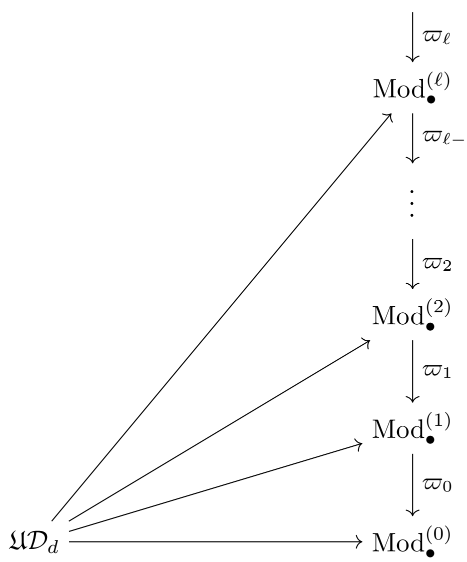
$$Q_\ell(M,A) \longrightarrow Q_k(M,A).$$

But its target is nilpotent of class at most k, so the subgroup $\Gamma_k(Q_\ell(M, A))$ must be sent to zero under this quotient. Hence it factors further through the quotient onto $Q_\ell(M, A)/\Gamma_k(Q_\ell(M, A))$, as required. □

Putting together the diagrams (5.11) and (5.10) (with its top row restricted to the subcategories $(-)^{(\ell)}$), we obtain:

$$\begin{array}{ccccccc} & \overset{F_{(Z,G,\ell)}}{\nearrow} & \mathrm{Cov}_\bullet^{(\ell)} & \xrightarrow{\mathrm{Lift}} & \mathrm{Top}_\bullet^{(\ell)} & \xrightarrow{H_i} & \mathrm{Mod}_\bullet^{(\ell)} \\ \mathfrak{U}\mathcal{D}_d & \Downarrow_\tau & \downarrow \varpi_k & /\!\!/ & \downarrow \varpi_k & \swarrow & \downarrow \varpi_k \\ & \underset{F_{(Z,G,k)}}{\searrow} & \mathrm{Cov}_\bullet^{(k)} & \xrightarrow[\mathrm{Lift}]{} & \mathrm{Top}_\bullet^{(k)} & \xrightarrow[H_i]{} & \mathrm{Mod}_\bullet^{(k)} \end{array} \tag{5.14}$$

Composing these functors and pasting together the natural transformations, this gives us a tower of homological representations $L_i(F_{(Z,G,\ell)})$ of the category $\mathfrak{U}\mathcal{D}_d$:



(5.15)

We emphasise that this diagram does not commute. Instead, the natural transformations of (5.14) paste together to give a natural transformation going "downwards" filling each of the triangles in (5.15).

Definition 5.25 The tower (5.15) of homological representations of $\mathfrak{U}\mathcal{D}_d$ is called the *pro-nilpotent homological representation* of $\mathfrak{U}\mathcal{D}_d$ associated to the input data (Z, G).

To justify this name, note that, if we restrict to a single object (M, A) of $\mathfrak{U}\mathcal{D}_d$, we obtain a tower of (possibly twisted) representations of the mapping class group $\pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, A))$ defined over the tower of groups $Q_\bullet(M, A)$. Since the group $Q_\ell(M, A)$ has nilpotency class at most ℓ by construction, the inverse limit of this tower of groups is a pro-nilpotent group.

Recall from §5.1.5 above that if the functor $F_{(Z,G,\ell)}$ is *Q-stable* on a subcategory $\langle \mathcal{G}, \mathcal{M} \rangle \subseteq \mathfrak{U}\mathcal{D}_d$ (in the sense of Definition 5.14), then the homological representation $L_i(F_{(Z,G,\ell)})$ restricted to $\langle \mathcal{G}, \mathcal{M} \rangle$ takes values in a subcategory of $\mathrm{Mod}_\bullet$ of the form $\mathrm{Mod}^{\mathrm{tw}}_{Q_\ell}$ for some fixed group Q_ℓ. Hence also in this case we have a tower of (possibly twisted) representations over a tower of groups $Q_\bullet$ whose inverse limit is a pro-nilpotent group.

In many specific examples, the tower of groups $Q_\bullet$ actually stops at a finite stage, so we only have finitely many different representations in the corresponding tower. However, there are *also* many examples in which the tower of groups $Q_\bullet$ does *not* stop, and hence we have a tower of representations that becomes richer at every stage. The question of whether and when the tower $Q_\bullet$ stops is investigated in many important examples related to surface braid groups and loop braid groups in the article [DPS21] by Darné and the authors.

Remark 5.26 In all of the discussion of this section, we may replace the ordinary twisted homology functor H_i with Borel-Moore twisted homology H_i^{BM}, and everything goes through in exactly the same way. In this case, of course, we also have to restrict the category $\mathrm{Top}_\bullet$ to its subcategory of proper maps.

5.2. Applications for motion groups

We apply the general construction of §5.1 to families of motion groups (*cf.* Definition 4.22) of the type $\{\mathrm{Mot}_{Y_n}(M) \mid n \in \mathbb{N}\}$, where M is a manifold of dimension $d \in \{2, 3\}$ and Y_n is a closed submanifold of M. For this purpose, we consider the restriction of the continuous functor $\mathring{F}_{(Z,G,\ell)} \colon \mathfrak{U}\mathcal{D}_d \to \mathrm{Cov}_\bullet$ of Proposition 5.8 to the appropriate full subcategory of the form $\langle \mathcal{G}, \mathcal{M} \rangle$ whose automorphism groups correspond to the family of motion groups: these are given by Lemma 3.5 applied to the examples of subgroupoids $\mathcal{G}$ and $\mathcal{M}$ described in §4.5–4.6. We will restrict to the following particular choices for the type of submanifold Z and the group G as inputs for the construction:

- Z is either a finite set $\underline{k}$ of size $k \geqslant 1$ in $\mathbb{R}^d$ or an unlink $\underline{k}\mathbb{S}^1$ with k components in $\mathbb{R}^3$;
- $G = \mathfrak{S}_k$ (when $Z = \underline{k}$) or $G = \mathrm{Diff}(\underline{k}\mathbb{S}^1)$ or $\mathrm{Diff}^+(\underline{k}\mathbb{S}^1)$ (when $Z = \underline{k}\mathbb{S}^1$).

Furthermore, the possibilities on the parameter ℓ on the lower central series degree will generally be restricted to $\ell \leqslant 3$, because the lower central series of the considered motion groups generally stop at Γ_2 or Γ_3. Taking the ground ring $\mathbb{A}$ to be $\mathbb{Z}$, the construction of §2.5.1 (or Definition 2.20 and Remark 2.23) produces homological functors for all $i \geqslant 1$:

$$L_i(\mathring{F}_{(Z,G,\ell)}) \colon \pi_0\langle \mathcal{G}, \mathcal{M} \rangle \longrightarrow \mathrm{Mod}_\bullet. \tag{5.16}$$

For $\ell = 0$, we obtain the representations corresponding to the action on the homology groups of the universal covers of the spaces $X(M, Y_n)$, and we recover the natural action of the motion groups on the homology groups of the spaces $X(M, Y_n)$ for $\ell = 1$; *cf.* Remark 5.6. For $\ell = 2$ or $\ell = 3$, as a consequence of some abelianisation computations, we will typically show that $Q_{(Z,G,\ell)}(X(M, Y_n))$ does not depend on n for $n \geqslant \mu_{(Z,G,\ell)}$, and denote this quotient of by $Q_{(Z,G,\ell)}$. In these cases we will deduce the following meta-proposition:

Proposition 5.27 *For all $i \geqslant 0$, trivialising the assignment for $n < \mu_{(Z,G,\ell)}$, the functor* (5.16) *defines homological functors*

$$L_i(\mathring{F}_{(Z,G,2)}) \colon \pi_0\langle \mathcal{G}, \mathcal{M} \rangle \longrightarrow \mathrm{Mod}_{\mathbb{Z}[Q_{(Z,G,2)}]} \quad \textit{and} \quad L_i(\mathring{F}_{(Z,G,3)}) \colon \pi_0\langle \mathcal{G}, \mathcal{M} \rangle \longrightarrow \mathrm{Mod}^{\mathrm{tw}}_{\mathbb{Z}[Q_{(Z,G,3)}]}.$$

Proof. Let $\langle \mathcal{G}, \mathcal{M} \rangle_{\geqslant \mu_{(Z,G,\ell)}}$ denote the full subcategory of $\langle \mathcal{G}, \mathcal{M} \rangle$ on all objects $n \geqslant \mu_{(Z,G,\ell)}$. By Proposition 5.16, the functor (5.16) restricts to

$$L_i(\mathring{F}_{(Z,G,2)}) \colon \pi_0\langle \mathcal{G}, \mathcal{M} \rangle_{\geqslant \mu_{(Z,G,2)}} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[Q_{(Z,G,2)}]}$$
$$L_i(\mathring{F}_{(Z,G,3)}) \colon \pi_0\langle \mathcal{G}, \mathcal{M} \rangle_{\geqslant \mu_{(Z,G,3)}} \longrightarrow \mathrm{Mod}^{\mathrm{tw}}_{\mathbb{Z}[Q_{(Z,G,3)}]}$$

for $\ell \in \{2,3\}$. By Lemma 5.17, we may extend them to functors over $\langle \mathcal{G}, \mathcal{M}\rangle$ by sending the objects $n < \mu_{(Z,G,\ell)}$ to the trivial module. $\square$

General alternatives: partitioned configurations. For all of the constructions presented in §5.2.1–§5.2.3, we can also define interesting alternatives by taking $G = 0_{\mathrm{Grp}}$ the trivial group and thus obtain $L_i(\mathring{F}_{(Z,0_{\mathrm{Grp}},2)})\colon \pi_0\langle \mathcal{G}, \mathcal{M}\rangle \to \mathrm{Mod}_\bullet$ and $L_i(\mathring{F}_{(Z,0_{\mathrm{Grp}},k)})\colon \pi_0\langle \mathcal{G}, \mathcal{M}\rangle \to \mathrm{Mod}_\bullet$ for each $i \geqslant 0$ and $k \geqslant 1$. This corresponds to taking *ordered* configuration spaces for the parameter $X(M,A)$ of §5.1.1 instead of *unordered* configuration spaces as done in §5.2.1–§5.2.3.

More generally, interpolating the parameter G between 0_{Grp} and $\mathrm{Diff}(Z)$, one may in particular take it such that we consider *partitioned* configurations. The associated lower central series of the corresponding fundamental group is in general more complicated: in particular, it does not always stop and therefore more sophisticated representations arise from these alternatives. This is illustrated in Remark 5.29 for the case of classical braid groups, and may be adapted easily for the other examples addressed in this paper. We do not detail all of these partitioned cases here for sake of simplicity; the relevant, more sophisticated transformation groups may be computed from the results of [DPS21].

5.2.1. Classical braid groups

We consider the restriction of the continuous functor $\mathring{F}_{(Z,G,\ell)}\colon \mathfrak{U}\mathcal{D}_2 \to \mathrm{Cov}_\bullet$ of Proposition 5.8 to the full subcategory $\mathfrak{U}\mathcal{B}r^{\mathbb{D}^2}$. Thus we have $M = \mathbb{D}^2$ the unit 2-disc and $Y_n = \underline{n}$ a set of $n \geqslant 0$ distinct points in its interior. Moreover, we take $Z = \underline{k}$ a set of $k \geqslant 1$ points in $\mathbb{R}^2$ and $G = \mathfrak{S}_k$. These choices determine a homological functor

$$L_i(\mathring{F}_{(\underline{k},\mathfrak{S}_k,\ell)})\colon \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathrm{Mod}_\bullet, \tag{5.17}$$

for any non-negative integers $i, k \geqslant 1$ and $\ell \geqslant 0$. We recall from [DPS21, Theorem 3.5] that:

- $\Gamma_2(\mathbf{B}_{k,n}) = \Gamma_3(\mathbf{B}_{k,n})$ for $k \in \mathbb{N} \smallsetminus \{2\}$ and $n \geqslant 3$;
- $\Gamma_i(\mathbf{B}_{2,n}) \neq \Gamma_{i+1}(\mathbf{B}_{2,n})$ for all $i \geqslant 1$.

Therefore, if $k \neq 2$, the construction produced for each $\ell \geqslant 2$ is equivalent to the one for $\ell = 2$, which is detailed in §5.2.1.1 below. In contrast, when $k = 2$ each choice of $\ell \geqslant 2$ provides a different construction, see §5.2.1.2, and these fit together into a *pro-nilpotent tower* as explained in §5.1.7.

5.2.1.1. Standard situation of the abelianisation

As a direct consequence of the computation for the abelianisations of partitioned braid groups of [DPS21, Proposition 3.4], we have that for all $n \geqslant 2$, $Q_{(\underline{k},\mathfrak{S}_k,2)}(\mathbb{D}_n) = \mathbb{Z}$ if $k = 1$ and $\mathbb{Z}^2$ if $k \geqslant 2$. We denote these quotients of by Q_k and deduce from Proposition 5.27:

Proposition 5.28 *For each $i \geqslant 0$ and $k \geqslant 1$, the functor (5.17) for $\ell = 2$ defines a homological functor $L_i(\mathring{F}_{(\underline{k},\mathfrak{S}_k,2)})\colon \mathfrak{U}\boldsymbol{\beta} \to \mathrm{Mod}_{\mathbb{Z}[Q_k]}$ trivialising the assignment for $n \leqslant 1$.*

Remark 5.29 Let $\mathbf{k} = (k_1, ..., k_r)$ be a partition of k with $k_1 = \cdots = k_s = 1$ and $k_{s+1}, \ldots, k_r \geqslant 2$. If we take $G = \mathfrak{S}_{\mathbf{k}} = \mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_r}$ instead of $\mathfrak{S}_k$, it follows from the computation of [DPS21, Proposition 3.4] that $Q_{(\underline{k},\mathfrak{S}_{\mathbf{k}},2)}(\mathbb{D}_n) = \mathbb{Z}^{(r(r+3)/2)-s}$ for all $n \geqslant 2$. We denote these quotients by $Q_{\mathbf{k}}$, and Proposition 5.8 with $\ell = 2$ defines a homological functor $L_i(\mathring{F}_{(\underline{k},\mathfrak{S}_{\mathbf{k}},2)})\colon \mathfrak{U}\boldsymbol{\beta} \to \mathrm{Mod}_{\mathbb{Z}[Q_{\mathbf{k}}]}$ trivialising the assignment for $n \leqslant 1$. These alternatives do not appear yet in the literature.

Bigelow's construction and the Lawrence-Bigelow representations. Bigelow introduced in [Big04] a general method to construct representations of all braid groups $\mathbf{B}_n$ starting from a representation of a particular braid group $\mathbf{B}_k$ for a fixed integer k. As a special case, it builds the well-known families of *Lawrence-Bigelow representations*, originally introduced by Lawrence [Law90] as representations of Hecke algebras. The most famous among these are the Burau representations, originally introduced in [Bur35], and the *Lawrence-Krammer-Bigelow* representations that Bigelow [Big01] and Krammer [Kra02] independently proved to be faithful. We will show below that the Lawrence-Bigelow representations are recovered by the constructions of the present section; see Theorem 5.31.

Remark 5.30 In fact, Bigelow's more general construction, which includes the Lawrence-Bigelow representations as a special case, is also recovered by our construction, using its more general version in §2.5.2 (incorporating a functorial fibrewise tensor product), instead of §2.5.1.[4] Fix an associative, unital ring R and an integer $k \geqslant 1$. We consider the following ingredients:

- The functor
$$\mathrm{Big}\colon \mathfrak{U}\mathcal{B}r^{\mathbb{D}^2} \longrightarrow {}_{\bullet}\mathrm{Cov}_{\bullet}$$
that sends n to $C_k(\mathbb{D}_n)$ together with the jointly surjective pair of quotients $\mathcal{W}\colon \mathbf{B}_k(\mathbb{D}_n) \to \mathbb{Z}$ (counting the total winding number of k points around n punctures) and $\mathcal{T}\colon \mathbf{B}_k(\mathbb{D}_n) \to \mathbf{B}_k$ (induced by the inclusion of $\mathbb{D}_n$ into $\mathbb{D}^2$). Recall from Proposition 4.8 that the morphism spaces of $\mathfrak{U}\mathcal{B}r^{\mathbb{D}^2}$ are certain embedding spaces $\mathrm{Emb}_{\mathfrak{U}\mathcal{B}r^{\mathbb{D}^2}}(\mathbb{D}_m, \mathbb{D}_n)$, whose elements induce maps of configuration spaces. One straightforwardly checks from the definitions that these maps commute with the quotients $\mathcal{W}$ and $\mathcal{T}$, so they preserve their kernels, making Big a valid functor into ${}_{\bullet}\mathrm{Cov}_{\bullet}$.
- Choose a left $R[\mathbf{B}_k]$-module V and let
$$\mathbf{V}\colon \mathfrak{U}\mathcal{B}r^{\mathbb{D}^2} \longrightarrow {}_{\bullet}\mathrm{Mod}_{\bullet}$$
be the constant functor at $(R[\mathbf{B}_k], R, V)$.

Using Definition 2.20, Bigelow's construction [Big04, §2] corresponds to the functor
$$L_k(\mathrm{Big}; \mathbf{V})\colon \mathfrak{U}\boldsymbol{\beta} \longrightarrow {}_{\bullet}\mathrm{Mod}_{\bullet}.$$
In other words, the *n-th term of Bigelow's construction* applied to the $\mathbf{B}_k$-representation V is the representation of $\mathbf{B}_n$ on the $R[\mathbb{Z}]$-module $L_k(\mathrm{Big}; \mathbf{V})(n)$.[5]

Assigning $R = \mathbb{Z}$ and $V = \mathbb{Z}[\mathbb{Z}]$ to be the $\mathbf{B}_k$-representation where $\mathbf{B}_k$ acts on $\mathbb{Z}$ through the abelianisation and left-multiplication, the functor $L_k(\mathrm{Big}; \mathbb{Z}[\mathbb{Z}])$ encodes the *k-th Lawrence-Bigelow* representations. In particular, the representations encoded in $L_1(\mathrm{Big}; \mathbb{Z}[\mathbb{Z}])$ are the *reduced Burau* representations, and those encoded in $L_2(\mathrm{Big}; \mathbb{Z}[\mathbb{Z}])$ are the *Lawrence-Krammer-Bigelow* representations; see [Big03; Big01]. By [PP02, Theorem 1.2], the tensor product $L_2(\mathrm{Big}; \mathbb{Z}[\mathbb{Z}]) \otimes_{\mathbb{Z}[\mathbb{Z}^2]} \mathbb{Q}(\mathbb{Z}^2)$ with the field of fractions $\mathbb{Q}(\mathbb{Z}^2)$ of $\mathbb{Z}[\mathbb{Z}^2]$ is isomorphic to the Lawrence-Krammer representation [Kra02] tensored with $\mathbb{Q}(\mathbb{Z}^2)$.

The following result may be verified directly by unwinding the constructions.

Theorem 5.31 *For each $k \geqslant 1$, the functor (5.17) with $i = k$ and $\ell = 2$ is isomorphic to the functor $L_k(\mathrm{Big}; \mathbb{Z}[\mathbb{Z}])$, and thus recovers the Lawrence-Bigelow representations on automorphism groups. We denote it $\mathfrak{LB}_k\colon \mathfrak{U}\boldsymbol{\beta} \to \mathrm{Mod}_{\mathbb{Z}[Q_k]}$ and call it the k-th* Lawrence-Bigelow functor.

5.2.1.2. Exceptional situation for $k = 2$

Recall from [DPS21, Theorem 3.5] that $\Gamma_i(\mathbf{B}_{2,n}) \neq \Gamma_{i+1}(\mathbf{B}_{2,n})$ for all $i \geqslant 1$. Applying the procedure of §5.1 thus provides an *infinite* pro-nilpotent tower of homological representations, i.e. a different homological representation for each $\ell \geqslant 2$.

We fix some $\ell \geqslant 3$ and $n \geqslant 2$. We consider the variant of the global homological representation construction using Borel-Moore homology and we restrict to taking the second homology group. The adaptation of the diagram (5.14) using Borel-Moore homology (*cf.* Remark 5.26) defines a $\mathbf{B}_n$-equivariant map $\eta_{\ell,n}\colon L_2^{BM}(F_{(\underline{2},\mathfrak{S}_2,\ell)})(n) \to L_2^{BM}(F_{(\underline{2},\mathfrak{S}_2,2)})(n)$.

A general property of [Big04, Lemma 3.1] describes the Borel-Moore homology of a configuration space of points in the punctured disc and thus provides isomorphisms
$$L_2^{BM}(F_{(\underline{2},\mathfrak{S}_2,\ell)})(n) \cong \mathbb{Z}[Q_{\ell,n}]^{\oplus n(n-1)/2} \text{ and } L_2^{BM}(F_{(\underline{2},\mathfrak{S}_2,2)})(n) \cong \mathbb{Z}[\mathbb{Z}^2]^{\oplus n(n-1)/2}.$$
Moreover, under these identifications, the map $\eta_{\ell,n}$ is the (necessarily surjective) map induced by a fixed quotient of groups $Q_{\ell,n} \twoheadrightarrow \mathbb{Z}^2$ on each summand.

[4] In fact, we use the mild generalisation of the construction of §2.5.2 mentioned at the beginning of Remark 2.25.

[5] In fact, to exactly recover Bigelow's definition in [Big04, §2], one should not use the version of our construction in which we apply ordinary (twisted) homology, but instead take the quotient of ordinary (twisted) homology given by its image in Borel-Moore (twisted) homology relative to the boundary; *cf.* Remark 2.21.

Faithfulness results. We first note the following general property on the faithfulness of a representation and its quotients.

Lemma 5.32 *Let (ρ, V) be a representation of a group G over a ring R. If any quotient (μ, W) of (ρ, V) is faithful, then so is (ρ, V).*

Proof. We denote by $\alpha\colon (\rho, V) \to (\mu, W)$ the quotient G-equivariant map. Let g be an element of G such that $\rho(g) = \mathrm{id}_V$. Then $\mu(g) \circ \alpha = \alpha$ and therefore $\mu(g) = \mathrm{id}_W$ since α is surjective. We deduce from the faithfulness of (μ, W) that $g = \mathrm{id}_G$, which ends the proof. ☐

Recall from [Big02, §4] that for each n, the $\mathbf{B}_n$-representation $L_2^{BM}(F_{(\underline{2},\mathfrak{S}_2,2)})(n)$ is faithful.

Corollary 5.33 *For each $\ell \geqslant 3$, the functor $L_2^{BM}(F_{(\underline{2},\mathfrak{S}_2,\ell)})$ encodes a faithful linear representation of $\mathbf{B}_n$ for each n.*

5.2.2. Surface braid groups

Let S be a compact, connected surface with boundary different from the 2-disc: it is therefore isomorphic to either $\Sigma_{g,1}$ or $\mathcal{N}_{h,1}$ for some g or $h \geqslant 1$. We consider the restriction of the continuous functor $\mathring{F}_{(Z,G,\ell)}\colon \mathfrak{U}\mathcal{D}_2 \to \mathrm{Cov}_\bullet$ of Proposition 5.8 to the full subcategory $\langle \mathcal{B}r^{\mathbb{D}^2}, \mathcal{B}r^S \rangle$. Thus we have $M = S$ and $Y_n = \underline{n}$ a set of $n \geqslant 0$ distinct points in its interior. Moreover, we take $Z = \underline{k}$ a set of $k \geqslant 1$ points in $\mathbb{R}^2$ and $G = \mathfrak{S}_k$. These choices determine a homological functor

$$L_i(\mathring{F}_{(\underline{k},\mathfrak{S}_k,\ell)})\colon \langle \boldsymbol{\beta}, \boldsymbol{\beta}^S \rangle \longrightarrow \mathrm{Mod}_\bullet \tag{5.18}$$

for any non-negative integers $i, k \geqslant 1$ and $\ell \geqslant 0$. We recall from [DPS21, Theorem 6.27, Propositions 6.35 and 6.41] that for any $n \geqslant 3$ and for $S = \Sigma_{g,1}$ or $\mathcal{N}_{h,1}$:

- $\Gamma_2(\mathbf{B}_{k,n}(S)) \neq \Gamma_3(\mathbf{B}_{k,n}(S)) = \Gamma_4(\mathbf{B}_{k,n}(S))$ if $k \geqslant 3$;
- $\Gamma_\ell(\mathbf{B}_{1,n}(S) \neq \Gamma_{\ell+1}(\mathbf{B}_{1,n}(S)$ if $S \neq \mathcal{N}_{1,1}$ and $\Gamma_\ell(\mathbf{B}_{2,n}(S) \neq \Gamma_{\ell+1}(\mathbf{B}_{2,n}(S)$ for all $\ell \geqslant 1$.

Hence we study the construction for $\ell = 2$ in §5.2.2.1 in all the situations and then we consider the further situation of $\ell = 3$ for orientable surfaces in §5.2.2.2. As in §5.2.1.2, the procedure of §5.1 for the particular parameters $k \in \{1, 2\}$ provides an *infinite* pro-nilpotent tower of homological representations, i.e. a different homological representation for each $\ell \geqslant 2$, which will however not be addressed in detail here.

5.2.2.1. Standard situation of the abelianisation

We deduce from the computation for the abelianisations of partitioned surface braid groups of [DPS21, Proposition 6.23] that for all $k \geqslant 1$, if $n \geqslant 2$ then $Q_{(\underline{k},\mathfrak{S}_k,2)}(S^{(n)}) \cong \mathbb{Z}^{p_S} \oplus (\mathbb{Z}/2)^{d_k}$ where

$$p_S = \begin{cases} 2g & \text{if } S = \Sigma_{g,1}, \\ h & \text{if } S = \mathcal{N}_{h,1}, \end{cases} \text{ and } d_k = \begin{cases} 0 & \text{if } k = 1, \\ 1 & \text{if } k \geqslant 2. \end{cases}$$

We denote this quotient by $Q_{(k,2)}(S)$ and deduce from Proposition 5.27 that:

Proposition 5.34 *For each $i \geqslant 0$ and $k \geqslant 1$, the functor (5.18) for $\ell = 2$ defines a homological functor $L_i(\mathring{F}_{(\underline{k},\mathfrak{S}_k,2)})\colon \langle \boldsymbol{\beta}, \boldsymbol{\beta}^S \rangle \to \mathrm{Mod}_{\mathbb{Z}[Q_{(k,2)}(S)]}$ trivialising the assignment for $n \leqslant 1$.*

5.2.2.2. Situation of $\ell = 3$

First of all, we note the following general result.

Lemma 5.35 *For all $k \geqslant 1$, if $n \geqslant 3$ then $Q_{(\underline{k},\mathfrak{S}_k,3)}(S^{(n)}) = Q_{(\underline{k},\mathfrak{S}_k,3)}(S^{(n+1)})$. For each k, we denote the common quotient by $Q_{(k,3)}(S)$.*

Proof. We fix $k \geqslant 0$ and $n \geqslant 3$. The presentation of the group $\mathbf{B}_{k,n}(S) \cong \mathbf{B}_k(S^{(n)}) \rtimes \mathbf{B}_n(S)$ is detailed in [BGG17, Proposition 3.2] for $S = \Sigma_{g,1}$, and for $S = \mathcal{N}_{h,1}$, the presentation is easily deduced from the presentation of $\mathbf{B}_n(\mathcal{N}_{h,1})$ (see [DPS21, Proposition A.15]) and the presentation of $B_n(\mathcal{N}_{h,1}^n)$ (see [DPS21, Proposition A.16]) using the method of presentation of an extension of [DPS21, §A.3]; see the proof of [DPS21, Proposition 6.31] for further details. The key properties of these presentations which we will now use for our proof are the same for orientable and non-orientable surfaces. We thus take up the notations and conventions of [BGG17, Proposition 3.2]. More precisely, we note that the elements that depend on n in these presentations are the set of braid generators $\{\sigma_1, \ldots, \sigma_{n-1}\}$ of $\mathbf{B}_n(S)$ and the set of the pure braid generators $\{\xi_1, \ldots, \xi_n\}$ of $\mathbf{B}_k(S^{(n)})$. Because of the commutativity of (5.4), we abuse the notation γ_3 to denote the projections onto the 2-nilpotent quotients.

First, since $\gamma_3([\sigma_i, [\sigma_{i+1}, \sigma_i]]) = 1$, it follows from the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ that $\gamma_3(\sigma_i) = \gamma_3(\sigma_{i+1})$ for all $i \in \{1, \ldots, n-2\}$. Moreover, we have the relations $\sigma_i\xi_i\sigma_i^{-1} = \xi_i^{-1}\xi_{i+1}\xi_i$, $\sigma_i\xi_{i+1}\sigma_i^{-1} = \xi_i$ and $\sigma_i\xi_j\sigma_i^{-1} = \xi_j$ if $j \notin \{i, i+1\}$; see [BGG17, Proposition 3.2, relations (c.3)] or the proof of [DPS21, Proposition 6.31] for further details. Then, since $\gamma_3(\sigma_i) = \gamma_3(\sigma_{i+1}) = \gamma_3(\sigma_{i+2})$, we deduce that $\gamma_3(\xi_{i+1}) = \gamma_3(\sigma_i^{-1}\xi_i\sigma_i) = \gamma_3(\sigma_{i+2}^{-1}\xi_i\sigma_{i+2}) = \gamma_3(\xi_i)$ for all $i \in \{1, \ldots, n-2\}$. Hence the presentation of $\mathbf{B}_{k,n}(S)/\mathit{\Gamma}_3(\mathbf{B}_{k,n}(S))$ is independent of n. In particular, this holds for $k = 0$, so $\mathbf{B}_n(S)/\mathit{\Gamma}_3(\mathbf{B}_n(S))$ is also independent of n. Thus the canonical morphism $\mathbf{B}_{k,n}(S)/\mathit{\Gamma}_3(\mathbf{B}_{k,n}(S)) \twoheadrightarrow \mathbf{B}_n(S)/\mathit{\Gamma}_3(\mathbf{B}_n(S))$ is independent of n for each $k \geqslant 1$, and hence so is its kernel $Q_{(\underline{k},\mathfrak{S}_k 3)}(S^{(n)})$. □

For $k, n \geqslant 3$, explicit calculations of $\mathbf{B}_{k,n}(S)/\mathit{\Gamma}_3(\mathbf{B}_{k,n}(S))$ and $\mathbf{B}_n(S)/\mathit{\Gamma}_3(\mathbf{B}_n(S))$ are provided by [BGG17, Corollaries 3.9(i) and 3.14(i)] for $S = \Sigma_{g,1}$ and by [DPS21, Propositions 6.9 and 6.31] for $S = \mathcal{N}_{h,1}$. In particular, for $k \geqslant 3$, we deduce that $Q_{(k,3)}(\Sigma_{g,1})$ is isomorphic to $\mathbb{Z} \times (\mathbb{Z}^{g+1} \rtimes \mathbb{Z}^g)$ and that $Q_{(k,3)}(\mathcal{N}_{h,1})$ is isomorphic to $\mathbb{Z}^{1+h} \oplus \mathbb{Z}/2\mathbb{Z}$. It then follows from Proposition 5.27 that:

Proposition 5.36 *For each $i \geqslant 0$ and $k \geqslant 1$, the functor (5.18) for $\ell = 3$ provides the homological functors (trivialising the assignment for $n \leqslant 2$):*

$$L_i(\mathring{F}_{(\underline{k},\mathfrak{S}_k,3)}) \colon \langle \boldsymbol{\beta}, \boldsymbol{\beta}^{\Sigma_{g,1}} \rangle \longrightarrow \mathrm{Mod}^{\mathrm{tw}}_{\mathbb{Z}[Q_{(k,3)}(\Sigma_{g,1})]} \quad \textit{for each } g \geqslant 1$$
$$L_i(\mathring{F}_{(\underline{k},\mathfrak{S}_k,3)}) \colon \langle \boldsymbol{\beta}, \boldsymbol{\beta}^{\mathcal{N}_{h,1}} \rangle \longrightarrow \mathrm{Mod}^{\mathrm{tw}}_{\mathbb{Z}[Q_{(k,3)}(\mathcal{N}_{h,1})]} \quad \textit{for each } h \geqslant 1.$$

In order to obtain *untwisted* representations, in other words a homological functor taking values in a category of the form $\mathrm{Mod}_{\mathbb{Z}[Q]}$ rather than $\mathrm{Mod}^{\mathrm{tw}}_{\mathbb{Z}[Q]}$, we need to take a quotient of $Q_{(k,3)}(S)$ on which the natural action of each $\mathbf{B}_n(S)$ is trivial (for $n \geqslant 3$).

Corollary 5.37 *For $k \geqslant 3$, we obtain homological functors for $S = \Sigma_{g,1}$ or $\mathcal{N}_{h,1}$ for each $g, h \geqslant 1$*

$$\langle \boldsymbol{\beta}, \boldsymbol{\beta}^S \rangle \longrightarrow \mathrm{Mod}_{\mathbb{Z}[Q^{\mathrm{u}}_{(k,3)}(S)]}$$

where $Q^{\mathrm{u}}_{(k,3)}(\Sigma_{g,1}) = \mathbb{Z}^{g+1} \rtimes \mathbb{Z}^g$ and $Q^{\mathrm{u}}_{(k,3)}(\mathcal{N}_{h,1}) = \mathbb{Z}^h \oplus \mathbb{Z}/2\mathbb{Z} = Q_{(k,2)}(\mathcal{N}_{h,1})$. In particular, for non-orientable surfaces, this representation is nothing but the one introduced in Proposition 5.34 defined by taking $\ell = 2$. These quotients are optimal in the sense sense that any other quotient of $Q_{(k,3)}(S)$ giving untwisted representations is a quotient of $Q^{\mathrm{u}}_{(k,3)}(S)$.

Proof. For $k \geqslant 3$, there is a $\mathbb{Z}$ summand in the quotient group $Q_{(k,3)}(S)$, generated by (the image of) the surface pure braid ξ where one of the first k strands winds once around one of the last n strands. For $n \geqslant 3$, it follows from the presentations of $\mathbf{B}_{k,n}(S)/\mathit{\Gamma}_3(\mathbf{B}_{k,n}(S))$ (given by [BGG17, Corollaries 3.9(i) and 3.14(i)] for $S = \Sigma_{g,1}$ and by [DPS21, Propositions 6.9 and 6.31] for $S = \mathcal{N}_{h,1}$) that the coinvariants of the action of $\mathbf{B}_n(S)$ on $Q_{(k,3)}(S)$ are given by killing the $\mathbb{Z}$ summand generated by ξ. □

Conjecture 5.38 For each $g \geqslant 1$, we conjecture that $\mathbf{B}_{1,n}(\Sigma_{g,1})/\mathit{\Gamma}_3(\mathbf{B}_{1,n}(\Sigma_{g,1})) \cong (\mathbb{Z}^2 \times \mathbb{Z}^{2g}) \rtimes \mathbb{Z}^{2g}$ and $\mathbf{B}_{2,n}(\Sigma_{g,1})/\mathit{\Gamma}_3(\mathbf{B}_{2,n}(\Sigma_{g,1})) \cong (\mathbb{Z}^3 \times \mathbb{Z}^{2g}) \rtimes \mathbb{Z}^{2g}$ for all $n \geqslant 3$. Therefore $Q_{(1,3)}(\Sigma_{g,1})$ would be equal to $\mathbb{Z}^{2g+1}$ and $Q_{(2,3)}(\Sigma_{g,1})$ would be equal to $\mathbb{Z} \times (\mathbb{Z}^{g+1} \rtimes \mathbb{Z}^g)$.

For each $h \geqslant 1$, we conjecture that $\mathbf{B}_{1,n}(\mathcal{N}_{h,1})/\mathit{\Gamma}_3(\mathbf{B}_{1,n}(\mathcal{N}_{h,1})) \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z}^h) \rtimes \mathbb{Z}^h$ and $\mathbf{B}_{2,n}(\mathcal{N}_{h,1})/\mathit{\Gamma}_3(\mathbf{B}_{2,n}(\mathcal{N}_{h,1})) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z} \times \mathbb{Z}^h) \rtimes \mathbb{Z}^h$ for all $n \geqslant 3$. Therefore $Q_{(1,3)}(\mathcal{N}_{h,1})$ would be isomorphic to $\mathbb{Z}^{1+h}$ and $Q_{(2,3)}(\mathcal{N}_{h,1})$ would be isomorphic to $\mathbb{Z}^{1+h} \oplus \mathbb{Z}/2\mathbb{Z}$.

Remark 5.39 Conjecturally, when $k \in \{1, 2\}$, there is again a $\mathbb{Z}$ summand in the quotient group $Q_{(k,3)}(S)$, generated by (the image of) the surface pure braid ξ where one of the first k strands winds once around one of the last n strands. Again, for $n \geqslant 3$, the coinvariants of the action of $\mathbf{B}_n(S)$ on $Q_{(k,3)}(S)$ are given by killing this $\mathbb{Z}$ summand. Thus, we obtain homological functors $\langle \boldsymbol{\beta}, \boldsymbol{\beta}^S \rangle \longrightarrow \mathrm{Mod}_{\mathbb{Z}[Q^{\mathrm{u}}_{(k,3)}(S)]}$ for $S = \Sigma_{g,1}$ or $\mathcal{N}_{h,1}$ for each $g, h \geqslant 1$, where:

- $Q^{\mathrm{u}}_{(k,3)}(\Sigma_{g,1})$ is $\mathbb{Z}^{2g}$ for $k = 1$ and $\mathbb{Z}^{g+1} \rtimes \mathbb{Z}^g$ for $k = 2$,
- $Q^{\mathrm{u}}_{(k,3)}(\mathcal{N}_{h,1})$ is $\mathbb{Z}^h$ for $k = 1$ and $\mathbb{Z}^h \oplus \mathbb{Z}/2\mathbb{Z}$ for $k = 2$.

The An-Ko representations. The procedure described above for orientable surfaces is in a sense a reinterpretation following [BGG17] of the work [AK10] to extend some homological representations from the classical braid groups to the surface braid groups.

Namely, the functor $L_k(\mathring{F}_{(\underline{k},\mathfrak{S}_k,3)})$ induces representation of $\mathbf{B}_n(\Sigma_{g,1})$ for any $k \geqslant 1$ and $n \geqslant 3$. The k-th An-Ko representation of $\mathbf{B}_n(\Sigma_{g,1})$ is a certain tensor product of $L_k(\mathring{F}_{(\underline{k},\mathfrak{S}_k,3)})$ with a quotient of $\mathbf{B}_{k,n}(\Sigma_{g,1})/\varGamma_3(\mathbf{B}_{k,n}(\Sigma_{g,1}))$; see [AK10, Theorem 3.2]. The group $Q_{(\underline{k},\mathfrak{S}_k,3)}$ is abstractly defined in [AK10] in terms of group presentations to satisfy certain technical homological constraints: [BGG17, §4] gives all the connections to the third lower central quotient. The general method applied in §5.2.2 underlines the mainspring of these groups, proving moreover that the use of the third lower central quotient is a key tool to define the homological representations and giving an alternative to the technical result [AK10, Lemma 3.1]. We refer the reader to [AK10, §3.B, pp. 273-274] for explicit computations for $k \in \{1,2,3\}$ of the matrices of the representation of $\mathbf{B}_n(\Sigma_{1,1})$. On the contrary, the untwisted versions introduced in Corollary 5.37 are specific to the present paper.

5.2.3. Loop braid groups

To apply the construction of §5.1 to extended loop braid groups, we consider restrictions of the continuous functor $\mathring{F}_{(Z,G,\ell)}\colon \mathfrak{U}\mathcal{D}_3 \to \mathrm{Cov}_\bullet$ of Proposition 5.8 to the full subcategory $\mathfrak{U}\mathcal{LB}'$ of $\mathfrak{U}\mathcal{D}_3$ (see §4.6), thus assigning $M = \mathbb{D}^3$ and $A = Y_n = \underline{n}\mathbb{S}^1$ a set of n disjoint, unlinked circles in its interior. For non-extended loop braid groups, we instead consider restrictions to $\mathfrak{U}\mathcal{LB} \subseteq \mathfrak{U}\mathcal{D}_3^+$ of the analogous continuous functor

$$\mathring{F}_{(Z,G,\ell)}\colon \mathfrak{U}\mathcal{D}_3^+ \longrightarrow \mathrm{Cov}_\bullet$$

given by the analogue of Proposition 5.8 for $\mathcal{D}_d^+$ instead of $\mathcal{D}_d$. This general construction is exactly parallel to the construction of the non-oriented version, the only difference being that, in diagram (5.4), embeddings of the manifold A are always considered modulo $\mathrm{Diff}^+(A)$, rather than the full diffeomorphism group $\mathrm{Diff}(A)$. The space $X(M,A)$ is therefore the same in this construction, but the quotient $Q(M,A)$ of its fundamental group is generally different.

Two choices for the submanifold Z naturally arise as relevant parameters to construct homological representations: either we consider a set of points (studied in §5.2.3.1 below), or we take Z to be an unlink (detailed in §5.2.3.2).

5.2.3.1. Using configurations of points

We take $Z = \underline{k}$ a set of $k \geqslant 1$ points in $\mathbb{R}^3$ and $G = \mathfrak{S}_k$. We therefore consider the functors

$$\mathring{F}'_{(\underline{k},\mathfrak{S}_k,\ell)}\colon \mathfrak{U}\mathcal{LB}' \longrightarrow \mathrm{Cov}_\bullet \quad \text{and} \quad \mathring{F}_{(\underline{k},\mathfrak{S}_k,\ell)}\colon \mathfrak{U}\mathcal{LB} \longrightarrow \mathrm{Cov}_\bullet$$

associated with a pair of non-negative integers (k,ℓ) where $k \geqslant 1$ and $\ell \geqslant 0$. By the construction of §2.5.1, these determine the following homological functors for all $i \geqslant 1$:

$$L_i(\mathring{F}'_{(\underline{k},\mathfrak{S}_k,\ell)})\colon \mathfrak{U}\mathcal{LB}' \longrightarrow \mathrm{Mod}_\bullet \qquad \text{and} \qquad L_i(\mathring{F}_{(\underline{k},\mathfrak{S}_k,\ell)})\colon \mathfrak{U}\mathcal{LB} \longrightarrow \mathrm{Mod}_\bullet. \tag{5.19}$$

We denote by $Q_{(\underline{k},\mathfrak{S}_k,\ell)}(\mathbb{D}^3_n)$ the quotient group $Q(M,A)$ in diagram (5.4) for $(Z,G,\ell) = (\underline{k},\mathfrak{S}_k,\ell)$ and recall from [DPS21, Theorem 5.8] that for any $n \geqslant 4$,

- $\varGamma_2 = \varGamma_3$ for the groups $\pi_1(C_k(\mathbb{D}^3_n)) \rtimes \mathbf{LB}_n$ and $\pi_1(C_k(\mathbb{D}^3_n)) \rtimes \mathbf{LB}'_n$ if $k = 1$ or $k \geqslant 3$;
- $\varGamma_\ell \neq \varGamma_{\ell+1}$ for all $\ell \geqslant 1$ for the group $\pi_1(C_2(\mathbb{D}^3_n)) \rtimes \mathbf{LB}_n$;
- $\varGamma_2 \neq \varGamma_3 = \varGamma_4$ for the group $\pi_1(C_2(\mathbb{D}^3_n)) \rtimes \mathbf{LB}'_n$.

Hence we just focus on the construction for $\ell = 2$ here. For the non-extended loop braid groups, the procedure of §5.1 for the particular parameter $k = 2$ provides an *infinite* pro-nilpotent tower of homological representations of the category $\mathfrak{U}\mathcal{LB}$, and thus a different representation for each $\ell \geqslant 2$, which will however not be addressed in detail here. For the extended loop braid groups, we also obtain one additional representation by taking $k = 2$ and $\ell = 3$, although this will not be discussed here.

We deduce from the computation for the abelianisations of tripartite loop braid groups of [DPS21, Proposition 5.7] that for all $n \geqslant 2$, $Q_{(\underline{k},\mathfrak{S}_k,2)}(\mathbb{D}^3_n) = \mathbb{Z}$ if $k = 1$ and $\mathbb{Z} \oplus \mathbb{Z}/2$ if $k \geqslant 2$, whereas $Q'_{(\underline{k},\mathfrak{S}_k,2)}(\mathbb{D}^3_n) = \mathbb{Z}/2$ if $k = 1$ and $(\mathbb{Z}/2)^2$ if $k \geqslant 2$. We deduce from Proposition 5.27 that:

Proposition 5.40 *For each $i \geqslant 0$ and $k \geqslant 1$, trivialising the assignment for $n \leqslant 1$, the functors* (5.19) *determine homological representations:*
(for $k = 1$) $\mathfrak{U}\mathcal{LB}' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}/2]}$ *and* $\mathfrak{U}\mathcal{LB} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}]}$*;*
(for $k \geqslant 2$) $\mathfrak{U}\mathcal{LB}' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}/2 \oplus \mathbb{Z}/2]}$ *and* $\mathfrak{U}\mathcal{LB} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}/2]}$*.*

Notation 5.41 We denote these homological representations by $L_i(\mathcal{LB}'_{k,2})$ and $L_i(\mathcal{LB}_{k,2})$ respectively. When $i = k$ we will also write them as $\mathfrak{L}_k(k, \mathcal{L\beta}')$ and $\mathfrak{L}_k(k, \mathcal{L\beta})$ respectively.

Among the families of representations constructed by Proposition 5.40, the easiest to understand are those corresponding to $i = k = 1$ and are explicitly computed in [PS21b]. We recall that the loop braid group $\mathbf{LB}_n$ admits a presentation given by generators $\{\sigma_i, \tau_i \mid 1 \leqslant i \leqslant n-1\}$, where $\{\sigma_1, \ldots, \sigma_{n-1}\}$ satisfy the relations of the classical braid group $\mathbf{B}_n$ and $\{\tau_1, \ldots, \tau_{n-1}\}$ those of the symmetric group $\mathfrak{S}_n$, together with three additional mixed relations; see [Dam17, Propositions 3.14 and 3.16] for details. We show in [PS21b] that the matrices of the representations $L_1(\mathcal{LB}_{1,2})(n) \colon \mathbf{LB}_n \to \mathrm{Aut}_{\mathbb{Z}[\mathbb{Z}]}(\mathbb{Z}[\mathbb{Z}]^{\oplus n-1})$ are those of:

- the reduced Burau representation of the braid group $\mathbf{B}_n$ for the generators $\{\sigma_1, \ldots, \sigma_{n-1}\}$;
- the standard representation of the symmetric group $\mathfrak{S}_n$ for the generators $\{\tau_1, \ldots, \tau_{n-1}\}$.

The matrices for the representations $L_1(\mathcal{LB}'_{1,2})(n)$ of the extended loop braid groups $\mathbf{LB}'_n$ over $R = \mathbb{Z}[\mathbb{Z}/2] = \mathbb{Z}[t^{\pm 1}]/(t^2 - 1)$ are more subtle, since the underlying R-module of the representation is not free in this case: it is $R^{n-1} \oplus R/(t-1)$. They can however be fully computed; see [PS21b].

On the other hand, the representations encoded by the other functors of Proposition 5.40 are new and will be studied in detail in the sequel paper [PS22a].

5.2.3.2. Using configurations of unlinks

We now set $Z = \underline{k}\mathbb{S}^1$ a k-component unlink in $\mathbb{R}^3$. We focus on two choices for the groups G, the full diffeomorphism group of $\underline{k}\mathbb{S}^1$ or its subgroup of orientation-preserving diffeomorphisms.

The oriented version. We take $G = \mathrm{Diff}^+(\underline{k}\mathbb{S}^1)$ and consider the homological functors

$$L_i(\mathring{F}'_{(\underline{k}\mathbb{S}^1, \mathrm{Diff}^+(\underline{k}\mathbb{S}^1), \ell)}) \colon \mathfrak{U}\mathcal{L\beta}' \longrightarrow \mathrm{Mod}_\bullet \quad \text{and} \quad L_i(\mathring{F}_{(\underline{k}\mathbb{S}^1, \mathrm{Diff}^+(\underline{k}\mathbb{S}^1), \ell)}) \colon \mathfrak{U}\mathcal{L\beta} \longrightarrow \mathrm{Mod}_\bullet \tag{5.20}$$

for $i, k \geqslant 1$ and $\ell \geqslant 0$. We recall from [DPS21, Theorem 5.8] that for any $n \geqslant 4$, the lower central series of the groups $\mathbf{LB}_{k,n}$ and $\pi_1(U_k^+(\mathbb{D}^3_n)) \rtimes \mathbf{LB}'_n$ satisfy:

- $\Gamma_2 = \Gamma_3$ when $k \in \mathbb{N} \smallsetminus \{2, 3\}$;
- $\Gamma_\ell \neq \Gamma_{\ell+1}$ for all $\ell \geqslant 1$ when $k \in \{2, 3\}$.

We detail below the construction produced for $\ell = 2$. Similarly to §5.2.1.2, the procedure of §5.1 for the particular parameters $k \in \{2, 3\}$ provides an *infinite* pro-nilpotent tower of homological representations of the categories $\mathfrak{U}\mathcal{L\beta}$ and $\mathfrak{U}\mathcal{L\beta}'$, i.e. a different homological representation for each $\ell \geqslant 2$, which will however not be addressed in detail here.

It follows from the abelianisation computations of [DPS21, Proposition 5.7] that for $n \geqslant 2$:

- $Q'_{(\underline{k}\mathbb{S}^1, \mathrm{Diff}^+(\underline{k}\mathbb{S}^1), 2)}(\mathbb{D}^3_n) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ if $k = 1$ and $\mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2$ if $k \geqslant 2$;
- $Q_{(\underline{k}\mathbb{S}^1, \mathrm{Diff}^+(\underline{k}\mathbb{S}^1), 2)}(\mathbb{D}^3_n) \cong \mathbb{Z}^2$ if $k = 1$ and $\mathbb{Z}^3 \oplus \mathbb{Z}/2$ if $k \geqslant 2$.

It then follows from Proposition 5.27 that:

Proposition 5.42 *For each $i \geqslant 0$ and $k \geqslant 1$, trivialising the assignment for $n \leqslant 1$, the functors* (5.20) *for $\ell = 2$ define homological functors:*
(for $k = 1$) $\mathfrak{U}\mathcal{LB}' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}/2]}$ *and* $\mathfrak{U}\mathcal{LB} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}^2]}$*;*
(for $k \geqslant 2$) $\mathfrak{U}\mathcal{LB}' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2]}$ *and* $\mathfrak{U}\mathcal{LB} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}^3 \oplus \mathbb{Z}/2]}$*.*

These representations all appear to be new and will be studied in detail in the sequel paper [PS22a].

The unoriented version. We take $G = \mathrm{Diff}(\underline{k}\mathbb{S}^1)$ and consider the homological functors

$$L_i(\mathring{F}'_{(\underline{k}\mathbb{S}^1, \mathrm{Diff}(\underline{k}\mathbb{S}^1), \ell)}) \colon \mathfrak{U}\mathcal{L\beta}' \longrightarrow \mathrm{Mod}_\bullet \quad \text{and} \quad L_i(\mathring{F}_{(\underline{k}\mathbb{S}^1, \mathrm{Diff}(\underline{k}\mathbb{S}^1), \ell)}) \colon \mathfrak{U}\mathcal{L\beta} \longrightarrow \mathrm{Mod}_\bullet \tag{5.21}$$

for $i, k \geqslant 1$ and $\ell \geqslant 0$. We recall from [DPS21, Theorem 5.8] that for any $n \geqslant 4$, the lower central series of the groups $\mathbf{LB}'_{k,n}$ and $\pi_1(U_k(\mathbb{D}^3_n)) \rtimes \mathbf{LB}_n$ satisfy:

- $\Gamma_2 = \Gamma_3$ when $k \in \mathbb{N} \smallsetminus \{1, 2, 3\}$;

- $\Gamma_\ell \neq \Gamma_{\ell+1}$ for all $\ell \geqslant 1$ when $k \in \{1,2,3\}$.

The construction produced for $\ell = 2$ is detailed below. Again, the procedure of §5.1 for the particular parameters $k \in \{1,2,3\}$ provides an *infinite* pro-nilpotent tower of homological representations of the categories $\mathfrak{U}\mathcal{L}\beta$ and $\mathfrak{U}\mathcal{L}\beta'$, i.e. a different representation for each $\ell \geqslant 2$, which will however not be addressed in detail here.

It follows from the computations of [DPS21, Proposition 5.7] that for $n \geqslant 2$:

- $Q'_{(\underline{k}\mathbb{S}^1,\mathrm{Diff}(\underline{k}\mathbb{S}^1),2)}(\mathbb{D}^3_n) \cong (\mathbb{Z}/2)^3$ if $k = 1$ and $(\mathbb{Z}/2)^5$ if $k \geqslant 2$;
- $Q_{(\underline{k}\mathbb{S}^1,\mathrm{Diff}(\underline{k}\mathbb{S}^1),2)}(\mathbb{D}^3_n) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2$ if $k = 1$ and $\mathbb{Z} \oplus (\mathbb{Z}/2)^4$ if $k \geqslant 2$.

We then deduce from Proposition 5.27 that:

Proposition 5.43 *For each $i \geqslant 0$ and $k \geqslant 1$, trivialising the assignment for $n \leqslant 1$, the functors* (5.21) *for $\ell = 2$ define homological functors:*

(for $k = 1$) $\mathfrak{U}\mathcal{L}\mathcal{B}' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[(\mathbb{Z}/2)^3]}$ *and* $\mathfrak{U}\mathcal{L}\mathcal{B} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)^2]}$*;*
(for $k \geqslant 2$) $\mathfrak{U}\mathcal{L}\mathcal{B}' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[(\mathbb{Z}/2)^5]}$ *and* $\mathfrak{U}\mathcal{L}\mathcal{B} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)^4]}$*.*

As above, these representations all appear to be new and will be studied in detail in the sequel paper [PS22a].

5.3. Global functors for mapping class groups

The construction in §5.1 of functors $\mathfrak{U}\mathcal{D}_d \to \mathrm{Mod}_\bullet$ is well-adapted for motion groups (a.k.a. *braided mapping class groups*, cf. Proposition 4.23). This is reflected in the fact that, *if* one restricts to a subcategory of $\mathfrak{U}\mathcal{D}_d$ of the form $\langle \mathcal{G}, \mathcal{M} \rangle$, where the automorphism groups of $\mathcal{M}$ are contained in the corresponding *braided* diffeomorphism groups, then Proposition 5.16 implies (under additional conditions) that the functor takes values in $\mathrm{Cov}_Q \subset \mathrm{Cov}_\bullet$ (after passing to π_0), for a fixed group Q. Hence, applying the general construction of §2, we obtain a functor into Mod_R for some fixed ring R, rather than just $\mathrm{Mod}_\bullet$. Thus the construction of §5.1 is effective for constructing families of *untwisted* representations of motion groups (braided mapping class groups). However, for (full) mapping class groups, this construction typically only gives twisted representations, and one must restrict to smaller subgroups in order to obtain genuine, untwisted representations.

In this subsection, we describe a variant of the construction of §5.1, based on the split short exact sequence (4.9) instead of (4.6), which has good properties for the *full* mapping class groups. More precisely, this variant satisfies an analogue of Proposition 5.16 where one does *not* have to assume that the automorphism groups of $\mathcal{M}$ are contained in the corresponding braided diffeomorphism groups; *cf.* Proposition 5.48 below.

Since the construction is very similar, we just sketch an outline and point out the differences from the construction of §5.1. The following is the analogue of Proposition 5.8.

Proposition 5.44 *For any integers $d \geqslant 2$ and $\ell \geqslant 0$, closed submanifold $Z \subset \mathbb{R}^d$ and open subgroup $G \leqslant \mathrm{Diff}(Z)$, there are well-defined functors*

$$\mathring{F}_{(Z,G,\ell)} \text{ and } F_{(Z,G,\ell)} \colon \mathfrak{U}\mathcal{D}_d \longrightarrow \mathrm{Cov}_\bullet, \tag{5.22}$$

constructed as described below.

The construction of the functor $\mathring{F}_{(Z,G,\ell)}$ on objects is similar to that of §5.1. The space $X(M,A)$ is defined in exactly the same way, but the quotient $\phi(M,A) \colon \pi_1(X(M,A)) \twoheadrightarrow Q(M,A)$ is defined using the following 6-term diagram instead of (5.4).

$$\begin{array}{ccccccccc}
 & & \pi_1(X(M,A)) & & & & & & \\
 & & \cong & & & & & & \\
1 & \longrightarrow & \pi_1(\mathcal{E}_G(Z,\mathring{M} \smallsetminus A)) & \longrightarrow & \pi_0(\mathrm{Diff}_{\mathrm{dec}}(M,A,Z|G)) & \underset{\leftarrow\,\text{-}\,\text{-}}{\longrightarrow} & \pi_0(\mathrm{Diff}_{\mathrm{dec}}(M,A)) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \phi(M,A)} & & \downarrow{\scriptstyle \gamma(M,A)} & & \downarrow{\scriptstyle \bar{\gamma}(M,A)} & & \\
1 & \longrightarrow & Q(M,A) & \longrightarrow & \dfrac{\pi_0(\mathrm{Diff}_{\mathrm{dec}}(M,A,Z|G))}{\Gamma_\ell(\pi_0(\mathrm{Diff}_{\mathrm{dec}}(M,A,Z|G)))} & \underset{\leftarrow\,\text{-}\,\text{-}}{\longrightarrow} & \dfrac{\pi_0(\mathrm{Diff}_{\mathrm{dec}}(M,A))}{\Gamma_\ell(\pi_0(\mathrm{Diff}_{\mathrm{dec}}(M,A)))} & \longrightarrow & 1
\end{array} \tag{5.23}$$

The top row is the split short exact sequence (4.9) of Corollary 4.19 when $G = \mathrm{Diff}(Z)$; more generally it is the middle row of diagram (4.12) for open subgroups $G \leqslant \mathrm{Diff}(Z)$. The rest of the diagram is constructed from this just as in §5.1, using Lemma 5.4. The construction of the functor on morphisms is then exactly as in §5.1.2, using the fact that the diagram (5.23) is functorial in the object (M, A).

The closed variant $F_{(Z,G,\ell)}$ of the functor (5.22) is constructed similarly, using the embedding space $\mathcal{E}'_G(Z, M \smallsetminus A)$ instead of $\mathcal{E}_G(Z, \mathring{M} \smallsetminus A)$ (cf. §5.1.3).

The observations in §5.1.4 hold also for the functors (5.22): they induce functors (not just semifunctors) on π_0, the closed version $F_{(Z,G,\ell)}$ takes values in $\mathrm{Cov}_\bullet^{\mathrm{pr}}$ and there is a natural homotopy equivalence $\mathring{F}_{(Z,G,\ell)} \Rightarrow F_{(Z,G,\ell)}$ between the two versions of (5.22).

The possible variants of the construction mentioned in §5.1.6, replacing the functorial quotient $G \mapsto G/\Gamma_\ell(G)$ by other quotients, apply similarly to the functors (5.22). Moreover, the constructions of §5.1.7 also carry over to these functors, and show that if we fix (Z, G) and allow ℓ to vary, the functors (5.22) form a “pro-nilpotent” tower of representations of $\mathfrak{U}\mathcal{D}_d$.

Remark 5.45 There is a natural morphism of diagrams (5.4) → (5.23) induced by the map of split short exact sequences from the top row to the middle row of diagram (4.12). This induces a natural transformation of functors

$$(5.6) \Rightarrow (5.22). \tag{5.24}$$

Two actions agree. The diagram (5.23) is functorial in (M, A) as an object of $\mathfrak{U}\mathcal{D}_d$, so there is an action of $\mathrm{Diff}_{\mathrm{dec}}(M, A) = \mathrm{Aut}_{\mathcal{D}_d}((M, A))$ on the bottom-left group $Q(M, A)$. Since $Q(M, A)$ is discrete, this factors through an action of $\pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, A))$ on $Q(M, A)$.

On the other hand, $\pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, A))$ is also the top-left group of (5.23), so it has another action on $Q(M, A)$ given either by lifting elements of $Q(M, A)$ along $\phi(M, A)$ and using the conjugation action of the semi-direct product on the top row, or equivalently by projecting along $\bar{\gamma}(M, A)$ to the bottom-right group of (5.23) and then using the conjugation action of the semi-direct product on the bottom row. The following is the analogue of Proposition 5.7.

Proposition 5.46 *The two actions of $\pi_0(\mathrm{Diff}_{\mathrm{dec}}(M, A))$ on $Q(M, A)$ described above are equal.*

Proof. This follows immediately from the first part of Lemma 4.25. □

The definition of *Q-stability* in the setting of this section is exactly analogous to Definition 5.14.

Definition 5.47 Let $\mathcal{M}$ and $\mathcal{G}$ be subgroupoids as in (5.5). We suppose that there is a group Q and there are identifications $Q(M, A) \cong Q$ for each object (M, A) of $\langle \mathcal{G}, \mathcal{M} \rangle$, such that, for each object (M', A') of $\mathcal{G}$, the homomorphism of groups

$$Q(M, A) \longrightarrow Q((M', A') \natural (M, A)),$$

induced by the canonical morphism $((M', A'), \mathrm{id})$ of $\langle \mathcal{G}, \mathcal{M} \rangle$, is equal to the identity under these identifications. In this case, we say that *the functor $F_{(Z,G,\ell)}$ is Q-stable on $\langle \mathcal{G}, \mathcal{M} \rangle$.*

The only difference between Definition 5.14 and Definition 5.47 is that, in the former, the group $Q(M, A)$ and the induced homomorphism $Q(M, A) \to Q((M', A') \natural (M, A))$ are those obtained from the 6-term diagram (5.4) (and its functoriality), whereas in the latter, they are obtained from the 6-term diagram (5.23) (and its functoriality).

The analogue of Lemma 5.15 is again immediate: Q-stability implies that $F_{(Z,G,\ell)}$ is equivalent to a functor taking values in $\mathrm{Cov}_Q^{\mathrm{tw}} \subset \mathrm{Cov}_\bullet$. Moreover, we have an analogue of Proposition 5.16, except that we drop the second assumption of Proposition 5.16 in the following.

Proposition 5.48 *Let $\mathcal{M}$ and $\mathcal{G}$ be subgroupoids of $\mathcal{D}_d$ such that $\mathcal{G}$ is closed under $\natural$ and $\mathcal{M}$ is closed under the action of $\mathcal{G}$ via $\natural$. Set $\ell = 2$, suppose that $F_{(Z,G,2)}$ is Q-stable on $\langle \mathcal{G}, \mathcal{M} \rangle$, and*

- *$\mathcal{G}$ is full in $\mathcal{D}_d$ and $\mathcal{M}$ is 0-full in $\mathcal{D}_d$,*
- *$|\mathcal{G}|$ is free as a monoid and $|\mathcal{M}|$ is free as a $|\mathcal{G}|$-set.*

Under these assumptions, the functor

$$\pi_0(F_{(Z,G,2)}|_{\langle \mathcal{G}, \mathcal{M} \rangle}) \colon \pi_0(\langle \mathcal{G}, \mathcal{M} \rangle) \longrightarrow \pi_0(\mathrm{Cov}_\bullet)$$

is equivalent to a functor with image contained in $\pi_0(\mathrm{Cov}_Q) \subset \pi_0(\mathrm{Cov}_\bullet)$.

Proof. The proof is exactly analogous to the proof of Proposition 5.16, except that we use Proposition 5.46, instead of Proposition 5.7, to understand the action of automorphisms of $\mathcal{M}$ on Q. Since Proposition 5.46 applies to the whole mapping class group $\pi_0(\mathrm{Diff}_{\mathrm{dec}}(M,A))$ (whereas Proposition 5.7 applies only to the subgroup $\pi_0(\mathrm{Diff}^{\mathrm{br}}_{\mathrm{dec}}(M,A))$, the braided mapping class group), we do not need to assume that the automorphisms of $\mathcal{M}$ are contained in the braided diffeomorphism groups $\mathrm{Diff}^{\mathrm{br}}_{\mathrm{dec}}(M,A)$, as we had to for Proposition 5.16. □

5.4. Applications for mapping class groups of surfaces

Although it is best adapted to motion groups, the general construction of §5.1 may also be used to construct representations of mapping class groups, and recovers several classical constructions. This is detailed in §5.4.1. We then apply in §5.4.2 the adapted method for mapping class groups of §5.3 to define other representations of these groups.

Recall from §4.4 the subgroupoid $\mathcal{M}_2^{\mathrm{t}}$ of $\mathcal{D}_2$ and let $\mathcal{M}_2^{\mathrm{t},+}$ (respectively $\mathcal{M}_2^{\mathrm{t},-}$) denote the full subgroupoids of $\mathcal{M}_2^{\mathrm{t}}$ with objects the orientable surfaces (respectively non-orientable surfaces together with the disc) with one boundary-component. Set $\mathcal{M}_2^{+} = \pi_0(\mathcal{M}_2^{\mathrm{t},+})$ and $\mathcal{M}_2^{-} = \pi_0(\mathcal{M}_2^{\mathrm{t},-})$.

5.4.1. Homological representations from motion group functors

A first idea to construct representations of the mapping class group of a d-manifold M is to use its action on covering spaces constructed as in §5.1. Let $\mathfrak{U}\mathcal{D}_d^{\emptyset}$ denote the full subcategory of $\mathfrak{U}\mathcal{D}_d$ on the objects $(M,\emptyset)$ and consider the restriction of the continuous functor $\mathring{F}_{(Z,G,\ell)}\colon \mathfrak{U}\mathcal{D}_d \to \mathrm{Cov}_{\bullet}$ of Proposition 5.8 to $\mathfrak{U}\mathcal{D}_d^{\emptyset}$. Note that this has the effect that the short exact sequences of the diagram (5.4) degenerate in the sense that the right-hand side is the trivial group.

We now restrict to dimension $d = 2$ and the groupoids $\mathcal{M}_2^{\mathrm{t},+}$ and $\mathcal{M}_2^{\mathrm{t},-}$ mentioned above. Let $Z = \underline{k}$ be a set of $k \geqslant 1$ points in $\mathbb{R}^2$. We thus consider the functors

$$\mathring{F}^{+}_{(\underline{k},G,\ell)}\colon \mathfrak{U}\mathcal{M}_2^{\mathrm{t},+} \longrightarrow \mathrm{Cov}_{\bullet} \quad \text{and} \quad \mathring{F}^{-}_{(\underline{k},G,\ell)}\colon \mathfrak{U}\mathcal{M}_2^{\mathrm{t},-} \longrightarrow \mathrm{Cov}_{\bullet} \tag{5.25}$$

for integers $\ell \geqslant 0$, $k \geqslant 1$ and subgroups $G \leqslant \mathfrak{S}_k$. We will consider the two extremal choices of $G = \mathfrak{S}_k$ and $G = \{\mathrm{id}\}$, corresponding to *unordered* configurations spaces (see §5.4.1.1) and *ordered* configurations spaces (see §5.4.1.2) respectively.

5.4.1.1. Using unordered configuration spaces

Similarly to the constructions of representations for motion groups in §5.2, we take $G = \mathfrak{S}_k$. The construction of §2.5.1 produces homological functors for all $i \geqslant 1$:

$$L_i(\mathring{F}^{+}_{(\underline{k},\mathfrak{S}_k,\ell)})\colon \mathfrak{U}\mathcal{M}_2^{+} \longrightarrow \mathrm{Mod}_{\bullet} \quad \text{and} \quad L_i(\mathring{F}^{-}_{(\underline{k},\mathfrak{S}_k,\ell)})\colon \mathfrak{U}\mathcal{M}_2^{-} \longrightarrow \mathrm{Mod}_{\bullet}. \tag{5.26}$$

For $\ell = 0$, we obtain the representations corresponding to the action on the homology groups of the universal covers of the spaces $X(M,\emptyset)$, and we recover the natural action of the mapping class groups on the homology groups of the spaces $X(M,\emptyset)$ for $\ell = 1$; *cf.* Remark 5.6. We now give more details of some representations encoded by $L_i(\mathring{F}^{+}_{(\underline{k},\mathfrak{S}_k,\ell)})$ when $\boldsymbol{k} \geqslant \mathbf{3}$.

We recall from [BGG08, Theorem 2] that for any $g \geqslant 1$ and $k \geqslant 3$, the lower central series of the group $\mathbf{B}_k(\Sigma_{g,1})$ satisfies $\mathit{\Gamma}_2 \neq \mathit{\Gamma}_3 = \mathit{\Gamma}_4$. Hence the two relevant cases to study are $\ell \in \{2,3\}$. We determine in each case the best subgroup of the mapping class group for which we obtain *untwisted* representations.

We fix a symplectic basis $\{A_1, B_1, \ldots, A_g, B_g\}$ for the first homology group of the surface $H_g := H_1(\Sigma_{g,1};\mathbb{Z})$ with respect to the algebraic intersection form $\omega_g\colon H_g \times H_g \to \mathbb{Z}$. The operation $(z,c)\cdot(z,c) = (z + z' + \omega_g(\lambda,\lambda'), \lambda + \lambda')$ for all $z, z' \in \mathbb{Z}$ and $\lambda, \lambda' \in H_g$ defines a central extension of H_g by $\mathbb{Z}$ that we denote by $\mathbb{Z} \times_{\omega_g} H_g$.

Lemma 5.49 *The abelianisation $\mathbf{B}_k(\Sigma_{g,1})/\Gamma_2(\mathbf{B}_k(\Sigma_{g,1}))$ is isomorphic to the product $\mathbb{Z}/2 \times H_g$. The third lower central quotient $\mathbf{B}_k(\Sigma_{g,1})/\Gamma_3(\mathbf{B}_k(\Sigma_{g,1}))$ is isomorphic to the central extension $\mathbb{Z} \times_{\omega_g} H_g$. In both situations, the action of the mapping class group $\mathbf{\Gamma}_{g,1}$ on the quotient H_g is the natural action $\mathfrak{a}_g$.*

Proof. For $\ell = 2$, the result directly follows from the presentation of $\mathbf{B}_k(\Sigma_{g,1})$ of [Bel04, Theorem 1]: the factor $\mathbb{Z}/2$ is the image of the braid generators $\{\sigma_1, \ldots, \sigma_{k-1}\}$ and the factor H_g is the image of the generators $\{a_1, b_1, \ldots, a_g, b_g\}$. For $\ell = 3$, by [BGG17, Corollary 3.14] the third lower central quotient $\mathbf{B}_k(\Sigma_{g,1})/\Gamma_3(\mathbf{B}_k(\Sigma_{g,1}))$ is isomorphic to the semi-direct product $(\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g$, the first factor $\mathbb{Z}$ is central and is generated by $\sigma := \gamma_3(\sigma_i)$ for all $i \in \{1, \ldots, k-1\}$, the second factor $\mathbb{Z}^g$ is generated by $\{a_i := \gamma_3(a_i) \mid i \in \{1, \ldots, g\}\}$, and the third factor $\mathbb{Z}^g$ is generated by $\{b_i := \gamma_3(b_i) \mid i \in \{1, \ldots, g\}\}$; for all $j \in \{1, \ldots, g\}$, the generator b_j acts trivially on a_i for $i \in \{1, \ldots, g\} \setminus \{j\}$ and for $i = j$ we have $a_j b_j = \sigma^2 b_j a_j$, in other words b_j sends a_j to $a_j + 2\sigma$ in additive notation. The isomorphism is given by sending σ to a generator of $\mathbb{Z}$ in the central extension, a_i to A_i and b_i to B_i for all $i \in \{1, \ldots, g\}$. The relation $a_j b_j = \sigma^2 b_j a_j$ is preserved through this morphism by the definition of the intersection form. $\square$

By Lemma 5.49, we must restrict to a subgroup of the Torelli group to obtain a trivial action both on the abelianisation and on the third lower central quotient. In particular the $\mathbb{Z}/2$ summand is generated by the image σ of the braid generators of $\mathbf{B}_n(\Sigma_{g,1})$. We recall from Corollary 4.19 that the splitting $\mathbf{\Gamma}_{g,1} \hookrightarrow \mathbf{\Gamma}^n_{g,1}$ is induced by the embedding of surfaces $\Sigma_{g,1} \hookrightarrow \Sigma^n_{0,1} \natural \Sigma_{g,1} \cong \Sigma^n_{g,1}$: this implies that $\mathbf{\Gamma}_{g,1}$ acts trivially on σ, since σ is supported in the subsurface $\Sigma^n_{0,1}$. Also the action of $\mathbf{\Gamma}_{g,1}$ on H_g is induced by $\mathfrak{a}_g$. Hence the result of Lemma 5.49 on $\mathbf{B}_k(\Sigma_{g,1})/\Gamma_2(\mathbf{B}_k(\Sigma_{g,1}))$ is an isomorphism of $\mathbf{\Gamma}_{g,1}$-modules, not just of abelian groups, if we view $\mathbb{Z}/2 \times H_g$ as a product of $\mathbb{Z}/2$ equipped with the trivial action and H_g as the symplectic $\mathbf{\Gamma}_{g,1}$-module. Therefore, for $\ell = 2$, the homological representations defined by $L_i(\mathring{F}^+_{(\underline{k}, \mathfrak{S}_k, 2)})$ are *untwisted*, i.e. they act through untwisted automorphisms of $\mathbb{Z}[\mathbb{Z}/2 \times H_g]$-modules, if we restrict to the Torelli groups.

For $\ell = 3$, we deduce that there exists a map $\kappa \colon \mathbf{\Gamma}_{g,1} \to \mathrm{Hom}(H_g, \mathbb{Z})$ so that the action of the mapping class group on the central extension $\mathbb{Z} \times_{\omega_g} H_g$ is described by the matrix

$$\begin{bmatrix} Id_{\mathbb{Z}} & \kappa \\ 0 & \mathfrak{a}_g \end{bmatrix}.$$

We denote by $\mathrm{Chill} \colon \mathbf{\Gamma}_{g,1} \to \mathrm{Hom}(H_g, \mathbb{Z})$ the *Chillingworth crossed homomorphism* [Chi72; Tra92] which describes the action of $\mathbf{\Gamma}_{g,1}$ on the winding numbers of the curves of $\Sigma_{g,1}$. Its kernel is called the *Chillingworth subgroup* and denoted by $\mathcal{C}_{g,1}$. By [Chi12, Corollary 4.8], it is the subgroup generated by the union of the *simply intersecting pair maps* and the *Johnson subgroup* [Joh83], which is the kernel of the natural map $\mathbf{\Gamma}_{g,1} \to \mathrm{Aut}(\pi_1(\Sigma_{g,1})/\Gamma_3(\pi_1(\Sigma_{g,1})))$.

Lemma 5.50 *The map κ is a crossed homomorphism and its kernel coincides with $\mathcal{C}_{g,1}$.*

Proof. Since the action of the mapping class group on the central extension is a morphism, we deduce that $\kappa(\varphi \circ \psi) = \kappa(\psi) + \kappa(\varphi)\mathfrak{a}_g(\psi)$ for all $\varphi, \psi \in \mathbf{\Gamma}_{g,1}$: this proves that κ is a crossed homomorphism. Moreover, by [Mor89, Proposition 6.4] we have $H^1(\mathbf{\Gamma}_{g,1}, H_g) \cong \mathbb{Z}$. Hence, up to principal crossed homomorphisms, we have $\mu \cdot \kappa = \lambda \cdot \mathrm{Chill}$ for integers $\mu, \lambda \in \mathbb{Z}$. In particular, we have $\mu \cdot \kappa = \lambda \cdot \mathrm{Chill}$ on the Torelli group, since principal crossed homomorphisms with values in H_g vanish on the Torelli group. It is easy to check that κ and Chill are both not the zero homomorphism on the Torelli group, so μ and λ are both non-zero integers. Thus $\ker(\kappa) = \ker(\mathrm{Chill}) = \mathcal{C}_{g,1}$. $\square$

Hence the homological representations defined by $L_i(\mathring{F}^+_{(\underline{k}, \mathfrak{S}_k, 3)})$ act by *untwisted* $\mathbb{Z}[\mathbb{Z} \times_{\omega_g} H_g]$-module automorphisms if we restrict to the Chillingworth subgroups.

Remark 5.51 There is in fact a quotient of $\mathbf{B}_k(\Sigma_{g,1})$ onto the *Heisenberg group* $\mathbb{Z} \times_{\omega_g} H_g$ for all $k \geqslant 2$ (not just $k \geqslant 3$); for $k = 2$ the kernel of this quotient lies strictly between Γ_2 and Γ_3. Homological representations arising from this quotient for $k \geqslant 2$ are studied in detail in [BPS21], where it is shown how this induces a family of untwisted representations of the Torelli group and of the stably universal central extension of the mapping class group.

5.4.1.2. Using ordered configuration spaces

Alternatively, we consider *ordered* configuration spaces of points on the surfaces: we take the same assignments as in §5.4.1.1 except that $G = \{\mathrm{id}\}$. By §2.5.1 we have homological functors $L_i(\mathring{F}^+_{(\underline{k}, \{\mathrm{id}\}, \ell)}) \colon \mathfrak{UM}^+_2 \to \mathrm{Mod}_\bullet$ and $L_i(\mathring{F}^-_{(\underline{k}, \{\mathrm{id}\}, \ell)}) \colon \mathfrak{UM}^-_2 \to \mathrm{Mod}_\bullet$ for all $i \geqslant 1$.

An interesting modification of this construction consists in removing the basepoint $p_0 \in \partial S$ from the surface and allowing the configuration points to be in the boundary of the surface. Namely, we use the closed variant F of §5.1.3 instead of the functor $\mathring{F}$ and consider the configurations in the surface $S \smallsetminus \{p_0\}$ for each object S. By Lemma 5.11, Borel-Moore homology may be applied for this alternative and has the advantage of being endowed with a natural free generating set; *cf.* [Big04, Lemma 3.1], [AK10, Lemma 3.3], [AP20, Theorem 6.6] or [PS21a, §3]. The representations arising from this variant for $\ell = 1$ have been studied by Moriyama [Mor07].

5.4.2. Homological representations from mapping class group functors

The quotient groups Q defining the representations constructed in §5.4.1 for the mapping class groups of the surfaces are typically *not* independent of the surface (i.e. we do not have *Q-stability* in the sense of Definition 5.14). The advantage of the method of §5.3 applied here is that it is much more effective for this property to be satisfied. We consider the restriction of the continuous functor $\mathring{F}_{(Z,G,\ell)} \colon \mathfrak{U}\mathcal{D}_2 \to \mathrm{Cov}_\bullet$ of Proposition 5.44 to the appropriate full subcategory of the form $\mathfrak{U}\mathcal{G}$ given by Lemma 3.5 applied to the examples of subgroupoids $\mathcal{G}$ of §4.4 (whose automorphism groups correspond to the family of mapping class groups). Moreover we take $Z = \underline{k}$ a set of $k \geqslant 1$ points in $\mathbb{R}^2$ and $G = \mathfrak{S}_k$.

The construction of §2.5.1 produces homological functors for all $i \geqslant 1$:

$$L_i(\mathring{F}_{(\underline{k},\mathfrak{S}_k,\ell)}) \colon \pi_0(\mathfrak{U}\mathcal{G}) \longrightarrow \mathrm{Mod}_\bullet \tag{5.27}$$

for $i, k \geqslant 1$ and $\ell \geqslant 0$. For $\ell = 0$, we obtain the representations corresponding to the action on the homology groups of the universal cover of the configuration spaces of points in the surface, and we recover the natural action of the mapping class groups on the homology groups of these configuration spaces for $\ell = 1$; *cf.* Remark 5.6. For $\ell = 2$, we will typically prove that the adapted Q-stability property (Definition 5.47) is satisfied. When this is the case, denoting by $Q_{(\underline{k},\mathfrak{S}_k,2)}$ the associated stable quotient, we deduce that:

Proposition 5.52 *We obtain a homological functor $L_i(\mathring{F}_{(\underline{k},\mathfrak{S}_k,2)}) \colon \pi_0(\mathfrak{U}\mathcal{G}) \to \mathrm{Mod}_{\mathbb{Z}[Q_{(\underline{k},\mathfrak{S}_k,2)}]}$ from the functor* (5.27) *for all $i \geqslant 0$.*

Proof. Let $\mathfrak{U}\mathcal{G}_{\mathrm{stab}}$ be the full subcategory of $\mathfrak{U}\mathcal{G}$ on all objects *except* those for which the obtained quotient is not isomorphic to $Q_{(\underline{k},\mathfrak{S}_k,2)}$. By Proposition 5.48, the functor (5.27) induces homological functors $L_i(\mathring{F}_{(\underline{k},\mathfrak{S}_k,2)}) \colon \pi_0(\mathfrak{U}\mathcal{G}_{\mathrm{stab}}) \to \mathrm{Mod}_{\mathbb{Z}[Q_{(\underline{k},\mathfrak{S}_k,2)}]}$. By Lemma 5.17, we extend them to functors on $\pi_0(\mathfrak{U}\mathcal{G})$ by sending the other objects to the trivial module. □

As far as the authors know, the representations encoded by the functors based on the above procedure and described in the following Propositions 5.54 and 5.57 appear to be new.

General alternatives: partitioned configurations. For all the constructions presented here, we could define interesting alternatives by interpolating the parameter G between the trivial group $\{\mathrm{id}\}$ and $\mathfrak{S}_k$: in particular, this includes considering *partitioned* configurations. The associated lower central series being more complicated (see [DPS21]), the representations that arise from these alternatives are more sophisticated, as in Remark 5.29 for classical braid groups. The following Lemmas 5.53 and 5.53 can easily be generalised in order to compute the associated transformation groups. Again we do not detail all these partitioned cases here for the sake of simplicity, but we note that all the methods explained here could be repeated mutatis mutandis.

5.4.2.1. For orientable surfaces

Consider the restriction of the continuous functor $\mathring{F}_{(\underline{k},\mathfrak{S}_k,\ell)} \colon \mathfrak{U}\mathcal{D}_2 \to \mathrm{Cov}_\bullet$ of Proposition 5.44 to the full subcategory $\mathfrak{U}\mathcal{M}_2^{\mathrm{t},+}$. By §2.5.1 we obtain a homological functor

$$L_i(\mathring{F}_{(\underline{k},\mathfrak{S}_k,\ell)}) \colon \mathfrak{U}\mathcal{M}_2^{+} \longrightarrow \mathrm{Mod}_\bullet \tag{5.28}$$

for $i, k \geqslant 1$ and $\ell \geqslant 0$. To obtain some restrictions on the possibilities for the parameter ℓ, we study the lower central series of the mapping class group $\mathbf{\Gamma}_{g,1}^{k}$. We compute its abelianisation:

Lemma 5.53 *For each $g \geqslant 3$, $(\mathbf{\Gamma}^k_{g,1})^{\mathrm{ab}}$ is trivial if $k=1$ and isomorphic to $\mathbb{Z}/2$ if $k \geqslant 2$.*

Proof. We recall from Corollary 4.19 that $\mathbf{\Gamma}^k_{g,1}$ is the semi-direct product $\mathbf{B}_k(\Sigma_{g,1}) \rtimes \mathbf{\Gamma}_{g,1}$. By abelianisation of a semi-direct product (see [GG09, Proposition 3.3] for instance), we deduce that $(\mathbf{\Gamma}^k_{g,1})^{\mathrm{ab}}$ is isomorphic to $(\mathbf{B}_k(\Sigma_{g,1})^{\mathrm{ab}})_{\mathbf{\Gamma}_{g,1}} \oplus (\mathbf{\Gamma}_{g,1})^{\mathrm{ab}}$, where $(\mathbf{B}_k(\Sigma_{g,1})^{\mathrm{ab}})_{\mathbf{\Gamma}_{g,1}}$ denotes the coinvariants of $\mathbf{B}_k(\Sigma_{g,1})^{\mathrm{ab}}$ under the natural action of $\mathbf{\Gamma}_{g,1}$ on $\Sigma_{g,1}$. We recall from [Kor02, Theorem 5.1] that the abelianisation of $(\mathbf{\Gamma}_{g,1})^{\mathrm{ab}}$ is trivial for any $g \geqslant 3$. It therefore just remains to study the left-hand summand. By [BGG17, Proposition 3.4], we know that

$$\mathbf{B}_k(\Sigma_{g,1})^{\mathrm{ab}} \;\cong\; \begin{cases} H_1(\Sigma_{g,1}) \oplus \mathbb{Z}/2 & k \geqslant 2, \\ H_1(\Sigma_{g,1}) & k = 1, \end{cases} \tag{5.29}$$

where the generator σ of the $\mathbb{Z}/2$ summand is the image of all the braid generators of $\mathbf{B}_k(\Sigma_{g,1})$. The splitting $\mathbf{\Gamma}_{g,1} \hookrightarrow \mathbf{\Gamma}^k_{g,1}$ of the exact sequence is induced by the embedding of surfaces $\Sigma_{g,1} \hookrightarrow \Sigma^k_{0,1}\natural\Sigma_{g,1} \cong \Sigma^k_{g,1}$, so $\mathbf{\Gamma}_{g,1}$ acts trivially on σ, since σ is supported in the subsurface $\Sigma^k_{0,1}$.

The action of $\mathbf{\Gamma}_{g,1}$ on the left-hand summand of (5.29) is induced by the natural action of $\mathbf{\Gamma}_{g,1}$ on the homology of the surface. Together with the fact that its action on σ is trivial, this in particular implies that (5.29) is a splitting of representations of $\mathbf{\Gamma}_{g,1}$, not just of abelian groups. The action of the Dehn twists $T_{A_i}, T_{B_i} \in \mathbf{\Gamma}_{g,1}$ on $H_1(\Sigma_{g,1}) = \langle A_1, \ldots, A_g, B_1, \ldots, B_g \rangle$ satisfies $T_{A_i}(B_i) - B_i = A_i$ and $T_{B_i}(A_i) - A_i = B_i$, so we have $H_1(\Sigma_{g,1})_{\mathbf{\Gamma}_{g,1}} = 0$. Putting this all together, we end the proof. □

In particular, the abelianisation $(\mathbf{\Gamma}^k_{g,1})^{\mathrm{ab}}$ is cyclic, generated by σ. It therefore follows from [DPS21, Corollary 2.2] that $\Gamma_2(\mathbf{\Gamma}^k_{g,1}) = \Gamma_3(\mathbf{\Gamma}^k_{g,1})$ for all $k \geqslant 1$ and $g \geqslant 3$. In the exceptional cases $g \in \{1,2\}$, there is an additional cyclic summand in $(\mathbf{\Gamma}^k_{g,1})^{\mathrm{ab}}$ generated by a Dehn twist T around a non-separating curve (of infinite order if $g=1$ and order 10 if $g=2$), by [Kor02, Theorem 5.1]. Since the two generators T and σ have representatives with disjoint support, the methods of [DPS21, §2] imply that, also in these exceptional cases, the lower central series stops at Γ_2, in other words $\Gamma_2(\mathbf{\Gamma}^k_{g,1}) = \Gamma_3(\mathbf{\Gamma}^k_{g,1})$. Hence we may restrict our study to the case of $\ell = 2$.

The quotient group $Q_{(\underline{k},\mathfrak{S}_k,2)}(\Sigma_{g,1})$ is isomorphic to the coinvariants $(\mathbf{B}_k(\Sigma_{g,1})^{\mathrm{ab}})_{\mathbf{\Gamma}_{g,1}}$; see the first two sentences of the proof above. The arguments of that proof imply that, for all $k \geqslant 1$ and $g \geqslant 1$, we have:

$$Q_{(\underline{k},\mathfrak{S}_k,2)}(\Sigma_{g,1}) \cong \begin{cases} 0 & \text{if } k = 1, \\ \mathbb{Z}/2 & \text{if } k \geqslant 2. \end{cases}$$

We denote this quotient by $Q_k(\Sigma)$ and deduce from Proposition 5.52 that:

Proposition 5.54 *For each $i \geqslant 0$ and $k \geqslant 1$, the functor (5.28) for $\ell = 2$ defines a homological functor $L_i(\mathring{F}_{(\underline{k},\mathfrak{S}_k,2)}) \colon \mathfrak{U}\mathcal{M}_2^+ \to \mathrm{Mod}_{\mathbb{Z}[Q_k(\Sigma)]}$.*

Remark 5.55 As far as the authors know, the representations arising in Proposition 5.54 are new. In particular, we prove in [PS21a, §3] that the representations defined by $L_k(\mathring{F}^{BM}_{(\underline{k},\mathfrak{S}_k,2)})$ are not *symplectic*, in the sense that they do not factor through the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$.

5.4.2.2. For non-orientable surfaces

Consider the restriction of the continuous functor $\mathring{F}_{(\underline{k},\mathfrak{S}_k,\ell)} \colon \mathfrak{U}\mathcal{D}_2 \to \mathrm{Cov}_\bullet$ of Proposition 5.44 to the full subcategory $\mathfrak{U}\mathcal{M}_2^{\mathrm{t},-}$. By §2.5.1 we obtain a homological functor

$$L_i(\mathring{F}_{(\underline{k},\mathfrak{S}_k,\ell)}) \colon \mathfrak{U}\mathcal{M}_2^- \longrightarrow \mathrm{Mod}_\bullet, \tag{5.30}$$

for $i, k \geqslant 1$ and $\ell \geqslant 0$.

Let $k \geqslant 0$ and $h \geqslant 5$. By [Stu10, Theorems 6.21 and 6.22], the abelianisation $(\boldsymbol{\mathcal{N}}^k_{h,1})^{\mathrm{ab}}$ is isomorphic to a direct sum of copies of $\mathbb{Z}/2$, generated by:

- a cross-cap slide y supported in a genus-2 subsurface,
- a Dehn twist T along a non-separating curve (when $h \in \{5,6\}$),
- a marked point slide c sending a marked point through the core of a cross-cap (when $k \geqslant 1$),
- a braid-type element σ that interchanges two marked points (when $k \geqslant 2$).

Using the same type of disjoint support arguments as [DPS21, §2], one may deduce from this that, for all $k \geqslant 3$ and $h \geqslant 6$ an *even* number, the lower central series of the mapping class group $\boldsymbol{\mathcal{N}}^k_{h,1}$ stops at Γ_2, in other words, $\Gamma_2(\boldsymbol{\mathcal{N}}^k_{h,1}) = \Gamma_3(\boldsymbol{\mathcal{N}}^k_{h,1})$. Hence we restrict our study to the case $\ell = 2$.

Lemma 5.56 *For $h \geqslant 2$, we have $H_1(\mathcal{N}_{h,1})_{\boldsymbol{\mathcal{N}}_{h,1}} \cong \mathbb{Z}/2$, generated by the core of a cross-cap.*

Proof. The first homology group $H_1(\mathcal{N}_{h,1})$ is isomorphic to $\mathbb{Z}^h$, generated by the cores of the cross-caps, which we denote by $\gamma_1, \ldots, \gamma_h$. The inclusion of surfaces $\Sigma_{0,h+1} \hookrightarrow \mathcal{N}_{h,1}$ given by gluing in h Möbius bands induces a homomorphism $\mathbf{B}_h \to \boldsymbol{\mathcal{N}}_{h,1}$. In particular, we have an element of $\boldsymbol{\mathcal{N}}_{h,1}$ that cyclically permutes the h cross-caps. This element induces the relations $\gamma_1 = \gamma_2 = \cdots = \gamma_h$ when we take co-invariants. Let y be the cross-cap slide that takes the first cross-cap through the core of one of the others. Then $y(\gamma_1) = -\gamma_1$, so this element induces the relation $(\gamma_1)^2 = \mathrm{id}$ when we take co-invariants. Thus we have shown that the co-invariants are either as claimed, or trivial; it remains to show that they are non-trivial.

First, assume that $h \geqslant 5$. From the identity

$$(\boldsymbol{\mathcal{N}}^k_{h,1})^{\mathrm{ab}} \cong (\mathbf{B}_k(\mathcal{N}_{h,1})^{\mathrm{ab}})_{\boldsymbol{\mathcal{N}}_{h,1}} \oplus (\boldsymbol{\mathcal{N}}_{h,1})^{\mathrm{ab}} \tag{5.31}$$

for $k = 1$, together with the results of Stukow quoted above, we deduce that

$$H_1(\mathcal{N}_{h,1})_{\boldsymbol{\mathcal{N}}_{h,1}} \cong (\mathbf{B}_1(\mathcal{N}_{h,1})^{\mathrm{ab}})_{\boldsymbol{\mathcal{N}}_{h,1}} \cong \mathbb{Z}/2,$$

in particular, the co-invariants are non-trivial.[6] To see that they are non-trivial for $h \in \{2,3,4\}$, we first consider the case $h = 1$. In this case, the mapping class group of the Möbius band is trivial, so we have $H_1(\mathcal{N}_{1,1})_{\boldsymbol{\mathcal{N}}_{1,1}} = H_1(\mathcal{N}_{1,1}) \cong \mathbb{Z}$. Next, consider the commutative diagram

$$\begin{array}{ccccc}
\mathbb{Z} & & \mathbb{Z}^h & & \mathbb{Z}^5 \\
\| & & \| & & \| \\
H_1(\mathcal{N}_{1,1}) & \longrightarrow & H_1(\mathcal{N}_{h,1}) & \longrightarrow & H_1(\mathcal{N}_{5,1}) \\
{\scriptstyle\cong}\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z} & \longrightarrow & ? & \longrightarrow & \mathbb{Z}/2
\end{array}$$

where $h \in \{2,3,4\}$ and the vertical maps are the quotients onto the respective co-invariants. The top horizontal maps are the standard inclusions and the right-hand vertical map sends each standard generator of $\mathbb{Z}^5$ onto the generator of $\mathbb{Z}/2$. Hence the map across the bottom of the diagram is surjective, so ? cannot be 0. $\square$

Using Lemma 5.56, we may deduce that we have Q-stability in the range $h \geqslant 2$, as follows. From the identity (5.31), we see that the quotient group $Q_{(\underline{k},\mathfrak{S}_k,2)}(\mathcal{N}_{h,1})$ is isomorphic to the co-invariants $(\mathbf{B}_k(\mathcal{N}_{h,1})^{\mathrm{ab}})_{\boldsymbol{\mathcal{N}}_{h,1}}$. Similarly to the orientable case, we have

$$\mathbf{B}_k(\mathcal{N}_{h,1})^{\mathrm{ab}} \cong \begin{cases} H_1(\mathcal{N}_{h,1}) \oplus \mathbb{Z}/2 & k \geqslant 2, \\ H_1(\mathcal{N}_{h,1}) & k = 1, \end{cases}$$

see [DPS21, Proposition 6.23]. Again, this is a splitting of representations of $\boldsymbol{\mathcal{N}}_{h,1}$ and the action on the $\mathbb{Z}/2$ summand is trivial. Combining this with Lemma 5.56, we deduce that

$$Q_{(\underline{k},\mathfrak{S}_k,2)}(\mathcal{N}_{h,1}) \cong \begin{cases} \mathbb{Z}/2 & \text{if } k = 1, \\ (\mathbb{Z}/2)^2 & \text{if } k \geqslant 2. \end{cases}$$

We denote this quotient by $Q_k(\mathcal{N})$ and deduce from Proposition 5.52 that:

Proposition 5.57 *For each $i \geqslant 0$ and $k \geqslant 1$, the functor (5.30) for $\ell = 2$ defines a homological functor $L_i(\mathring{F}_{(\underline{k},\mathfrak{S}_k,2)})\colon \mathfrak{UM}_2^- \to \mathrm{Mod}_{\mathbb{Z}[Q_k(\mathcal{N})]}$.*

As far as the authors know, the representations encoded by the functors of Proposition 5.57 do not appear in the literature, and are therefore new.

[6] An alternative argument, which works for $h \geqslant 3$, is to consider Stukow's generating set for the mapping class group $\boldsymbol{\mathcal{N}}_{h,1}$ in [Stu10, Theorem 5.2], and study how these generators act on $\gamma_1 = \cdots = \gamma_h$. There are only three that act non-trivially: t_{f_1} if the genus h is odd and t_{c_r} and $t_{b_{r+1}}$ if the genus $h = 2r + 2$ is even. In each case, one may verify that these elements do not introduce any new relations in the co-invariants.

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