The Burau representations of loop braid groups

Martin Palmer and Arthur Soulié

23$^{\text{rd}}$ September 2021

Abstract

We give a simple topological construction of the Burau representations of the loop braid groups. There are four versions: defined either on the non-extended or extended loop braid groups, and in each case there is an unreduced and a reduced version. Three are not surprising, and one could easily guess the correct matrices to assign to generators. The fourth is more subtle, and does not seem combinatorially obvious, although it is topologically very natural.

Introduction

Loop braid groups appear in many guises in topology and group theory. They may be seen geometrically as fundamental groups of trivial links in $\mathbb{R}^3$, diagrammatically as equivalence classes of *welded braids* (closely related to virtual braids and virtual knot theory), algebraically as subgroups of automorphism groups of free groups or combinatorially via explicit group presentations.

Loop braid groups have been studied, from the topological viewpoint of motions of trivial links in $\mathbb{R}^3$, by Dahm [Dah62], Goldsmith [Gol81], Brownstein and Lee [BL93] and Jensen, McCammond and Meier [JMM06]. In parallel, the *symmetric automorphism groups* and the *braid-permutation groups* (subgroups of $\mathrm{Aut}(F_n)$, which may also be interpreted in terms of welded braids) were studied by McCool [McC86], Collins [Col89] and Fenn, Rimányi and Rourke [FRR97]. The two points of view were then brought together in [BWC07] by finding common group presentations. Loop braid groups, as well as related groups of "wickets", have also been studied more recently by Brendle and Hatcher [BH13]. For a detailed survey of the many different facets of loop braid groups, see Damiani's survey [Dam17].

The definition that we shall use is the following.

Definition 1 Let $\mathbb{D}^3$ denote the closed unit ball in $\mathbb{R}^3$ and choose a trivial n-component link U_n in its interior. Let $\mathrm{Emb}(U_n, \mathbb{D}^3)$ denote the set of all smooth embeddings of U_n into the interior of $\mathbb{D}^3$, equipped with the smooth Whitney topology, and write $\mathrm{Emb}^u(U_n, \mathbb{D}^3)$ for the path-component containing the inclusion (the superscript ^{u} stands for "unknotted and unlinked"). There is a natural action of the diffeomorphism group $\mathrm{Diff}(U_n) \cong \mathrm{Diff}(\mathbb{S}^1) \wr \mathfrak{S}_n$ on this space, and we define

$$\mathcal{E}(U_n, \mathbb{D}^3) \coloneqq \mathrm{Emb}^u(U_n, \mathbb{D}^3)/\mathrm{Diff}(U_n).$$

The n-th *extended loop braid group* is the fundamental group $\mathbf{LB}'_n \coloneqq \pi_1(\mathcal{E}(U_n, \mathbb{D}^3))$. Similarly, we define

$$\mathcal{E}^+(U_n, \mathbb{D}^3) \coloneqq \mathrm{Emb}^u(U_n, \mathbb{D}^3)/\mathrm{Diff}^+(U_n),$$

where Diff^+ denotes orientation-preserving diffeomorphisms, and the n-th (non-extended) *loop braid group* is the fundamental group $\mathbf{LB}_n \coloneqq \pi_1(\mathcal{E}^+(U_n, \mathbb{D}^3))$.

Thus elements of $\mathbf{LB}'_n$ are thought of as loops of n-component unlinks in $\mathbb{R}^3$, and elements of $\mathbf{LB}_n$ are thought of as loops of *oriented* n-component unlinks in $\mathbb{R}^3$. Since $\text{Diff}^+(U_n)$ is an index-2^n subgroup of $\text{Diff}(U_n)$, the natural quotient map

$$\mathcal{E}^+(U_n, \mathbb{D}^3) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{E}(U_n, \mathbb{D}^3)$$

is a 2^n-sheeted covering map, and thus induces an injection

$$\mathbf{LB}_n \lhook\joinrel\longrightarrow \mathbf{LB}'_n \tag{1}$$

of fundamental groups. Thus we view the (non-extended) loop braid group $\mathbf{LB}_n$ as a subgroup (of index 2^n) of the extended loop braid group $\mathbf{LB}'_n$.

Generators. We fix a basepoint for $\mathcal{E}^+(U_n, \mathbb{D}^3)$ where the n circles are arranged on the xy-plane in a row from left to right, as pictured in Figure 4. With respect to this basepoint, the loop braid group $\mathbf{LB}_n$ is generated by the elements $\tau_1, \ldots, \tau_{n-1}$ and $\sigma_1, \ldots, \sigma_{n-1}$ illustrated in Figure 1. The elements τ_i and σ_i involve only the i-th and $(i+1)$-st loops, which are exchanged; for τ_i, no loop passes through the other; for σ_i, the i-th loop passes through the $(i+1)$-st loop. The extended loop braid group $\mathbf{LB}'_n$ is generated by these elements together with the elements $\rho_1, \ldots, \rho_n$, also illustrated in Figure 1. For finite presentations of $\mathbf{LB}_n$ and $\mathbf{LB}'_n$ using these generators, see Fenn, Rimanyi and Rourke [FRR97, §1] and Brendle and Hatcher [BH13, Propositions 3.3 and 3.7]. We note that there are many conflicting conventions for the names of these generators in the literature; in particular, our notation is consistent with [FRR97] but inconsistent with [BH13].
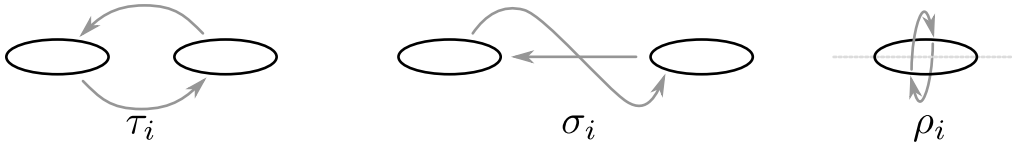

Figure 1 The loop braid group $\mathbf{LB}_n$ is generated by the loops of loops $\tau_1, \ldots, \tau_{n-1}$ and $\sigma_1, \ldots, \sigma_{n-1}$. Together with $\rho_1, \ldots, \rho_n$, these generate the extended loop braid group $\mathbf{LB}'_n$.

Burau representations of classical braid groups. The classical braid groups $\mathbf{B}_n$ are the fundamental groups of the configuration spaces $C_n(\mathbb{R}^2)$ of points in the plane. One of the oldest interesting representations of $\mathbf{B}_n$ is the *Burau representation* [Bur35]

$$\mathbf{B}_n \longrightarrow \text{GL}_n(\mathbb{Z}[t^{\pm 1}]), \tag{2}$$

which was defined originally by assigning explicit matrices to the standard generators of $\mathbf{B}_n$, but which is most naturally understood as a homological representation, as follows. The braid group $\mathbf{B}_n$ is naturally isomorphic to the *mapping class group* $\text{MCG}(\mathbb{D}_n^2) = \pi_0(\text{Diff}_\partial(\mathbb{D}^2, Q_n))$, the group of isotopy classes of diffeomorphisms of the 2-disc that act by the identity on its boundary and that preserve a subset Q_n of n points in its interior. In this way, $\mathbf{B}_n$ acts (up to homotopy) on the complement $\mathbb{D}_n^2 = \mathbb{D}^2 \smallsetminus Q_n$. There is a projection $\pi_1(\mathbb{D}_n^2) \twoheadrightarrow \mathbb{Z}$ sending a loop to the sum of its winding numbers around each of the points Q_n, and it turns out that the $\mathbf{B}_n$ action on $\mathbb{D}_n^2$ lifts to the corresponding regular covering space $\pi \colon \widetilde{\mathbb{D}}_n^2 \twoheadrightarrow \mathbb{D}_n^2$ and commutes with the deck transformations. The induced $\mathbf{B}_n$ action on the first homology $H_1(\widetilde{\mathbb{D}}_n^2)$ therefore respects its structure as a module over the group-ring of the deck transformation group, $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t^{\pm 1}]$, so we obtain a representation

$$\mathbf{B}_n \longrightarrow \text{Aut}_{\mathbb{Z}[t^{\pm 1}]}\big(H_1(\widetilde{\mathbb{D}}_n^2)\big).$$

The homology group $H_1(\widetilde{\mathbb{D}}_n^2)$ is in fact a free $\mathbb{Z}[t^{\pm 1}]$-module of rank $n-1$, so choosing a free basis we may rewrite this as

$$\mathbf{B}_n \longrightarrow \text{GL}_{n-1}(\mathbb{Z}[t^{\pm 1}]). \tag{3}$$

This is the *reduced Burau representation*. To obtain the unreduced Burau representation (2), we consider instead the induced $\mathbf{B}_n$ action on the *relative* first homology $H_1(\widetilde{\mathbb{D}}_n^2, \pi^{-1}(*))$, where $*$ is

a basepoint in the boundary of the disc. This is now a free $\mathbb{Z}[t^{\pm 1}]$-module of rank n, so choosing a free basis we obtain (2). The canonical map $H_1(\widetilde{\mathbb{D}}_n^2) \to H_1(\widetilde{\mathbb{D}}_n^2, \pi^{-1}(*))$ is injective, so the reduced Burau representation (3) is a subrepresentation of the Burau representation (2).

Choosing appropriate ordered free generating sets for $H_1(\widetilde{\mathbb{D}}_n^2)$ and $H_1(\widetilde{\mathbb{D}}_n^2, \pi^{-1}(*))$ over $\mathbb{Z}[t^{\pm 1}]$, the representations (2) and (3) may be written explicitly as

$$\sigma_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 1-t & 1 \\ t & 0 \end{bmatrix} \oplus I_{n-i-1} \qquad \text{and} \qquad \sigma_i \longmapsto I_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus I_{n-i-2} \qquad (4)$$

respectively. We note that the Burau representation is sometimes defined using the transposes of these matrices, such as in [KT08], but this is not an essential difference, since the Burau representation is equivalent to its transpose. For more details of these representations, see [KT08].

From classical braids to loop braids. There is an obvious map

$$t \colon C_n(\mathbb{R}^2) \longrightarrow \mathcal{E}^+(U_n, \mathbb{D}^3) \qquad (5)$$

given by replacing each point in the given configuration with a small circle, oriented positively in $\mathbb{R}^2$, and then including this unlinked configuration of circles into $\mathbb{R}^3$. On fundamental groups, this induces a homomorphism $\mathbf{B}_n \to \mathbf{LB}_n$ sending the standard generators of $\mathbf{B}_n$ to the elements $\tau_1, \ldots, \tau_{n-1}$ of $\mathbf{LB}_n$. In particular, this map factors through the projection $\mathbf{B}_n \to \mathfrak{S}_n$ onto the symmetric group on n letters. There is also a more interesting map

$$s \colon C_n(\mathbb{R}^2) \longrightarrow \mathcal{E}^+(U_n, \mathbb{D}^3) \qquad (6)$$

defined as follows. Let us identify $\mathbb{R}^2$ with the right-hand xz-plane (the half where the x-coordinate is positive) and the interior of $\mathbb{D}^3$ with $\mathbb{R}^3$. Given a configuration of n points in the right-hand xz-plane, we produce an n-component unlink by rotating the configuration about the z-axis, tracing out n circles while doing so, which all lie in planes parallel to the xy-plane; see Figure 2. We orient these circles positively with respect to the parallel copy of the xy-plane in which they lie.
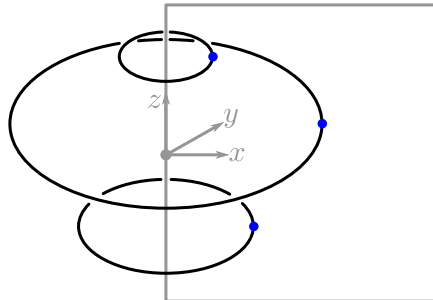

Figure 2   Given a configuration of (blue) points in the right-hand xz-plane, we rotate about the z-axis as shown to produce a configuration of unlinked circles in $\mathbb{R}^3$.

To see that the induced homomorphism on fundamental groups is injective, let us take a basepoint in $C_n(\mathbb{R}^2)$ where the configuration points are arranged in a line along the x-axis; this corresponds to a basepoint of $\mathcal{E}^+(U_n, \mathbb{D}^3)$ with n concentric circles in the xy-plane, centred at the origin. The standard generators of $\mathbf{B}_n$ are sent to loops of the form illustrated on the left-hand side of Figure 3. Changing the basepoint of $\mathcal{E}^+(U_n, \mathbb{D}^3)$ to the one chosen earlier, with *non-concentric* circles on the xy-plane, this corresponds to the loop on the right-hand side of Figure 3, which is the generator σ_i of $\mathbf{LB}_n$. By [BH13, Proposition 4.3], the group homomorphism $\mathbf{B}_n \to \mathbf{LB}_n$ sending the standard generators of $\mathbf{B}_n$ to the elements $\sigma_1, \ldots, \sigma_{n-1} \in \mathbf{LB}_n$ is injective. As we have just seen, the map (6) realises this homomorphism at the space level, and so:

Proposition 2 *The map* (6) *induces an injection on* π_1.

Figure 3   The image of the i-th standard generator of $\mathbf{B}_n$ under the map s_* (left) corresponds, by a change of basepoint, to the element σ_i of $\mathbf{LB}_n$ (right).

Burau representations of loop braid groups. A natural question is whether, and how, one may extend the Burau representations (2) and (3) along the inclusions

$$\mathbf{B}_n \overset{(6)_*}{\hookrightarrow} \mathbf{LB}_n \overset{(1)}{\hookrightarrow} \mathbf{LB}'_n. \tag{7}$$

The unreduced Burau representation (2) has been extended to $\mathbf{LB}_n$ by Vershinin [Ver01], using a presentation of $\mathbf{LB}_n$ and by assigning explicit matrices to generators. (More precisely, Vershinin's representation of $\mathbf{LB}_n$ restricts to the *transpose* of the unreduced Burau representation of $\mathbf{B}_n$, according to our conventions.)

Our approach will instead be topological, analogous to the description above of the classical Burau representations of $\mathbf{B}_n$, via the homology of covering spaces. In each case, we will find a natural generating set of the relevant homology group, and calculate the matrix that each standard generator of the loop braid group is sent to.

Notation 3 We write $R = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}]$ and $S = \mathbb{Z}[\mathbb{Z}/2] = \mathbb{Z}[t^{\pm 1}]/(t^2 - 1)$.

In this notation, the unreduced and reduced Burau representations are R-linear actions $\mathbf{B}_n \curvearrowright R^{\oplus n}$ and $\mathbf{B}_n \curvearrowright R^{\oplus n-1}$ respectively. The case of the non-extended loop braid groups is straightforward:

Theorem 4 *These R-linear actions extend to R-linear actions $\mathbf{LB}_n \curvearrowright R^{\oplus n}$ and $\mathbf{LB}_n \curvearrowright R^{\oplus n-1}$. Explicit matrices are given in equations* (15) *and* (17) *respectively.*

To extend further to the extended loop braid groups is a little more subtle. We must first reduce modulo $t^2 - 1$, in other words tensor $- \otimes_R S$ to obtain S-linear actions $\mathbf{LB}_n \curvearrowright S^{\oplus n}$ and $\mathbf{LB}_n \curvearrowright S^{\oplus n-1}$. The unreduced Burau representation then extends directly:

Theorem 5 *The S-linear action $\mathbf{LB}_n \curvearrowright S^{\oplus n}$ extends to an S-linear action $\mathbf{LB}'_n \curvearrowright S^{\oplus n}$. Explicit matrices are given in equations* (15) *and* (21).

The reduced Burau representation does not extend directly; instead:

Theorem 6 *The S-linear action $\mathbf{LB}_n \curvearrowright S^{\oplus n-1}$ is a subrepresentation of an S-linear action $\mathbf{LB}_n \curvearrowright S^{\oplus n-1} \oplus S/(t-1)$, and this extends to an S-linear action $\mathbf{LB}'_n \curvearrowright S^{\oplus n-1} \oplus S/(t-1)$. Explicit matrices are given in Table 1 on page 12.*

We emphasise that these extensions of the Burau representations to $\mathbf{LB}_n$ and $\mathbf{LB}'_n$ are precisely those that arise naturally via actions on first homology groups of covering spaces, mirroring the topological construction of the classical Burau representations. As a partial summary, we have

$$
\begin{array}{cccc}
{\color{blue}x_1,\dots,x_{n-1}} & {\color{blue}x_1,\dots,x_{n-1}} & {\color{blue}x_1,\dots,x_{n-1},y} & {\color{blue}a_1,\dots,a_n} \\
R^{\oplus n-1} \twoheadrightarrow & S^{\oplus n-1} \hookrightarrow & S^{\oplus n-1} \oplus S/(t-1) \hookrightarrow & S^{\oplus n} \\
\| & \| & \| & \| \\
H_1(\widetilde{\mathbb{D}}^3_n; \mathbb{Z}) & H_1(\widetilde{\mathbb{D}}^3_n; \mathbb{Z}) \otimes_R S & H_1(\widehat{\mathbb{D}}^3_n; \mathbb{Z}) & H_1(\widehat{\mathbb{D}}^3_n, \{v, tv\}; \mathbb{Z}),
\end{array}
\tag{8}
$$

where from left to right we have (1) the reduced Burau representation of $\mathbf{LB}_n$ over $R = \mathbb{Z}[t^{\pm 1}]$ (Theorem 4), (2) its reduction modulo $t^2 - 1$ over $S = \mathbb{Z}[t^{\pm 1}]/(t^2 - 1)$, (3) the inclusion of (2) into the reduced Burau representation of $\mathbf{LB}'_n$ over S (Theorem 6) and (4) its further inclusion into the unreduced Burau representation of $\mathbf{LB}'_n$ (Theorem 5). In each case, an ordered generating set corresponding to the direct sum decomposition is given in blue.

4

Remark 7 The matrices of the representations in Theorems 4 and 5 are the "obvious" matrices that one may guess by analogy with the matrices for the classical (reduced and unreduced) Burau representations. However, the matrices for the reduced Burau representation of the extended loop braid groups, from Theorem 6, do not seem combinatorially or algebraically obvious. However, they arise very naturally *topologically*. Two additional subtleties in this case are the appearance of torsion in the $S/(t-1)$ summand and the *non-locality* of the matrices for the ρ_i generators.

Remark 8 It is stated in Theorem 6 that the $\mathbf{LB}_n$-representation $S^{\oplus n-1}$ is a subrepresentation of an $\mathbf{LB}_n$-representation $S^{\oplus n-1} \oplus S/(t-1)$. We remark that it is however *not* a direct summand of the $\mathbf{LB}_n$-representation $S^{\oplus n-1} \oplus S/(t-1)$.

See §4 for further remarks on reducibility, faithfulness and other properties of these representations.

Remark 9 The Burau representations of the classical braid groups $\mathbf{B}_n$ form the first of an infinite family of *Lawrence-Bigelow* representations [Law90; Big04], and the Burau representations of the loop braid groups $\mathbf{LB}_n$ (extended loop braid groups $\mathbf{LB}'_n$) may be extended, in more than one way, to an analogous infinite family of representations; see [PS19]. Explicit bases and matrices for these representations will be the subject of forthcoming work.

1. Action on the homology of covering spaces

Let φ be a diffeomorphism of the 3-ball $\mathbb{D}^3$ that restricts to the identity near the boundary. We may restrict φ to our chosen unlink U_n in the interior of $\mathbb{D}^3$ to obtain a new embedding $U_n \hookrightarrow \mathbb{D}^3$. Since φ is isotopic to a diffeomorphism that acts by the identity on U_n (it may be isotoped to act by the identity on a larger and larger collar neighbourhood of the boundary, until this collar neighbourhood contains U_n), this new embedding is isotopic to the inclusion, hence an element of $\text{Emb}^u(U_n, \mathbb{D}^3)$. We therefore have a restriction map

$$\text{Diff}_\partial(\mathbb{D}^3) \longrightarrow \text{Emb}^u(U_n, \mathbb{D}^3), \tag{9}$$

where $\text{Diff}_\partial(\mathbb{D}^3)$ denotes the topological group of diffeomorphisms that are the identity on a neighbourhood of the boundary, equipped with the smooth Whitney topology. The map (9) is a locally trivial fibration [Pal60; Cer61; Lim63] and the quotient map

$$\text{Emb}^u(U_n, \mathbb{D}^3) \longrightarrow\!\!\!\!\!\!\rightarrow \text{Emb}^u(U_n, \mathbb{D}^3)/\text{Diff}(U_n) = \mathcal{E}(U_n, \mathbb{D}^3) \tag{10}$$

is also a locally trivial fibration [BF81] (see also [Pal21, §4] for both of these). Putting together (9) and (10), we have a locally trivial fibration

$$\text{Diff}_\partial(\mathbb{D}^3) \longrightarrow \mathcal{E}(U_n, \mathbb{D}^3). \tag{11}$$

If we modify (10) to quotient only by $\text{Diff}^+(U_n)$, it remains a locally trivial fibration, and together with (9) we obtain a locally trivial fibration

$$\text{Diff}_\partial(\mathbb{D}^3) \longrightarrow \mathcal{E}^+(U_n, \mathbb{D}^3). \tag{12}$$

Together with Hatcher's proof [Hat83] of the Smale conjecture, this implies the following, where $\text{Diff}_\partial(\mathbb{D}^3, U_n) \leqslant \text{Diff}_\partial(\mathbb{D}^3)$ is the subgroup of diffeomorphisms that preserve U_n (setwise) and $\text{Diff}_\partial(\mathbb{D}^3, U_n^+) \leqslant \text{Diff}_\partial(\mathbb{D}^3)$ is the subgroup of diffeomorphisms that preserve U_n and its orientation.

Lemma 10 *There are isomorphisms*

$$\mathbf{LB}'_n = \pi_1(\mathcal{E}(U_n, \mathbb{D}^3)) \cong \pi_0(\text{Diff}_\partial(\mathbb{D}^3, U_n))$$
$$\mathbf{LB}_n = \pi_1(\mathcal{E}^+(U_n, \mathbb{D}^3)) \cong \pi_0(\text{Diff}_\partial(\mathbb{D}^3, U_n^+)).$$

Proof. The topological group $\text{Diff}_\partial(\mathbb{D}^3)$ is contractible, in particular simply-connected, by [Hat83], and these isomorphisms then follow from the long exact sequences associated to (11) and (12). $\square$
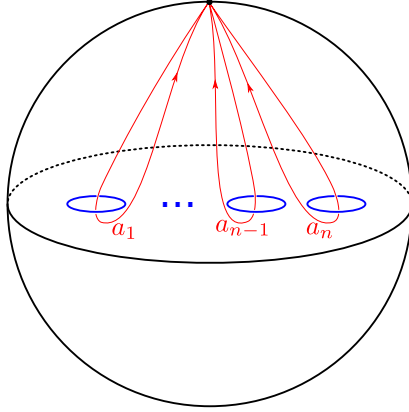
Figure 4   The unlink-complement $\mathbb{D}_n^3$ with free generators $a_1, \ldots, a_n$ for $\pi_1(\mathbb{D}_n^3) \cong F_n$.

Notation 11 We will abbreviate $\mathbb{D}_n^3 = \mathbb{D}^3 \smallsetminus U_n$, where U_n is the n-component unlink in the interior of $\mathbb{D}^3$ chosen previously. See Figure 4 for an illustration of a particular choice.

By the mapping class group interpretation of loop braid groups (Lemma 10), the group $\mathbf{LB}_n'$ (and hence also its subgroup $\mathbf{LB}_n$) acts, up to homotopy, on the unlink-complement $\mathbb{D}_n^3$ by diffeomorphisms (in particular, homeomorphisms).

Remark 12 We will speak of actions of mapping class groups up to homotopy, which induce (strict) actions on homology. An alternative, equivalent viewpoint would be that the corresponding *diffeomorphism group* acts (strictly) at the level of spaces, and then observing that its induced action on homology factors through the mapping class group, since homology groups are discrete.

Note that the fundamental group of $\mathbb{D}_n^3$ is the free group F_n on n generators. This is easy to see: the unlink-complement $\mathbb{D}_n^3 \subseteq \mathbb{D}^3$ deformation retracts onto a wedge of n circles and n copies of the 2-sphere. The n circles $a_1, \ldots, a_n$ are shown in Figure 4. Now let

$$\phi \colon \pi_1(\mathbb{D}_n^3) \longrightarrow \mathbb{Z}$$

be the surjective homomorphism defined by $\phi(a_i) = 1$ for all $i = 1, \ldots, n$ and let

$$\phi' \colon \pi_1(\mathbb{D}_n^3) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

be the composition of ϕ with the unique surjection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$.

Definition 13 We denote by $\widetilde{\mathbb{D}}_n^3$ the regular covering space corresponding to $\ker(\phi)$ and by $\widehat{\mathbb{D}}_n^3$ the regular covering space corresponding to $\ker(\phi')$. We therefore have regular coverings

$$\eta \colon \widetilde{\mathbb{D}}_n^3 \longrightarrow \mathbb{D}_n^3 \qquad \text{and} \qquad \eta' \colon \widehat{\mathbb{D}}_n^3 \longrightarrow \mathbb{D}_n^3$$

whose deck transformations groups are $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively.

In general, if a group G acts (up to homotopy) on a based space X and we choose a surjection $\psi \colon \pi_1(X) \twoheadrightarrow Q$ that is invariant under the induced action of G on $\pi_1(X)$, then the G-action on X lifts uniquely to the regular covering space corresponding to ψ and commutes with the action of Q by deck transformations.

Let us first take $G = \mathbf{LB}_n$ and $X = \mathbb{D}_n^3$ with basepoint $* \in \partial \mathbb{D}^3 = \partial \mathbb{D}_n^3$. We note that the quotient ϕ is invariant under the action of $\mathbf{LB}_n$: for this it suffices to check that each generator τ_i, σ_i of $\mathbf{LB}_n$ sends each generator a_j of $\pi_1(\mathbb{D}_n^3)$ to an element in $\phi^{-1}(1)$, and this follows since, up to conjugation, τ_i and σ_i simply permute the generators a_j. We therefore have an induced action (up to homotopy) of $\mathbf{LB}_n$ on $\widetilde{\mathbb{D}}_n^3$ commuting with the deck transformation action of $\mathbb{Z}$. Thus the first integral homology groups

$$H_1\big(\widetilde{\mathbb{D}}_n^3; \mathbb{Z}\big) \qquad \text{and} \qquad H_1\big(\widetilde{\mathbb{D}}_n^3, \eta^{-1}(*); \mathbb{Z}\big) \tag{13}$$
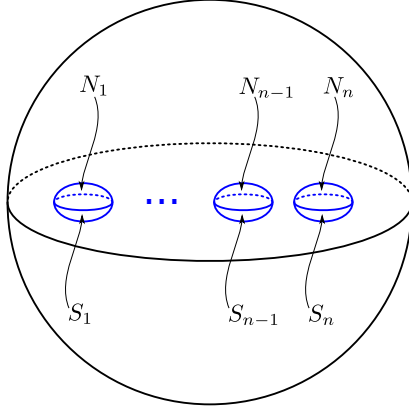
Figure 5 The complement $\mathring{\mathbb{D}}^3_n$ of the interiors and equators of n closed little 3-discs ("lens shapes") in the interior of the closed unit 3-disc $\mathbb{D}^3$. The boundary of $\mathring{\mathbb{D}}^3_n$ decomposes as the disjoint union of $2n+1$ components: $\partial\mathring{\mathbb{D}}^3_n = \partial\mathbb{D}^3 \sqcup N_1 \sqcup \ldots \sqcup N_n \sqcup S_1 \sqcup \ldots \sqcup S_n$.

are $\mathbb{Z}[\mathbb{Z}]$-modules via the deck transformation action, and are $\mathbf{LB}_n$-representations over $\mathbb{Z}[\mathbb{Z}]$ via the lifted $\mathbf{LB}_n$-action on $\widetilde{\mathbb{D}}^3_n$.

Definition 14 The $\mathbf{LB}_n$-representations (13) are the *reduced* and the *unreduced Burau representations* of loop braid groups over $\mathbb{Z}[\mathbb{Z}] = R$.

Let us now take $G = \mathbf{LB}'_n$ and again $X = \mathbb{D}^3_n$ with basepoint $* \in \partial\mathbb{D}^3 = \partial\mathbb{D}^3_n$. This time ϕ is not invariant under the induced action of $\mathbf{LB}'_n$, since, for example, the generator ρ_i sends $a_i \in \phi^{-1}(1)$ to $a_i^{-1} \in \phi^{-1}(-1)$. However, the deeper quotient ϕ' (namely ϕ reduced mod 2) is clearly invariant under the action of $\mathbf{LB}'_n$. We therefore have an induced action (up to homotopy) of $\mathbf{LB}'_n$ on $\widehat{\mathbb{D}}^3_n$ commuting with the deck transformation action of $\mathbb{Z}/2\mathbb{Z}$. Thus the first integral homology groups

$$H_1\big(\widehat{\mathbb{D}}^3_n; \mathbb{Z}\big) \qquad \text{and} \qquad H_1\big(\widehat{\mathbb{D}}^3_n, (\eta')^{-1}(*); \mathbb{Z}\big) \tag{14}$$

are $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$-modules via the deck transformation action, and are $\mathbf{LB}'_n$-representations over $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ via the lifted $\mathbf{LB}'_n$-action on $\widehat{\mathbb{D}}^3_n$.

Definition 15 The $\mathbf{LB}'_n$-representations (14) are the *reduced* and the *unreduced Burau representations* of extended loop braid groups over $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] = S$.

Let us make these covering spaces more concrete by building explicit models for each of them. We embed n pairwise disjoint closed 3-discs into the interior of the unit 3-disc $\mathbb{D}^3$ as pictured in Figure 5, so that each little 3-disc looks like a "lens shape" and the union of their equators is precisely the n-component unlink that we fixed earlier. Let $\mathring{\mathbb{D}}^3_n$ denote $\mathbb{D}^3_n$ minus the interiors of these n little 3-discs, equivalently, $\mathbb{D}^3$ minus the interiors and equators of the n little 3-discs. Also, write N_i for the open northern hemisphere of the boundary of the i-th little 3-disc, and write S_i for the open southern hemisphere of the boundary of the i-th little 3-disc. Now consider

$$\mathbb{Z} \times \mathring{\mathbb{D}}^3_n$$

and glue $\{j\} \times N_i$ to $\{j-1\} \times S_i$ via the homeomorphism $N_i \cong S_i$ given by reflection in the plane passing through the equator. This is an explicit model for $\widetilde{\mathbb{D}}^3_n$. Similarly, we may consider

$$\mathbb{Z}/2\mathbb{Z} \times \mathring{\mathbb{D}}^3_n$$

and glue as before, where j is now considered mod 2. This is an explicit model for $\widehat{\mathbb{D}}^3_n$.

2. Matrices for non-extended loop braid groups

We first consider the $\mathbf{LB}_n$-representations (13). The calculations of these representations are unsurprising, but they are a useful warm-up to the slightly more subtle ones in the next section, for the $\mathbf{LB}'_n$-representations (14).
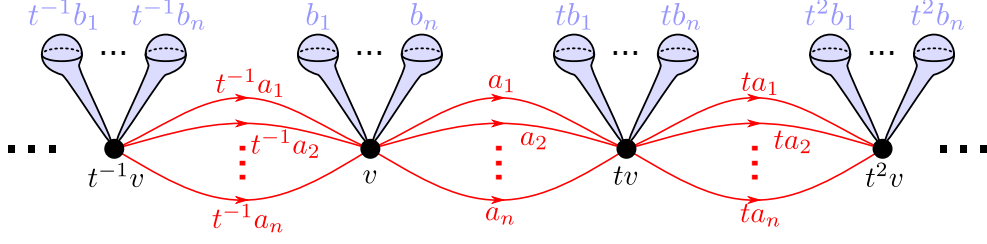
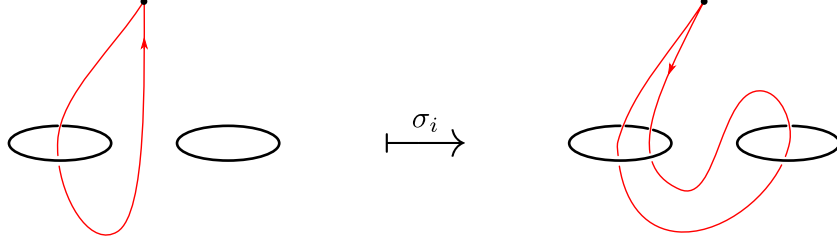Figure 6   The deformation retract X of the $\mathbb{Z}$-covering $\widetilde{\mathbb{D}}_n^3$.

Figure 7   The action of $\sigma_i \in \mathbf{LB}_n$ on the homological generator a_i. The right-hand loop may be decomposed (writing from the left to right) as $a_i \cdot t a_{i+1} \cdot t \bar{a}_i$.

The modules.   As noted above, the unlink-complement $\mathbb{D}_n^3$ deformation retracts onto a wedge of n circles and n copies of the 2-sphere. This deformation retraction lifts to a deformation retraction of the covering space $\widetilde{\mathbb{D}}_n^3$ onto the space pictured in Figure 6. This is an infinite 2-dimensional cell complex X with vertices indexed by $\mathbb{Z}$, with exactly n edges between consecutive vertices (and none between non-consecutive vertices) and with exactly n copies of the 2-sphere wedged onto each vertex. Its fundamental group is freely generated by $t^k.(a_2\bar{a}_1), \ldots \ldots, t^k.(a_n\bar{a}_{n-1})$ for all $k \in \mathbb{Z}$, where $\bar{a}$ denotes the reverse of a path a. Abelianising and writing $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}]$, we see that its first homology is freely generated, as a $\mathbb{Z}[t^{\pm 1}]$-module, by $x_1 \coloneqq a_2\bar{a}_1, \ldots \ldots, x_{n-1} \coloneqq a_n\bar{a}_{n-1}$.

The relative homology group $H_1(\widetilde{\mathbb{D}}_n^3, \eta^{-1}(*); \mathbb{Z})$ is isomorphic to the first homology of X relative to its set of vertices (since these vertices are fixed by the deformation retraction described above), which is freely generated, as a $\mathbb{Z}[t^{\pm 1}]$-module, by $a_1, \ldots, a_n$. Summarising, we have natural isomorphisms

$$H_1\big(\widetilde{\mathbb{D}}_n^3, \eta^{-1}(*); \mathbb{Z}\big) \cong \mathbb{Z}[t^{\pm 1}]\{a_1, \ldots, a_n\}$$
$$H_1\big(\widetilde{\mathbb{D}}_n^3; \mathbb{Z}\big) \cong \mathbb{Z}[t^{\pm 1}]\{x_1, \ldots, x_{n-1}\},$$

and the canonical homomorphism $H_1(\widetilde{\mathbb{D}}_n^3; \mathbb{Z}) \to H_1(\widetilde{\mathbb{D}}_n^3, \eta^{-1}(*); \mathbb{Z})$ is given under these identifications by $x_i \mapsto a_{i+1} - a_i$.

The unreduced representation.   It is easy to calculate visually how the $\mathbf{LB}_n$ generators τ_i and σ_i act on the homological generators a_j. Clearly τ_i simply interchanges a_i and a_{i+1}. On the other hand, σ_i acts by

$$\sigma_i(a_i) = (1-t).a_i + t.a_{i+1} \qquad \sigma_i(a_{i+1}) = a_i \qquad \sigma_i(a_j) = a_j \text{ (for } j \notin \{i, i+1\}),$$

where the first formula comes from the fact that, at the fundamental group level, σ_i sends the loop a_i to the loop $a_i \cdot t a_{i+1} \cdot t \bar{a}_i$, which may be read off from Figure 7. Thus we see that the matrices for the unreduced Burau representation $\mathbf{LB}_n \to \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$ are given by

$$\tau_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_{n-i-1} \qquad \text{and} \qquad \sigma_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 1-t & 1 \\ t & 0 \end{bmatrix} \oplus I_{n-i-1}. \qquad (15)$$

Remark 16 These are precisely the transposes of the matrices used in [Ver01]. Related to this, we note that, as observed in [Ibr21, Theorem 3.2], one may extend the (transpose of the) unreduced
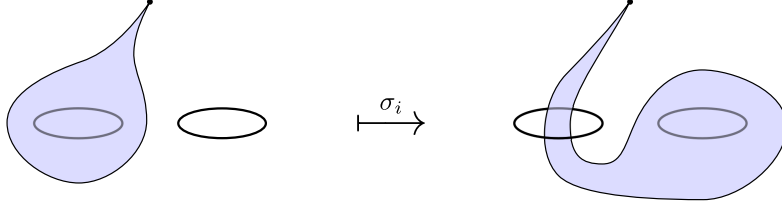
8

Figure 8   The action of $\sigma_i \in \mathbf{LB}_n$ on the element $b_i \in \pi_2(\widetilde{\mathbb{D}}_n^3)$. The right-hand side is homotopic to $a_i \cdot b_{i+1}$, where $\cdot$ denotes the action of π_1 on π_2.
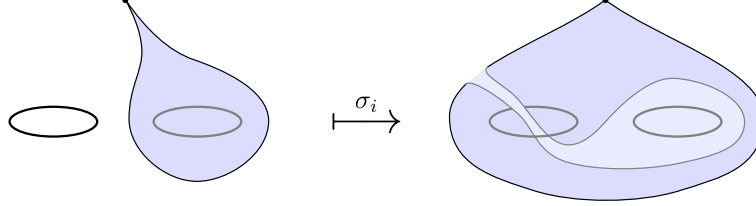

Figure 9   The action of $\sigma_i \in \mathbf{LB}_n$ on the element $b_{i+1} \in \pi_2(\widetilde{\mathbb{D}}_n^3)$. The right-hand side is homotopic to $b_i + b_{i+1} - a_i \cdot b_{i+1}$, where $\cdot$ denotes the action of π_1 on π_2.

Burau representation to the virtual braid group $\mathbf{VB}_n \to \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}, u^{\pm 1}])$ by

$$\tau_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 0 & u^{-1} \\ u & 0 \end{bmatrix} \oplus I_{n-i-1} \qquad \text{and} \qquad \sigma_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix} \oplus I_{n-i-1}. \qquad (16)$$

This factors through the projection $\mathbf{VB}_n \twoheadrightarrow \mathbf{LB}_n$ if one sets $u = 1$, but in general it does not. It would be interesting to find a topological construction of this representation, in the sense of the present paper, although it is unclear how this could be done, as we are unaware of a topological interpretation of the virtual braid group as a motion group, analogous to the realisation of the loop braid group as the group of motions of an oriented trivial link in $\mathbb{R}^3$.

The reduced representation.   Using this computation of the unreduced representation, and the explicit formula $(x_i \mapsto a_{i+1} - a_i)$ for the inclusion of the reduced representation into the unreduced one, it is an easy exercise to read off the following explicit formulas for the reduced Burau representation $\mathbf{LB}_n \to \mathrm{GL}_{n-1}(\mathbb{Z}[t^{\pm 1}])$:

$$\tau_i \longmapsto I_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus I_{n-i-2} \qquad \text{and} \qquad \sigma_i \longmapsto I_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus I_{n-i-2}. \qquad (17)$$

If $i = 1$ or $i = n - 1$, one should ignore the "I_{-1}" on the left or right, and instead remove the left column and top row, respectively the right column and bottom row, from the displayed matrix.

Observe that, when restricted to the σ_i generators, the formulas (15) and (17) are precisely the matrices (4) defining the unreduced and reduced Burau representations of the classical braid groups. This concludes the proof of Theorem 4.

Action on the second homology.   The other non-trivial homology group of the covering space $\widehat{\mathbb{D}}_n^3 \simeq X$ is in degree two, where we have $H_2(\widehat{\mathbb{D}}_n^3; \mathbb{Z}) \cong \mathbb{Z}[t^{\pm 1}]\{b_1, \ldots, b_n\}$, where b_i are illustrated in blue in Figure 6. The generator $\tau_i \in \mathbf{LB}_n$ clearly acts by swapping the homological generators b_i and b_{i+1}. The generator $\sigma_i \in \mathbf{LB}_n$ acts as illustrated in Figures 8 and 9. It sends b_i, considered as an element of $\pi_2(\widetilde{\mathbb{D}}_n^3)$, to $a_i \cdot b_{i+1}$, where $\cdot$ denotes the canonical action of $\pi_1(\widetilde{\mathbb{D}}_n^3)$ on $\pi_2(\widetilde{\mathbb{D}}_n^3)$. Inspecting Figure 6, we see that, viewed as an element of $H_2(\widetilde{\mathbb{D}}_n^3; \mathbb{Z})$, this is tb_{i+1}. Similarly, σ_i sends b_{i+1}, considered as an element of $\pi_2(\widetilde{\mathbb{D}}_n^3)$, to $b_i + b_{i+1} - a_i \cdot b_{i+1}$, which is $b_i + (1-t).b_{i+1}$ as an element of $H_2(\widetilde{\mathbb{D}}_n^3; \mathbb{Z})$. Thus, with respect to the ordered basis $(b_1, \ldots, b_n)$ of $H_2(\widetilde{\mathbb{D}}_n^3; \mathbb{Z})$, this

Figure 10 The deformation retract of the double covering $\widehat{\mathbb{D}}_n^3$ corresponding to quotienting Figure 6 by the action of t^2.

representation $\mathbf{LB}_n \to GL_n(\mathbb{Z}[t^{\pm 1}])$ is given by

$$\tau_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_{n-i-1} \qquad \text{and} \qquad \sigma_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ t & 1-t \end{bmatrix} \oplus I_{n-i-1}. \qquad (18)$$

Reversing the ordering, i.e. using instead the ordered basis $(b_n, \ldots, b_1)$, we obtain

$$\tau_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_{n-i-1} \qquad \text{and} \qquad \sigma_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix} \oplus I_{n-i-1}, \qquad (19)$$

which is the transpose of the unreduced Burau representation (15) of $\mathbf{LB}_n$, and agrees with the matrices used in [Ver01].

3. Matrices for extended loop braid groups

The modules. As in §2, the deformation retraction of the unlink-complement $\mathbb{D}_n^3$ onto a wedge of circles and 2-spheres lifts to a deformation retraction of its covering $\widehat{\mathbb{D}}_n^3$ onto the space pictured in Figure 10. This is a finite 2-dimensional cell complex with two vertices $\{v, tv\}$, $2n$ edges between them and with n copies of the 2-sphere wedged onto each vertex. Its fundamental group is freely generated by the $2n-1$ loops

$$x_1 = a_2 \bar{a}_1, \ldots \ldots, x_{n-1} = a_n \bar{a}_{n-1}$$
$$tx_1 = ta_2\, t\bar{a}_1, \ldots \ldots, tx_{n-1} = ta_n\, t\bar{a}_{n-1}$$
$$y := a_n\, ta_n.$$

Hence its first homology $H_1(\widehat{\mathbb{D}}_n^3; \mathbb{Z})$ is generated, as an abelian group, by the same $2n-1$ loops, viewed as homology classes. The first $2n-2$ of these classes generate a free module of rank $n-1$ over $S = \mathbb{Z}[\mathbb{Z}/2] = \mathbb{Z}[t^{\pm 1}]/(t^2-1)$, whereas the last element y generates a summand isomorphic to $\mathbb{Z}$ viewed as a *trivial* $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$-module, in other words $S/(t-1)$.

The relative homology group $H_1(\widehat{\mathbb{D}}_n^3, (\eta')^{-1}(*); \mathbb{Z})$ is isomorphic to the first homology of this complex relative to its vertices $\{t, tv\}$. This is much simpler: it is a free module over $S = \mathbb{Z}[t^{\pm 1}]/(t^2-1)$ of rank n, generated by $a_1, \ldots, a_n$. Summarising, we have natural isomorphisms

$$H_1\big(\widehat{\mathbb{D}}_n^3, (\eta')^{-1}(*); \mathbb{Z}\big) \cong S\{a_1, \ldots, a_n\}$$
$$H_1\big(\widetilde{\mathbb{D}}_n^3; \mathbb{Z}\big) \cong S\{x_1, \ldots, x_{n-1}\} \oplus S/(t-1)\{y\}.$$

The canonical homomorphism $H_1(\widehat{\mathbb{D}}_n^3; \mathbb{Z}) \to H_1(\widehat{\mathbb{D}}_n^3, (\eta')^{-1}(*); \mathbb{Z})$ is given under these identifications by $x_i \mapsto a_{i+1} - a_i$ and $y \mapsto (1+t)a_n$. As a matrix, this is:

$$\begin{bmatrix} -1 & 0 & 0 & & & \\ 1 & -1 & 0 & & & \\ 0 & 1 & -1 & & & \\ & & & \ddots & & \\ & & & & -1 & 0 \\ & & & & 1 & 1+t \end{bmatrix}. \qquad (20)$$
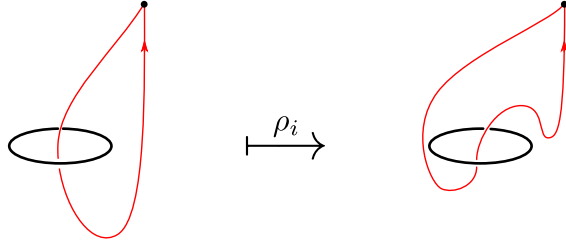
Figure 11   The action of $\rho_i \in \mathbf{LB}'_n$ on the homological generator a_i. The right-hand loop is $t\bar{a}_i$. Note that it is not simply $\bar{a}_i$, since this is a path from tv to v, whereas $\rho_i(a_i)$ is a path from $v = t^2 v$ to tv.

Note that this matrix describes an injective homomorphism. (Viewed as an endomorphism of $S^{\oplus n}$, it is of course *not* injective, since $1 + t$ is a zero-divisor. But its kernel is precisely the submodule $0^{\oplus n-1} \oplus (t-1)$ of $S^{\oplus n}$ so once we replace the domain with $S^{\oplus n-1} \oplus S/(t-1)$ it becomes injective.)

The unreduced representation.   The action of the $\mathbf{LB}'_n$ generators τ_i and σ_i on the homological generators a_j is exactly as in §2, and given by the matrices (15), considered now over the ring $S = \mathbb{Z}[t^{\pm 1}]/(t^2 - 1)$ instead of $R = \mathbb{Z}[t^{\pm 1}]$. The $\mathbf{LB}'_n$ generator ρ_i acts trivially on a_j for $j \neq i$ and sends a_i to $-ta_i$. This last formula comes from the fact that, at the fundamental group level, ρ_i sends the loop a_i to the loop $t\bar{a}_i$, which may be read off from Figure 11. Thus we see that the matrices for the unreduced Burau representation $\mathbf{LB}'_n \to \mathrm{GL}_n(S)$ are given by

$$(15) \qquad \text{and} \qquad \rho_i \longmapsto I_{i-1} \oplus \begin{bmatrix} -t \end{bmatrix} \oplus I_{n-i}. \tag{21}$$

In particular, the restriction of this representation of $\mathbf{LB}'_n$ to the generators τ_i and σ_i (i.e. to $\mathbf{LB}_n$) is the reduction modulo t^2 of the representation (15) of Theorem 4. This concludes the proof of Theorem 5.

The reduced representation.   It is now a purely algebraic exercise, using the formulas (15) and (21) for the unreduced Burau representation, together with the explicit description (20) of the inclusion, to deduce explicit formulas for the reduced Burau representation

$$\mathbf{LB}'_n \longrightarrow \mathrm{Aut}_S(S^{\oplus n-1} \oplus S/(t-1)).$$

These are given in Table 1, where we abbreviate $\delta := 1 + t$. Note that the matrices for the extended generators ρ_i are, in a sense, "non-local".

Remark 17   Since these matrices describe automorphisms of $S^{\oplus n-1} \oplus S/(t-1)$, each entry above the bottom row should be considered as an element of S, whereas each element of the bottom row should be considered as an element of $S/(t-1) \cong \mathbb{Z}$. In other words, we set $t = 1$ on the bottom row. More precisely, the entries in the bottom row lie in $\mathrm{Hom}_S(S, S/(t-1)) \cong S/(t-1)$, except the bottom-right entry, which lies in $\mathrm{Hom}_S(S/(t-1), S/(t-1)) \cong S/(t-1)$. The entries in the right-hand column (except the bottom-right entry) lie in $\mathrm{Hom}_S(S/(t-1), S) \cong (1+t)S = \delta S \subset S$. (The S-modules $S/(t-1)$ and δS are abstractly isomorphic, but they are related differently to S.)

In particular, the restriction of this representation of $\mathbf{LB}'_n$ to the group generators τ_i and σ_i and to the homological generators $x_1, \ldots, x_{n-1}$ is equal to the reduction modulo t^2 of the representation (17) of Theorem 4. This concludes the proof of Theorem 6.

Remark 18   It is an amusing exercise to verify explicitly that the matrices in Table 1 indeed satisfy all of the relations of the extended loop braid group $\mathbf{LB}'_n$, as described for example in [BH13, §3] or [Dam17, §3]. (Warning: the papers [BH13], [Dam17] and the present paper pairwise disagree on notation for the three families of generators of $\mathbf{LB}'_n$.) One should bear in mind that braid words are written from left to right in [BH13] and [Dam17] (as is usual for composition of loops), whereas matrix multiplication goes from right to left (as is usual for function composition), so in fact the matrices in Table 1 satisfy the *opposite* of the relations of $\mathbf{LB}'_n$ described in [BH13; Dam17].

(Technically, this means that we have constructed a representation of the opposite group $(\mathbf{LB}'_n)^{\mathrm{op}}$, but, except for computations, we ignore this subtlety, since this is abstractly isomorphic to $\mathbf{LB}'_n$.) For an explicit verification, using Sage, that the matrices in Table 1 satisfy the 40 relations of the extended loop braid group $\mathbf{LB}'_n$ in the case $n = 4$, see the supplementary materials for this paper available at mdp.ac/papers/loop-Burau.

	$i = 1$	$2 \leqslant i \leqslant n - 2$		$i = n - 1$	
τ_i	$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \oplus \mathrm{I}_{n-2}$	$\mathrm{I}_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus \mathrm{I}_{n-i-1}$		$\mathrm{I}_{n-3} \oplus \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & -\delta \\ 0 & 0 & 1 \end{bmatrix}$	
σ_i	$\begin{bmatrix} -t & 1 \\ 0 & 1 \end{bmatrix} \oplus \mathrm{I}_{n-2}$	$\mathrm{I}_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus \mathrm{I}_{n-i-1}$		$\mathrm{I}_{n-3} \oplus \begin{bmatrix} 1 & 0 & 0 \\ t & -t & -\delta \\ 0 & 0 & 1 \end{bmatrix}$	
	$i = 1$	$2 \leqslant i \leqslant n - 1$		$i = n$	
ρ_i	$\begin{bmatrix} -t & 0 & \cdots & 0 \\ -\delta & & & \\ \vdots & & \mathrm{I}_{n-1} & \\ -\delta & & & \\ 1 & & & \end{bmatrix}$	$\mathrm{I}_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \delta & -t & 0 & \cdots & 0 \\ \delta & -\delta & & & \\ \vdots & \vdots & & \mathrm{I}_{n-i} & \\ \delta & -\delta & & & \\ -1 & 1 & & & \end{bmatrix}$		$\mathrm{I}_{n-2} \oplus \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$	

Table 1   Explicit matrices for the reduced Burau representation of the extended loop braid group $\mathbf{LB}'_n$.
Notation: $\delta = 1 + t$. All entries lie in $S = \mathbb{Z}[t^{\pm 1}]/(t^2 - 1)$, except for the bottom row, where they lie in $S/(t - 1) \cong \mathbb{Z}$, in other words we set $t = 1$ on the bottom row.

4. Properties

Irreducibility. The unreduced Burau representations (15) of $\mathbf{LB}_n$ and (21) of $\mathbf{LB}'_n$ are clearly reducible, since they contain the reduced Burau representations (17) and (Table 1) respectively. The reduced Burau representation (17) of $\mathbf{LB}_n$ becomes irreducible when we pass to the field of fractions $\mathbb{Q}(t)$ of $\mathbb{Z}[t^{\pm 1}]$. This follows because its restriction to the symmetric group $\mathfrak{S}_n \subset \mathbf{LB}_n$ is the standard $(n - 1)$-dimensional representation of $\mathfrak{S}_n$, which is irreducible over any field.

On the other hand, for the reduced Burau representation (Table 1) of $\mathbf{LB}'_n$, we cannot directly pass to a field of fractions, since its ground ring $S = \mathbb{Z}[t^{\pm 1}]/(t^2 - 1)$ is not an integral domain. Instead, we may tensor over S with $\mathbb{Q}$, setting either $t = -1$ or $t = 1$. In the first case, the additional $S/(t - 1)$ summand is killed and we obtain an $(n - 1)$-dimensional representation of $\mathbf{LB}'_n$ over $\mathbb{Q}$, which is irreducible, again because its restriction to $\mathfrak{S}_n \subset \mathbf{LB}'_n$ is the standard representation of $\mathfrak{S}_n$. In the second case, we obtain an n-dimensional representation:

$$\mathbf{LB}'_n \longrightarrow \mathrm{GL}_n(\mathbb{Q}). \tag{22}$$

Lemma 19 *The representation* (22) *of* $\mathbf{LB}'_n$ *is irreducible.*

Proof. Suppose that $V \subseteq \mathbb{Q}^n$ is a non-trivial subrepresentation; we will show that $V = \mathbb{Q}^n$. Write $v = \alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1} + \beta y$.

Step 1. It suffices to find $v \in V$ with $v \neq 0$ and $\beta = 0$.
First note that $\mathrm{span}_{\mathbb{Q}}\{x_1, \ldots, x_{n-1}\}$ is an irreducible subrepresentation (it is irreducible since its restriction to $\mathfrak{S}_n$ is the standard representation of $\mathfrak{S}_n$). Thus V must contain $\mathrm{span}_{\mathbb{Q}}\{x_1, \ldots, x_{n-1}\}$. But then it must also contain $x_{n-1} - \rho_n(x_{n-1}) = y$, and so $V = \mathbb{Q}^n$.

Step 2. It suffices to find $v \in V$ with $v \neq 0$ and $\alpha_{n-2} - 2\alpha_{n-1} - 2\beta \neq 0$.
For such a v, we have $\tau_{n-1}(v) - v = (\alpha_{n-2} - 2\alpha_{n-1} - 2\beta)x_{n-1}$, and we are done by Step 1.

Step 3. It suffices to find $v \in V$ with $v \neq 0$ and $\alpha_{n-1} + 2\beta \neq 0$.
For such a v, we have $\rho_n(v) - v = -(\alpha_{n-1} + 2\beta)y$, and we are done by Step 2.

Step 4. Let $v \in V$ be a non-zero vector. By the previous steps, we may assume that its coefficients satisfy $\alpha_{n-2} = \alpha_{n-1} = -2\beta$ and $\beta \neq 0$. We then have $\tau_{n-2}(v) - v = (\alpha_{n-3} + 2\beta)x_{n-2}$, so we are done by Step 1 unless $\alpha_{n-3} = -2\beta$. On the other hand, if $\alpha_{n-3} = -2\beta$, we have $\tau_{n-3}(v) - v = (\alpha_{n-4} + 2\beta)x_{n-3}$, so again we are done by Step 1 unless $\alpha_{n-4} = -2\beta$. Repeating this a further $n - 5$ times, we see that we are done unless $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = -2\beta$. But in this case we have $\tau_1(v) - v = 2\beta x_1$, and we are done by Step 1. $\qquad\square$

Remark 20 The restriction of the representation (22) to $\mathfrak{S}_n \subset \mathbf{LB}'_n$ is isomorphic to the regular representation of $\mathfrak{S}_n$. To see this, note that $\mathrm{span}_{\mathbb{Q}}\{x_1, \ldots, x_{n-1}\}$ is a subrepresentation isomorphic to the standard representation of $\mathfrak{S}_n$, and the quotient is a trivial 1-dimensional representation. Thus, by Maschke's theorem, $(22)|_{\mathfrak{S}_n}$ is isomorphic to the sum of the standard representation and a trivial 1-dimensional representation, which is isomorphic to the regular representation of $\mathfrak{S}_n$.

Faithfulness.   The classical (unreduced) Burau representation $\mathbf{B}_n \to \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$ is known to be faithful for $n \leqslant 3$ and unfaithful for $n \geqslant 5$ [Moo91; LP93; Big99]. By contrast, the unreduced Burau representation (15): $\mathbf{LB}_n \to \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$ is unfaithful for all $n \geqslant 2$, by [Bar05, Lemmas 4 and 5]. The unreduced Burau representations for $\mathbf{LB}_n$ and $\mathbf{LB}'_n$ fit together in the commutative square

$$\begin{array}{ccc} \mathbf{LB}_n & \xrightarrow{\;\;(15)\;\;} & \mathrm{GL}_n(R) \\[2pt] {\scriptstyle\mathrm{incl.}}\big\downarrow & & \big\downarrow{\scriptstyle -\otimes_R S} \\[2pt] \mathbf{LB}'_n & \xrightarrow{\;\;(21)\;\;} & \mathrm{GL}_n(S), \end{array} \qquad (23)$$

(recall that $R = \mathbb{Z}[t^{\pm 1}]$ and $S = R/(t^2 - 1)$) so the unreduced Burau representation (21): $\mathbf{LB}'_n \to \mathrm{GL}_n(S)$ is also unfaithful for all $n \geqslant 2$. In fact, the kernel of the composition $\mathbf{LB}_n \to \mathrm{GL}_n(S)$ across the diagonal of (23) is larger than the kernel of (15): $\mathbf{LB}_n \to \mathrm{GL}_n(R)$ since, for example, $(\tau_1\sigma_1)^2$ is sent to $\begin{bmatrix} t^2 & 0 \\ 0 & 1 \end{bmatrix} \oplus \mathrm{I}_{n-2}$.

The transpose of the Burau representation.   As mentioned in the introduction, the (unreduced) Burau representation of the classical braid group $\mathbf{B}_n$ is equivalent to its transpose. Explicitly, conjugation by the diagonal matrix $\mathrm{Diag}(1, t, \ldots, t^{n-1})$ passes between the Burau representation and its transpose. On the other hand:

Lemma 21 *The unreduced Burau representation* (15) *of the loop braid group* $\mathbf{LB}_n$ *is not equivalent to its transpose.*

Proof. Let (15) $=$ Bur. The aim is to show that there is no invertible matrix $M \in \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$ such that $\mathrm{Bur}(g)M = M\mathrm{Bur}(g)^{\mathrm{t}}$ for all $g \in \mathbf{LB}_n$, where $(-)^{\mathrm{t}}$ denotes the transpose of a matrix. For $n = 2$, the equations $\mathrm{Bur}(\sigma_1)M = M\mathrm{Bur}(\sigma_1)^{\mathrm{t}}$ and $\mathrm{Bur}(\tau_1)M = M\mathrm{Bur}(\tau_1)^{\mathrm{t}}$ imply that M is of the form $\begin{bmatrix} a & -a \\ -a & a \end{bmatrix}$, which is not invertible. For $n \geqslant 3$, we will prove by induction on n the statement that the only matrix M satisfying $\mathrm{Bur}(g)M = M\mathrm{Bur}(g)^{\mathrm{t}}$ for all $g \in \mathbf{LB}_n$ is the zero matrix. We begin with the base case $n = 3$. Applying the argument for $n = 2$ to the top-left and bottom-right 2×2 blocks of M, we deduce that M must be of the form $\begin{bmatrix} a & -a & b \\ -a & a & -a \\ c & -a & a \end{bmatrix}$. Given this, the equation $\mathrm{Bur}(\tau_1)M = M\mathrm{Bur}(\tau_1)^{\mathrm{t}}$ then implies that $b = c = -a$, and the equation $\mathrm{Bur}(\sigma_1)M = M\mathrm{Bur}(\sigma_1)^{\mathrm{t}}$ implies that $a = at$, thus $a = 0$ and M is the zero matrix. For the inductive step, we may apply the inductive hypothesis to the top-left and bottom right $(n-1) \times (n-1)$ blocks of M to see that the entries of M are all zero except possibly the top-right and bottom-left entries. But then the equation $\mathrm{Bur}(\tau_1)M = M\mathrm{Bur}(\tau_1)^{\mathrm{t}}$ implies that these are zero too. $\qquad\square$

Thus the representations (15) and (15)$^{\mathrm{tr}}$ are non-equivalent representations of $\mathbf{LB}_n$. However, we have constructed both of these topologically: (15) is the action of $\mathbf{LB}_n$ on the first homology group $H_1(\widetilde{\mathbb{D}}_n^3, \eta^{-1}(*); \mathbb{Z})$ and (15)$^{\mathrm{tr}}$ is the action of $\mathbf{LB}_n$ on the second homology group $H_2(\widetilde{\mathbb{D}}_n^3; \mathbb{Z})$, as shown at the end of §2.

References

[Bar05]   V. G. Bardakov. *Extending representations of braid groups to the automorphism groups of free groups*. *J. Knot Theory Ramifications* 14.8 (2005), pp. 1087–1098.

[BF81]   E. Binz and H. R. Fischer. *The manifold of embeddings of a closed manifold. Differential geometric methods in mathematical physics (Proc. Internat. Conf., Tech. Univ. Clausthal, Clausthal-Zellerfeld, 1978)*. Vol. 139. Lecture Notes in Phys. With an appendix by P. Michor. Springer, Berlin-New York, 1981, pp. 310–329.

[BH13]   T. E. Brendle and A. Hatcher. *Configuration spaces of rings and wickets*. *Comment. Math. Helv.* 88.1 (2013), pp. 131–162.

[Big04]   S. Bigelow. *Homological representations of the Iwahori-Hecke algebra*. *Proceedings of the Casson Fest*. Vol. 7. Geom. Topol. Monogr. Geom. Topol. Publ., Coventry, 2004, pp. 493–507.

[Big99]   S. Bigelow. *The Burau representation is not faithful for $n = 5$*. *Geom. Topol.* 3 (1999), pp. 397–404.

[BL93]   A. Brownstein and R. Lee. *Cohomology of the group of motions of n strings in 3-space*. *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*. Vol. 150. Contemp. Math. Amer. Math. Soc., Providence, RI, 1993, pp. 51–61.

[Bur35]   W. Burau. *Über Zopfgruppen und gleichsinnig verdrillte Verkettungen. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*. Vol. 11. 1. Springer. 1935, pp. 179–186.

[BWC07]   J. C. Baez, D. K. Wise and A. S. Crans. *Exotic statistics for strings in 4D BF theory*. *Adv. Theor. Math. Phys.* 11.5 (2007), pp. 707–749.

[Cer61]   J. Cerf. *Topologie de certains espaces de plongements*. *Bull. Soc. Math. France* 89 (1961), pp. 227–380.

[Col89]   D. J. Collins. *Cohomological dimension and symmetric automorphisms of a free group*. *Comment. Math. Helv.* 64.1 (1989), pp. 44–61.

[Dah62]   D. M. Dahm. *A generalization of braid theory*. Thesis (Ph.D.)–Princeton University. ProQuest LLC, Ann Arbor, MI, 1962, p. 59.

[Dam17]   C. Damiani. *A journey through loop braid groups*. *Expo. Math.* 35.3 (2017), pp. 252–285.

[FRR97]   R. Fenn, R. Rimányi and C. Rourke. *The braid-permutation group*. *Topology* 36.1 (1997), pp. 123–135.

[Gol81]   D. L. Goldsmith. *The theory of motion groups*. *Michigan Math. J.* 28.1 (1981), pp. 3–17.

[Hat83]   A. E. Hatcher. *A proof of the Smale conjecture, $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$*. *Ann. of Math. (2)* 117.3 (1983), pp. 553–607.

[Ibr21]   A. Ibrahim. *On matrix representations for generalized braid groups*. ArXiv:2109.03583. 2021.

[JMM06]   C. Jensen, J. McCammond and J. Meier. *The integral cohomology of the group of loops*. *Geom. Topol.* 10 (2006), pp. 759–784.

[KT08]   C. Kassel and V. Turaev. *Braid groups*. Vol. 247. Graduate Texts in Mathematics. With the graphical assistance of Olivier Dodane. Springer, New York, 2008, pp. xii+340.

[Law90]   R. J. Lawrence. *Homological representations of the Hecke algebra*. *Comm. Math. Phys.* 135.1 (1990), pp. 141–191.

[Lim63]   E. L. Lima. *On the local triviality of the restriction map for embeddings*. *Comment. Math. Helv.* 38 (1963), pp. 163–164.

[LP93]   D. D. Long and M. Paton. *The Burau representation is not faithful for $n \geqslant 6$*. *Topology* 32.2 (1993), pp. 439–447.

[McC86]   J. McCool. *On basis-conjugating automorphisms of free groups*. *Canad. J. Math.* 38.6 (1986), pp. 1525–1529.

[Moo91]   J. A. Moody. *The Burau representation of the braid group B_n is unfaithful for large n*. *Bull. Amer. Math. Soc. (N.S.)* 25.2 (1991), pp. 379–384.

[Pal21]   M. Palmer. *Homological stability for moduli spaces of disconnected submanifolds, I*. *Algebr. Geom. Topol* 21-3 (2021), pp. 1371–1444.

[Pal60]   R. S. Palais. *Local triviality of the restriction map for embeddings*. *Comment. Math. Helv.* 34 (1960), pp. 305–312.

[PS19]   M. Palmer and A. Soulié. *Topological representations of motion groups and mapping class groups – a unified functorial construction*. arxiv:1910.13423. 2019.

[Ver01]   V. V. Vershinin. *On homology of virtual braids and Burau representation*. Vol. 10. 5. Knots in Hellas '98, Vol. 3 (Delphi). 2001, pp. 795–812.

Martin Palmer, *Institutul de Matematică Simion Stoilow al Academiei Române, 21 Calea Griviței, 010702 București, România.* Email address: `mpanghel@imar.ro`

Arthur Soulié, *University of Glasgow, School of Mathematics and Statistics, 132 University Pl, Glasgow G12 8TA, United Kingdom.* Email address: `artsou@hotmail.fr`