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Polynomiality of surface braid and mapping class group representations

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Abstract

A wide family of homological representations of surface braid groups and mapping class groups of surfaces was developed in [PS21]. These representations are naturally defined as functors on a category whose automorphism groups are the family of groups under consideration, and whose richer structure may be used to prove twisted homological stability results — subject to the condition that the functor is *polynomial*. We prove that many of these *homological representation functors* are polynomial, including those extending the Lawrence-Bigelow representations of the classical braid groups. In particular, we carry out general computations of the homological representation modules by using Borel-Moore homology and qualitative properties of the group actions. These polynomiality results also have applications for representation theoretic questions.

Introduction

The representation theory of *surface braid groups* and of *mapping class groups of surfaces* has been the subject of intensive study over the last decades; see Birman and Brendle's survey [BB05, §4] or Margalit's expository paper [Mar19] for instance. As many mathematical objects, these groups naturally come in *families* in the sense that they are canonically indexed by non-negative integers, forming a collection of the type $\{G_n\}_{n\in\mathbb{N}}$ equipped with morphisms $G_n \to G_m$ whenever $n \leqslant m$. We consider the families of classical braid groups $\mathbf{B}_n$, of surface braid groups $\mathbf{B}_n(\Sigma_{g,1})$ and $\mathbf{B}_n(\mathcal{N}_{h,1})$ where $\Sigma_{g,1}$ and $\mathcal{N}_{h,1}$ are respectively orientable and non-orientable compact surfaces with one boundary component and genus $g, h \geqslant 1$, and of the mapping class groups $\mathbf{\Gamma}_{g,1}$ and $\boldsymbol{\mathcal{N}}_{h,1}$ of the surfaces $\Sigma_{g,1}$ and $\mathcal{N}_{h,1}$; see §1.1.2 for proper recollections on these groups. Each of these families of groups has an associated groupoid $\mathcal{M}$: a category whose objects are indexed by non-negative integers, whose automorphism group for the object n is G_n and with no further morphisms.

A convenient idea to study and mildly simplify the representation theories of these families of groups consists in considering *families* of representations. This means collecting one representation for each group G_n and requiring these gathered representations to satisfy compatibility conditions with respect to the morphisms $G_n \to G_m$ for $n \leqslant m$. The natural mathematical objects encoding this procedure are functors $\langle \mathcal{G}, \mathcal{M}\rangle \to R$-Mod, where R-Mod denotes the category of R-modules (for R a ring). Here $\langle \mathcal{G}, \mathcal{M}\rangle$ denotes a category obtained via a construction due to Quillen from the groupoid $\mathcal{M}$ and an auxiliary braided monoidal groupoid $\mathcal{G}$ acting on it; see §1.1.1. The category $\langle \mathcal{G}, \mathcal{M}\rangle$ has more morphisms that allow us to encode the compatibilities. A general functorial machinery is defined in [PS21] in order to construct such functors. In particular, we introduce a large range of families of representations for the aforementioned families of groups.

In the present paper, we prove *polynomiality* results for the functors constructed in [PS21] for surface braid groups and mapping class groups of surfaces; see Theorems B, C and D. These results require a deep understanding of the structures of the representations (see Theorem A) and have powerful applications for homological stability and representation theory of the families of groups that we consider (see Theorems E, F and G).

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Homological representation functors. The general construction of [PS21] and its application for surface braid groups and mapping class groups are recalled in §1.2 and §1.3. At the level of a family of groups $\{G_n\}_{n\in\mathbb{N}}$, the key idea consists in defining a regular cover of a configuration space of points on a surface $\mathcal{S}_n$, on which a topological lift of the group G_n acts naturally. This definition essentially depends on two parameters: a integer $\ell \geqslant 0$ corresponding to the level of some lower central series and a partition $\mathbf{k}$ of the number $k \geqslant 1$ of configuration points. It is however independent of the index n of the group. We denote by $Q_{\mathbf{k},\ell}(\mathcal{S})$ the corresponding deck transformation group. The family of representations is then defined by the action of each G_n on the homology groups of configuration spaces on the surface $\mathcal{S}_n$ with local coefficients induced from that regular covering. Furthermore, although each one of these homology groups is naturally equipped with a $\mathbb{Z}[Q_{\mathbf{k},\ell}(\mathcal{S})]$-module structure, the action of G_n does not always commute with the action of $Q_{\mathbf{k},\ell}(\mathcal{S})$, in which case we say that the representations are *twisted*; see Definition 1.4. Therefore, the output of the machinery of [PS21] is in general a functor $\langle \mathcal{G}, \mathcal{M}\rangle \to \mathbb{Z}$-Mod, that is generically denoted by $\mathfrak{L}_{(\mathbf{k},\ell)}$ and called a *homological representation functor*.

We note that, when the group $Q_{(\mathbf{k},\ell)}(\mathcal{S})$ is infinite (which is the case in many of our examples; see §1.3.1–1.3.3), the representations encoded by the functor $\mathfrak{L}_{(\mathbf{k},\ell)}$ may be of infinite rank (over $\mathbb{Z}$). In this case, however, there exists an optimal quotient $Q^u_{(\mathbf{k},\ell)}(\mathcal{S})$ of $Q_{\mathbf{k},\ell}(\mathcal{S})$ inducing a local coefficient system commuting with the action of each group G_n; see (1.9). This alteration defines a homological representation functor $\mathfrak{L}^u_{(\mathbf{k},\ell)} \colon \langle \mathcal{G}, \mathcal{M}\rangle \to \mathbb{Z}[Q^u_{(\mathbf{k},\ell)}(\mathcal{S})]$-Mod that is said to be *untwisted*. The representations encoded by the functor $\mathfrak{L}^u_{(\mathbf{k},\ell)}$ are then defined over $\mathbb{Z}[Q^u_{(\mathbf{k},\ell)}(\mathcal{S})]$ and are of *finite* rank. Furthermore, an interesting variation of the construction consists in considering the configuration spaces in the "dual" surface $\check{\mathcal{S}}_n$ defined by blowing up each puncture and removing the original boundary $\partial\mathcal{S}_n$; see Definition 2.12. The construction then repeats to define another homological representation functor $\mathfrak{L}^v_{(\mathbf{k},\ell)} \colon \langle \mathcal{G}, \mathcal{M}\rangle \to \mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})]$-Mod, called the *vertical-type alternative* to $\mathfrak{L}_{(\mathbf{k},\ell)}$. This terminology comes from the module structure of these alternatives and their relations to the dual representations of those encoded by $\mathfrak{L}_{(\mathbf{k},\ell)}$; see §2.3 and Facts 4.15 and 4.19. Finally, we may modify the module structures of the homological representation functors by some *change of rings operation* on the ground ring $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})]$; see §1.2.2.

We study homological representation functors defined in §1.3 for surface braid groups and mapping class groups for each pair of integers $\ell \geqslant 1$ and $k \geqslant 1$, and each partition $\mathbf{k} \vdash k$. Namely:

- For classical braid groups, the $(\mathbf{k},\ell)$ homological representation functor is denoted by $\mathfrak{LB}_{(\mathbf{k},\ell)}$ and is called the $(\mathbf{k},\ell)$-*Lawrence-Bigelow functor*. This is because the functor $\mathfrak{LB}_{(k,2)}$ encodes the well-known k-th family of the representations introduced by Lawrence and Bigelow in [Law90; Big04]; see Example 1.12. For instance, the functor $\mathfrak{LB}_{(1,2)}$ encodes the family of *Burau* representations [Bur35], while the functor $\mathfrak{LB}_{(2,2)}$ that of the *Lawrence-Krammer-Bigelow* representations [Big01; Kra02].
- For surface braid groups, the $(\mathbf{k},\ell)$ homological representation functors are respectively denoted by $\mathfrak{L}_{(\mathbf{k},\ell)}(\Sigma_{g,1})$ for the orientable surface $\Sigma_{g,1}$ and $\mathfrak{L}^u_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ and $\mathfrak{L}_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ for the non-orientable surface $\mathcal{N}_{h,1}$. The encoded representations are generally new, except those of $\mathfrak{L}_{(k,3)}(\Sigma_{g,1})$ which are related to the work of An and Ko [AK10]; see Example 1.13.
- For the mapping class groups of surfaces, the $(\mathbf{k},\ell)$ homological representation functors are respectively denoted by $\mathfrak{L}_{(\mathbf{k},\ell)}(\mathbf{\Gamma})$ for orientable surfaces and $\mathfrak{L}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})$ for non-orientable surfaces. Apart from the representations encoded by $\mathfrak{L}_{(\{1,\ldots,1\},1)}(\mathbf{\Gamma})$, which were introduced by Moriyama [Mor07] (see Example 1.15), the other families of representations are new.

Computations of the representations. First of all, we develop techniques to compute the modules and group actions of homological representation functors.

Theorem A (Lemma 2.1, §2.2 and §3.3) *For each one of the aforementioned homological representation functors $\mathfrak{L}_{(\mathbf{k},\ell)}$, the G_n-modules $\mathfrak{L}_{(\mathbf{k},\ell)}(n)$, $\mathfrak{L}^u_{(\mathbf{k},\ell)}(n)$, $\mathfrak{L}^v_{(\mathbf{k},\ell)}(n)$ and $\mathfrak{L}^{u,v}_{(\mathbf{k},\ell)}(n)$ are free $Q_{\mathbf{k},\ell}(\mathcal{S})$-modules of finite rank. Their bases are geometrically modelled by distributions, respecting the partition $\mathbf{k}$, of the configuration points on some subgraphs of the surface $\mathcal{S}_n$; see §2.3. We also have certain qualitative properties for the group actions on these bases; see §3.3.*

Theorem A relies on the use of *Borel-Moore homology* of configuration spaces in the definition of the homological representation functors; see §§2.1–2.3. We may however repeat all our results for homological representation functors defined using *classical* homology assuming that their local systems satisfy *genericity* conditions; see §2.4 and Lemma 3.4. In addition to the results of Theorem A, we may explicitly compute the group actions of the surface braid groups and mapping class groups on these modules. For instance, these formulas have been computed in [PS22, §4.3] for the representations encoded by the functors $\mathfrak{LB}_{(\mathbf{k},\ell)}$ when $\ell \geqslant 3$. Such technical work is not required for the study of polynomiality; for this purpose, the qualitative properties of §3.3 are sufficient. However, these formulas will be elaborated in [PS]. Also, the results of Theorem A naturally generalise at a much greater level of generality than that of configurations of points on surfaces: for instance it holds for configurations on higher-dimensional analogues of surfaces or for configurations on complements of links; see Lemma 2.1 and Examples 2.5 and 2.6. These cases are beyond the scope of the present paper but will be studied in forthcoming work.

Polynomiality results. For each category $\langle \mathcal{G}, \mathcal{M} \rangle$ considered in this paper, there are various fundamental notions of *polynomiality* on the objects in the category $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, R\text{-Mod})$. The first definitions of polynomial functors date back to Eilenberg and Mac Lane in [EM54] for functors on module categories. This notion has progressively been extended to deal with a more general framework, and has been the object of intensive study because of its applications in representation theory (see Djament, Touzé and Vespa [DTV21]), group cohomology (see Franjou, Friedlander, Scorichenko and Suslin [FFSS99]) and homological stability with twisted coefficients (see Randal-Williams and Wahl [RW17]). In particular, Djament and Vespa [DV19, §1] introduce the notion of *strong* polynomial functors in the context of a functor category $\mathbf{Fct}(\mathcal{M}, \mathcal{A})$, where $\mathcal{M}$ is a (small) symmetric monoidal category where the unit is an initial object and $\mathcal{A}$ is a Grothendieck category, which recovers and extends all the classical concepts of polynomial functors. Furthermore, the notion of weak polynomial functor is first introduced in [DV19, §1] and reflects more accurately the stable behaviour of the objects of the category $\mathbf{Fct}(\mathcal{M}, \mathcal{A})$; see [DV19, §5] and Djament [Dja17]. The notions of strong and weak polynomial functors are then extended in [Sou22, §4] to the larger setting where $\mathcal{M}$ is a full subcategory of a pre-braided monoidal category where the unit is an initial object. Also the notion of *very strong* polynomial functor in this context is introduced there: it is closely related to the notion of *coefficient systems of finite degree* of Randal-Williams and Wahl [RW17]. All these notions of polynomiality straightforwardly extend to the slightly more general context of the present paper; see §3.1. We also define the notion of *split* polynomial functor, a particular kind of very strong polynomial functor, following an analogous notion from [RW17]; see also [Pal17] for a comparison of the various instances of polynomial functors.

The central results of this paper are the following polynomiality properties, which we prove for the aforementioned homological representation functors. From now on, we consider integers $\ell \geqslant 1$ and $k \geqslant 1$, and a partition $\mathbf{k} \vdash k$. For classical braid groups, we prove:

Theorem B (See §4.1) *The $(1,\ell)$-Lawrence-Bigelow functor $\mathfrak{LB}_{(1,\ell)}$ (which is isomorphic to its untwisted version $\mathfrak{LB}^u_{(1,\ell)}$, see §1.3.1) is strong polynomial of degree 2 and weak polynomial of degree 1. For $k \geqslant 2$, the $(\mathbf{k},\ell)$-Lawrence-Bigelow functor $\mathfrak{LB}_{(\mathbf{k},\ell)}$ and its untwisted version $\mathfrak{LB}^u_{(\mathbf{k},\ell)}$ are both very strong and weak polynomial of degree k. These results still hold after any (non-zero) change of rings operation. In contrast, the vertical-type alternative $(\mathbf{k},\ell)$-Lawrence-Bigelow functors $\mathfrak{LB}^v_{(\mathbf{k},\ell)}$ and $\mathfrak{LB}^{u,v}_{(\mathbf{k},\ell)}$ are not strong polynomial, but they are weak polynomial of degree 0.*

Theorem B recovers the previous results of [RW17, Ex. 4.3] about the unreduced Burau functor $\mathfrak{LB}_{(1,2)}$ and of [Sou19, Prop. 3.33] about the Lawrence-Bigelow functor $\mathfrak{LB}_{(2,2)}$. Furthermore, for the surface braid groups, we prove:

Theorem C (See §4.2) *The functors $\mathfrak{L}^u_{(\mathbf{k},\ell)}(\Sigma_{g,1})$ and $\mathfrak{L}_{(\mathbf{k},\ell)}(\Sigma_{g,1})$ for orientable surfaces, and $\mathfrak{L}^u_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ and $\mathfrak{L}_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ for non-orientable surfaces, are very strong and weak polynomial of degree k. These results still hold after any (non-zero) change of rings operation. In contrast, the vertical-type alternatives $\mathfrak{L}^{u,v}_{(\mathbf{k},\ell)}(\Sigma_{g,1})$, $\mathfrak{L}^v_{(\mathbf{k},\ell)}(\Sigma_{g,1})$, $\mathfrak{L}^{u,v}_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ and $\mathfrak{L}^v_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ are not strong polynomial, but they are weak polynomial of degree 0.*

As far as the authors know, none the results of Theorem C appear to have been known so far. Finally, for mapping class groups of surfaces, we prove:

Theorem D (See §4.3) *Each one of the functors* $\mathfrak{L}^u_{(\mathbf{k},\ell)}(\boldsymbol{\Gamma})$, $\mathfrak{L}_{(\mathbf{k},\ell)}(\boldsymbol{\Gamma})$, $\mathfrak{L}^v_{(\mathbf{k},1)}(\boldsymbol{\Gamma})$ *and* $\mathfrak{L}^v_{(\mathbf{k},2)}(\boldsymbol{\Gamma})$ *for orientable surfaces,* $\mathfrak{L}^u_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})$, $\mathfrak{L}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})$, $\mathfrak{L}^v_{(\mathbf{k},1)}(\boldsymbol{\mathcal{N}})$ *and* $\mathfrak{L}^v_{(\mathbf{k},2)}(\boldsymbol{\mathcal{N}})$ *for non-orientable surfaces, is split and weak polynomial of degree* k*. These results still hold after any (non-zero) change of rings operation.*

This recovers the previous result of Ivanov [Iva93, §2.8] (see also Boldsen [Bol12, Ex. 4.5]) for $\mathfrak{L}^u_{(1,2)}(\boldsymbol{\Gamma}) = \mathfrak{L}_{(1,2)}(\boldsymbol{\Gamma})$, which encodes the natural action of $\boldsymbol{\Gamma}_{g,1}$ on $H_1(\Sigma_{g,1};\mathbb{Z})$.

Methods: the common strategy for the proofs of Theorems B, C and D consists in unearthing key short exact sequences for homological representation functors; see (4.1) and (4.2) for classical braid groups, (4.5) for surface braid groups, and the isomorphisms (4.6) and (4.7) induced from split short exact sequences for mapping class groups of surfaces. They relate each functor $\mathfrak{L}_{(\mathbf{k},\ell)}$ and its shift by some translation functor to the functors of type $\mathfrak{L}_{(\mathbf{k}',\ell)}$ where $\mathbf{k}'$ is obtained from $\mathbf{k}$ by deleting 1 in one partition block. None of these sequences have appeared yet in the literature and they provide new connections between homological representation functors. All the proofs share a common philosophy: the representation structures correspond to group actions on some type of graphs, the sequences are guessed from this geometrical viewpoint and one then has to check a few key relations to complete the proof.

Applications. Beyond being fundamental properties, the various types of polynomiality for functors happen to be very useful for many questions.

First of all, one of the main motivations for our interest in very strong polynomial functors is their twisted homological stability properties. We refer to §5.3 for proper recollections on the notion of twisted homological stability. Our fundamental reference for this is the work of Randal-Williams and Wahl [RW17], where the current optimal framework for twisted homological stability for surface braid groups and mapping class groups is established. The key condition on the twisted coefficients for their setting is that they are given by some type of strong polynomial functor; see §5.3. Combining these results with the above polynomiality properties, we deduce that:

Theorem E (See §5.3) *There is homological stability with twisted coefficients given by any of the homological representation functors of Theorems B, C, D that are very strong or split polynomial.*

Since the representation theories of the families of groups that we study in this paper are wild and an active research topic, the strong polynomial functors associated with these groups are not well-understood. Hence, Theorem E extends the scope of twisted homological stability to more sophisticated sequences of representations.

Furthermore, our weak polynomiality results help to organise the representation theory of the families of groups as follows. Weak polynomial functors of degree less than or equal to some $d \in \mathbb{N}$ form a category $\mathcal{P}ol_d(\langle \mathcal{G}, \mathcal{M}\rangle)$ that is *localising* in $\mathcal{P}ol_{d+1}(\langle \mathcal{G}, \mathcal{M}\rangle)$: this allows us to define quotient categories (see (5.5)), which provide an organising tool for families of representations; see §5.4. We thus gain from Theorems B, C and D a better understanding of these quotient categories, and these in turn catalogue the homological representation functors into a kind of *family* of functors:

Theorem F (See §5.4) *Considering a family of groups with associated category* $\langle \mathcal{G}, \mathcal{M}\rangle$*, we denote by* $\mathfrak{L}_{(\mathbf{k},\ell)}$ *any functor of Theorems B, C and D. There is a sequence of quotient categories*

$$\cdots \xleftarrow{\mathcal{P}_d} \mathcal{P}ol_d(\langle \mathcal{G}, \mathcal{M}\rangle)/\mathcal{P}ol_{d-1}(\langle \mathcal{G}, \mathcal{M}\rangle) \xleftarrow{\mathcal{P}_{d+1}} \mathcal{P}ol_{d+1}(\langle \mathcal{G}, \mathcal{M}\rangle)/\mathcal{P}ol_d(\langle \mathcal{G}, \mathcal{M}\rangle) \xleftarrow{\mathcal{P}_{d+2}} \cdots$$

where each $\mathcal{P}_d$ *is naturally defined from polynomial functor theory in §3.1, such that* $\mathfrak{L}_{(\mathbf{k},\ell)}$ *is non-trivial only in the quotient category* $\mathcal{P}ol_k(\langle \mathcal{G}, \mathcal{M}\rangle)/\mathcal{P}ol_{k-1}(\langle \mathcal{G}, \mathcal{M}\rangle)$*. Moreover, for each partition* $\mathbf{k}' \vdash k-1$ *obtained from* $\mathbf{k}$ *by subtracting* 1 *in one partition block, the functor* $\mathfrak{L}_{(\mathbf{k}',\ell)}$ *is a direct summand of* $\mathcal{P}_d(\mathfrak{L}_{(\mathbf{k},\ell)})$*.*

Therefore the quotients of Theorem F provide a new classifying tool to organise polynomial functors and more generally the representation theories of families of groups. It also simplifies them in a reasonable way. Indeed, although these polynomiality properties naturally appear for many functors, there are also some well-known examples of families of representations, typically those whose module structure is not tame enough, whose induced functors are not polynomial. For instance, the quotients of Theorem F do not contain some wild representations such as the exponential ones (for example, the quantum representation functor $\mathfrak{Ver}\colon \mathfrak{U}\beta \to \mathbb{L}\text{-Mod}$, the functors defined from the Magnus representations or those defined from the discrete Heisenberg group [BPS21]; see Corollary 5.2 and Remark 5.3) or some of the vertical-type alternatives of the homological representation functors (see Theorems 4.5 and 4.13). Therefore, although the polynomiality properties are not universal, they are common enough to handle a large range of interesting families of representations, while simplifying the representation theories by eliminating some other families whose behaviour is too wild.

Finally, the techniques that we use to prove polynomiality allow us in addition to deduce some faithfulness results for some of the braid group homological representations; see §5.1. Namely, using the short exact sequences (4.1) and (4.2) that we discover in order to deal with polynomiality, we deduce the following result from the faithfulness properties of Bigelow [Big02; Big01]:

Theorem G (see §5.1) *Let $n \geqslant 0$ and $k \geqslant 2$ be integers and $\mathbf{k} = \{k_1; \ldots ; k_r\} \vdash k$ be a partition such that $k_l = 2$ for at least one $1 \leqslant l \leqslant r$. Then, using the canonical injection $\mathbf{B}_n \hookrightarrow \mathbf{B}_{n+1}$, the $(\mathbf{k}, \ell)$-Lawrence-Bigelow $\mathbf{B}_n$-representations $\mathfrak{LB}_{(\mathbf{k},\ell)}(1+n)$ and $\mathfrak{LB}^u_{(\mathbf{k},\ell)}(1+n)$ are faithful for each $n \geqslant 0$ and $\ell \geqslant 2$.*

Outline. The paper is organised as follows. In §1, we recollect the categorical framework and the construction of [PS21] to properly introduce the homological representation functors. We then study the structures of the representations in §2, proving the key result Theorem A about the underlying modules of the representations. The general theory of polynomial functors is recalled in §3, where we also establish some qualitative properties of the representations by diagrammatic arguments (§3.3). The next section §4 is devoted to the proofs of Theorems B, C and D. Finally, we explain in §5 the applications of these polynomiality properties, proving Theorems E, F and G.

General notations. We denote by $\mathbb{N}$ the set of non-negative integers. Let $\mathcal{C}$ be a small category. We use the abbreviation $\mathrm{ob}(\mathcal{C})$ to denote the set of objects of $\mathcal{C}$. For $\mathcal{D}$ a category and $\mathcal{C}$ a small category, we denote by $\mathbf{Fct}(\mathcal{C}, \mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$. For X a space, $\mathring{X}$ denotes its interior. For R a non-zero unital ring, we denote by R-Mod the category of left R-modules. For G a group and M a G-module, we denote by $\mathrm{Aut}_G(M)$ the group of G-module automorphisms of M. When $G = \mathbb{Z}$, we omit it from the notation as long as there is no ambiguity. We denote by $\mathfrak{S}_n$ the symmetric group on a set of n elements. For an integer $n \geqslant 1$, a *partition of* n means an r-tuple $\mathbf{n} = \{n_1, \ldots, n_r\}$ of integers $n_i \geqslant 1$ (for some $r \geqslant 1$ called the *length* of $\mathbf{n}$) such that $n = \sum_{1 \leqslant i \leqslant r} n_i$. For simplicity, we denote the trivial partition $\mathbf{n} = \{n\}$ by n. The lower central series of a group G is the descending chain of subgroups $\{\Gamma_l(G)\}_{l \geqslant 0}$ defined by $\Gamma_1(G) := G$ and $\Gamma_{l+1}(G) := [G, \Gamma_l(G)]$, the subgroup of G generated by the commutators $[g, h]$ for $g \in G$, $h \in \Gamma_l(G)$.

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Contents

1. Background on homological representation functors

This section recollects the construction of homological representation functors introduced in [PS21]; see §1.2. We first recall the underlying categorical framework in §1.1 and then detail in §1.3 the outputs of the construction of [PS21] for the families of groups studied in this paper.

1.1. Categorical framework for families of groups

We introduce here the categorical framework that is central to this paper to handle families of groups. Namely, we present an appropriate groupoid encoding each of the considered families of groups in §1.1.2. These groupoids will always be braided monoidal or modules over a braided monoidal category. This allows us to apply in each case the Quillen bracket construction recalled in §1.1.1. The resulting category is "richer" in the sense that it has more morphisms, and is the appropriate one both for the constructions of compatible representations of [PS21] and to define properties of polynomiality of homological representation functors; see §3.1.

Preliminaries on categorical tools. We refer to [Mac98] for a complete introduction to the notions of strict monoidal categories and modules over them. We generically denote a strict monoidal category by $(\mathcal{C}, \natural, 0)$, where $\mathcal{C}$ is a category, $\natural$ is the monoidal product and 0 is the monoidal unit. If it is braided, then its braiding is denoted by $b^{\mathcal{C}}_{A,B} \colon A\natural B \xrightarrow{\sim} B\natural A$ for all objects A and B of $\mathcal{C}$. A left-module $(\mathcal{M}, \sharp)$ over a (strict) monoidal category $(\mathcal{C}, \natural, 0)$ is a category $\mathcal{M}$ with a functor $\sharp \colon \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ that is unital and associative. For instance, a monoidal category $(\mathcal{C}, \natural, 0)$ is equipped with a left-module structure over itself, induced by the monoidal product $\natural \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Considering the category of (small skeletal strict) braided monoidal groupoids $\mathfrak{BrG}$, there is always an arbitrary binary choice for the convention of the braiding. We may pass from one to the other by the following inversion of the braiding operator. Let $(-)^{\dagger} \colon \mathfrak{BrG} \to \mathfrak{BrG}$ be the endofunctor defined on each object $(\mathcal{G}, \natural, 0)$ by $(\mathcal{G}, \natural, 0)^{\dagger} = (\mathcal{G}, \natural, 0)$ as a monoidal groupoid but whose braiding is defined by the inverse of that of $(\mathcal{G}, \natural, 0)$, i.e. $b^{\mathcal{G}^{\dagger}}_{A,B} := (b^{\mathcal{G}}_{B,A})^{-1}$.

1.1.1. The Quillen bracket construction

In this section, we describe a useful categorical framework for handling families of groups: the *bracket construction* due to Quillen, which is a particular case of a more general construction described in [Gra76, p.219]. **Throughout §1, we fix an object $(\mathcal{G}, \natural, 0)$ of $\mathfrak{BrG}$ and a (small strict) left-module $(\mathcal{M}, \natural)$ over $\mathcal{G}$.**

The *Quillen bracket construction* $\langle \mathcal{G}, \mathcal{M} \rangle$ on the left-module $(\mathcal{M}, \natural)$ over the groupoid $(\mathcal{G}, \natural, 0)$

is the category with the same objects as $\mathcal{M}$ and whose morphisms are given by:

$$\mathrm{Hom}_{\langle\mathcal{G},\mathcal{M}\rangle}(X,Y)=\underset{\mathcal{G}}{\mathrm{colim}}[\mathrm{Hom}_{\mathcal{M}}(-\natural X,Y)].$$

Thus, a morphism from X to Y in $\langle\mathcal{G},\mathcal{M}\rangle$ is denoted by $[A,\varphi]\colon X\to Y$: it is an equivalence class of pairs (A,φ), where A is an object of $\mathcal{G}$ and $\varphi\colon A\natural X\to Y$ is a morphism in $\mathcal{M}$. Also, for two morphisms $[A,\varphi]\colon X\to Y$ and $[B,\psi]\colon Y\to Z$ in $\langle\mathcal{G},\mathcal{M}\rangle$, the composition is defined by $[B,\psi]\circ[A,\varphi]=[B,Y\natural A,\psi\circ(\mathrm{id}_B\natural\varphi)]$. There is a faithful canonical functor $\mathcal{M}\hookrightarrow\langle\mathcal{G},\mathcal{M}\rangle$ defined as the identity on objects and sending $\phi\in\mathrm{Hom}_{\mathcal{M}}(X,Y)$ to $[0,\phi]$. If $\mathcal{M}$ is a groupoid, if $(\mathcal{G},\natural,0)$ has *no zero divisors* (i.e. $A\natural B\cong 0$ if and only if $A\cong B\cong 0$ for $A,B\in\mathrm{Obj}(\mathcal{G})$) and if $\mathrm{Aut}_{\mathcal{G}}(0)=\{\mathrm{id}_0\}$, then $\mathcal{M}$ is the maximal subgroupoid of $\langle\mathcal{G},\mathcal{M}\rangle$; see [RW17, Prop. 1.7]. All these properties are satisfied in all the situations of this paper (see §1.1.2), and we assume them from now on.

Monoidal structure. The category $\langle\mathcal{G},\mathcal{M}\rangle$ inherits a (strict) monoidal product as follows. All the following discussion is a verbatim generalisation of [RW17, Prop. 1.8]. The monoidal product $\natural$ extends to $\langle\mathcal{G},\mathcal{M}\rangle$ by letting for $[X,\varphi]\in\mathrm{Hom}_{\langle\mathcal{G},\mathcal{M}\rangle}(A,B)$ and $[Y,\psi]\in\mathrm{Hom}_{\langle\mathcal{G},\mathcal{M}\rangle}(C,D)$:

$$[X,\varphi]\natural[Y,\psi]=[X\natural Y,(\varphi\natural\psi)\circ(\mathrm{id}_X\natural(b^{\mathcal{G}}_{A,Y})^{-1}\natural\mathrm{id}_C)].\tag{1.1}$$

Moreover, if $\mathcal{M}=\mathcal{G}$, the unit 0 of the monoidal structure is an initial object in $\langle\mathcal{G},\mathcal{G}\rangle$. We denote by $\iota_X\colon 0\to X$ the unique morphism from 0 to a given object X of $\langle\mathcal{G},\mathcal{G}\rangle$. Also, the category $(\langle\mathcal{G},\mathcal{G}\rangle,\natural,0)$ is *pre-braided* monoidal in the sense of [RW17, Def. 1.5]: namely, its maximal subgroupoid $(\mathcal{G},\natural,0)$ is braided monoidal and the braiding $b^{\langle\mathcal{G},\mathcal{G}\rangle}_{A,B}\colon A\natural B\to B\natural A$ satisfies the relation $b^{\langle\mathcal{G},\mathcal{G}\rangle}_{A,B}\circ(\mathrm{id}_A\natural\iota_B)=\iota_B\natural\mathrm{id}_A\colon A\to B\natural A$ for all $A,B\in\mathrm{Obj}(\langle\mathcal{G},\mathcal{G}\rangle)$.

Extensions along the Quillen bracket construction. The following result provides a way to extend a functor on the category $\mathcal{M}$ to a functor with $\langle\mathcal{G},\mathcal{M}\rangle$ as source category. Its proof is a mutatis mutandis generalisation of that of [Sou22, Lem. 1.2].

Lemma 1.1 *Let $\mathcal{C}$ be a category and F an object of $\mathbf{Fct}(\mathcal{M},\mathcal{C})$. Assume that there exist a morphism $\eta_{Q,P}\colon F(P)\to F(Q\natural P)$ for each $Q\in\mathrm{Obj}(\mathcal{G})$ and $P\in\mathrm{Obj}(\mathcal{M})$, such that $\eta_{Y,X\natural A}\circ\eta_{X,A}=\eta_{Y\natural X,A}$ for all $X,Y\in\mathrm{Obj}(\mathcal{G})$ and $A\in\mathrm{Obj}(\mathcal{M})$, and $\eta_{0,B}=\mathrm{id}_{F(B)}$ for all $B\in\mathrm{Obj}(\mathcal{M})$. Then, the assignments $F([X,f])=F(f)\circ\eta_{X,A}$ to all morphisms $[X,f]\colon A\to X\natural A$ of $\langle\mathcal{G},\mathcal{M}\rangle$ extend the functor $F\colon\mathcal{M}\to\mathcal{C}$ to a functor $F\colon\langle\mathcal{G},\mathcal{M}\rangle\to\mathcal{C}$ if and only if for all $X\in\mathrm{Obj}(\mathcal{G})$ and $A\in\mathrm{Obj}(\mathcal{M})$, for all $f'\in\mathrm{Aut}_{\mathcal{G}}(X)$ and $f''\in\mathrm{Aut}_{\mathcal{M}}(A)$, the following relation holds:*

$$F([X,\mathrm{id}_{X\natural A}])\circ F(f'')=F(f'\natural f'')\circ F([X,\mathrm{id}_{X\natural A}]).\tag{1.2}$$

Similarly, we have the following criterion for extending a morphism of the category $\mathbf{Fct}(\mathcal{M},\mathcal{C})$ to a morphism in the category $\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{C})$. The proof is a verbatim adaptation of that of [Sou19, Lem. 1.12].

Lemma 1.2 *Let $\mathcal{C}$ be a category, F and G be objects of $\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{C})$ and $\eta\colon F\to G$ a natural transformation in $\mathbf{Fct}(\mathcal{M},\mathcal{C})$. The restriction $\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{C})\to\mathbf{Fct}(\mathcal{M},\mathcal{C})$ is obtained by pre-composing by the canonical inclusion $\mathcal{M}\hookrightarrow\langle\mathcal{G},\mathcal{M}\rangle$. Then, η is a natural transformation in the category $\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{C})$ if and only if for all $X,Y\in\mathrm{Ob}(\mathcal{M})$ such that $Y\cong A\natural X$ with $A\in\mathrm{Ob}(\mathcal{G})$:*

$$\eta_Y\circ F([A,\mathrm{id}_Y])=G([A,\mathrm{id}_Y])\circ\eta_X.\tag{1.3}$$

1.1.2. Categories for surface braid groups and mapping class groups

We now recollect the suitable categories for our work, which are inspired by [RW17, §5.6]. This is done in §1.1.2.1 for the mapping class groups of surfaces and in §1.1.2.2 for (surface) braid groups. We also take this opportunity to recollect the various definitions and properties of these families of groups. The content of this section is classical knowledge; see [RW17, §5.6] (for the original definitions of the categories) and [Sou22, §3.1] (for the skeletal versions) for further technical justifications of the properties and definitions.

1.1.2.1. For mapping class groups of surfaces

The decorated surface groupoid $\hat{\mathcal{M}}_2$ is introduced in [RW17, §5.6] and defined as follows. Its objects are the decorated surfaces (S,k,I), where S is a smooth connected compact surface with one boundary component ∂S, with a finite set of $k \geqslant 0$ points removed from the interior of S (in other words with *punctures*) together with a parametrised interval $I\colon [-1,1] \hookrightarrow \partial S$ in the boundary. When there is no ambiguity, we omit k and I from the notation for convenience. Let $\mathrm{Diff}_I(S,\mathbf{k})$ be the group of diffeomorphisms of the surface $\bar{S}$ obtained from S by filling in each puncture with a *marked point*, which restrict to the identity on a neighbourhood of the parametrised interval I and fixing the k marked points setwise by stabilising the partition $\mathbf{k}$ as a subset. When the surface S is orientable, the orientation on S is induced by the orientation of I, and then the isotopy classes of $\mathrm{Diff}_I(S,\mathbf{k})$ automatically preserve that orientation. The (auto)morphisms of $\hat{\mathcal{M}}_2$ are the *mapping class groups* of $(S,\mathbf{k},I)$ denoted by $\mathrm{MCG}(S,\mathbf{k})$, i.e. the isotopy classes $\pi_0\mathrm{Diff}_I(S,\mathbf{k})$.

By [RW17, §5.6.1], the boundary connected sum $\natural$ induces a braided monoidal structure on $\hat{\mathcal{M}}_2$ as follows. For a parametrised interval I, the left half-interval $[-1,0] \hookrightarrow \partial S$ of I is denoted by I^- and the right half-interval $[0,1] \hookrightarrow \partial S$ of I is denoted by I^+, defining $I = I^- \cup I^+$. For two decorated surfaces (S_1,I_1) and (S_2,I_2), the boundary connected sum $(S_1,I_1)\natural(S_2,I_2)$ is defined to be the surface $(S_1\natural S_2, I_1\natural I_2)$, where $S_1\natural S_2$ obtained by gluing S_1 and S_2 along the half-interval I_1^+ and the half-interval I_2^-, and $I_1\natural I_2 = I_1^- \cup I_2^+$. The braiding $b^{\hat{\mathcal{M}}_2}_{S_1,S_2}$ of the monoidal structure is the half Dehn twist with respect to the separating curve $I_2^- = I_1^+$ that exchanges the two summands S_1 and S_2; see [RW17, Fig. 2]. In order to strictify this monoidal structure, we formally adjoin a new strict monoidal unit $\mathbb{I}$ to $\hat{\mathcal{M}}_2$ and then apply [Sch01, Th. 4.3], which says that one may force $\natural$ to be strictly associative, without changing the underlying category or the unit object, by making careful choices of concrete (set-theoretic) realisations of $S_1\natural S_2$ for each S_1 and S_2. Let us denote the resulting *strict* braided monoidal groupoid by $(\mathcal{M}_2, \natural, \mathbb{I}, b^{\mathcal{M}_2}_{-,-})$.

Now fix a once-punctured disc $\mathbb{D}_1$, a torus with one boundary component $\mathbb{T}$ and a Möbius strip $\mathbb{M}$. We will often denote $\mathbb{D}_1^{\natural s}$ by $\mathbb{D}_s$ for concision. Let $\mathcal{M}_2^+$ and $\mathcal{M}_2^-$ be the full subgroupoids of $\mathcal{M}_2$ on the objects $\{\mathbb{T}^{\natural g}\}_{g\in\mathbb{N}}$ and $\{\mathbb{M}^{\natural h}\}_{h\in\mathbb{N}}$ respectively, where $\mathbb{T}^{\natural 0} = \mathbb{M}^{\natural 0} = \mathbb{I}$. The strict braided monoidal structure on $\mathcal{M}_2$ restricts to a strict braided monoidal structure on each of $\mathcal{M}_2^+$ and $\mathcal{M}_2^-$. We also note that these two subgroupoids are small, skeletal, have no zero divisors and that $\mathrm{Aut}_{\mathcal{M}_2^+}(\mathbb{T}^{\natural 0}) = \mathrm{Aut}_{\mathcal{M}_2^-}(\mathbb{M}^{\natural 0}) = \{id_{\mathbb{I}}\}$. We denote the mapping class groups $\mathrm{MCG}(\mathbb{T}^{\natural g})$ and $\mathrm{MCG}(\mathbb{M}^{\natural h})$ by $\mathbf{\Gamma}_{h,1}$ and $\boldsymbol{\mathcal{N}}_{h,1}$ respectively.

1.1.2.2. For braid goups on surfaces

Let S be a compact, connected, smooth surface with one boundary component. There exist integers $g \geqslant 0$ and $h \geqslant 0$ and a homeomorphism $S \cong \mathbb{T}^{\natural g}\natural\mathbb{M}^{\natural h}$ using the notations of §1.1.2.1. If the surface is orientable (i.e. $h=0$), then g is unique and we prefer to denote $S \cong \mathbb{T}^{\natural g}$ by $\Sigma_{g,1}$. We also denote $S \cong \mathbb{M}^{\natural h}$ by $\mathcal{N}_{h,1}$.

There are several ways to introduce (partitioned) surface braid groups; see for example [DPS22, §6.2–6.3] for a detailed overview. For a partition $\mathbf{n} = \{n_1,\ldots,n_r\} \vdash n$, we denote by $C_{\mathbf{n}}(S)$ the $\mathbf{n}$-configuration space $\{(x_1,\ldots,x_n) \in S^{\times n} \mid x_i \neq x_j \text{ if } i \neq j\}/\mathfrak{S}_{\mathbf{n}}$ of n points in the surface S, with $\mathfrak{S}_{\mathbf{n}} := \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_r}$. The $\mathbf{n}$*-partitioned braid group on n strings on the surface S* is the fundamental group of this configuration space: $\mathbf{B}_{\mathbf{n}}(S) = \pi_1(C_{\mathbf{n}}(S), c_0)$, where c_0 is a configuration in the boundary of S. The braid groups on the 2-disc $\mathbb{D}^2$ are the *classical braid groups*; we omit $\mathbb{D}^2$ from the notation in this case. Full presentations of these groups are recalled in [PS23, Prop. 2.2].

The family of classical braid groups is associated with the small skeletal groupoid $\boldsymbol{\beta}$, with objects the non-negative integers n and morphisms $\mathrm{Hom}_{\boldsymbol{\beta}}(n,m) = \mathbf{B}_n$ if $n=m$ and the empty set otherwise. The composition of morphisms $\circ$ in the groupoid $\boldsymbol{\beta}$ corresponds to the group operation of the braid groups. We recall from [Mac98, Chapter XI, §4] that $\boldsymbol{\beta}$ has a canonical strict monoidal product $\natural\colon \boldsymbol{\beta}\times\boldsymbol{\beta} \to \boldsymbol{\beta}$ defined by the usual addition for the objects and laying two braids side by side for the morphisms. The object 0 is the unit of this monoidal product. The strict monoidal groupoid $(\boldsymbol{\beta},\natural,0)$ is braided: the braiding is defined for all non-negative integers n and m such that $n+m \geqslant 2$ by $b^{\boldsymbol{\beta}}_{n,m} = (\sigma_m \circ \cdots \circ \sigma_2 \circ \sigma_1) \circ \cdots \circ (\sigma_{n+m-1} \circ \cdots \circ \sigma_{n+1} \circ \sigma_n)$, where each σ_i denotes the i-th Artin generator. Finally, we note that $(\boldsymbol{\beta},\natural,0)$ has no zero divisors and that $\mathrm{Aut}_{\boldsymbol{\beta}}(0) = \{\mathrm{id}_0\}$.

Similarly, let $\boldsymbol{\beta}^S$ be the groupoid with objects the non-negative integers and morphisms given by $\mathrm{Hom}_{\boldsymbol{\beta}^S}(n,m) = \mathbf{B}_n(S)$ if $n = m$ and the empty set otherwise. For each S, there is a canonical strict left $\boldsymbol{\beta}$-module structure on $\boldsymbol{\beta}^S$: the associative, unital functor $\natural\colon \boldsymbol{\beta} \times \boldsymbol{\beta}^S \to \boldsymbol{\beta}^S$ is defined by addition on objects and, on morphisms, by the maps of configuration spaces $C_n(S) \times C_m(\mathbb{D}^2) \to C_{n+m}(S)$ induced by a choice of homeomorphism $S\natural\mathbb{D}^2 \cong S$ whose restriction to the left-hand summand is a self-embedding of S that is isotopic to the identity.

1.1.3. Topological lifts

We construct in [PS21, §4] topologically-enriched versions of all of the above discrete groupoids that recover the discrete groupoids after applying π_0 to all morphism spaces (which are certain diffeomorphism groups). We denote these "topological lifts" of the discrete groupoids by a superscript $(-)^{\mathrm{t}}$. We also give a topologically-enriched version of the Quillen bracket construction [PS21, §3.1], which under mild conditions commutes with applying π_0 to morphism spaces [PS21, Lem. 3.6]. Thus we also obtain topologically-enriched versions of the Quillen bracket constructions of the above modules over braided monoidal groupoids, whose morphism spaces may be identified with certain embedding spaces [PS21, Prop. 4.8].

1.2. Construction of homological representation functors

Here, we first recollect the general machinery of [PS21] to construct homological representation functors for families of groups; see §1.2.1. We then explain in §1.2.2 some key manipulations of the transformation groups associated to these homological representation functors.

1.2.1. The general recipe

In this section, we sum up the construction of homological representation functors for surface braid groups and mapping class groups of surfaces; see Construction 1.7. We refer to [PS21, §2, §5] for further details. This general theory may however be avoided by some ad hoc constructions; see Remark 1.9. Hence, although we prefer following the framework of [PS21] because it is more complete and natural, the results of the present paper do not depend on this reference.

Framework. Let $\mathcal{G}$ be a small strict braided monoidal groupoid and $\mathcal{M}$ a $\mathcal{G}$-module defined in §1.1.2. We denote by $\{G_n\}_{n\in\mathbb{N}}$ the family of groups encoded as the automorphisms of $\mathcal{M}$. These come equipped with canonical injections induced by the monoidal structure $\mathrm{id}_1\natural(-)_n\colon G_n \hookrightarrow G_{n+1}$. The first key ingredient to define homological representations is to find a family of spaces on which the family $\{G_n\}_{n\in\mathbb{N}}$ acts. In the case of surface braid groups, this will involve considering "partitioned versions" of the groups G_n, which we define in a general context:

Definition 1.3 Let $\mathtt{G}_k$ be a group equipped with a surjection $\mathfrak{s}_k\colon \mathtt{G}_k \twoheadrightarrow \mathfrak{S}_k$. Given a partition $\mathbf{k} = \{k_1,\ldots,k_r\} \vdash k$, for $j \leqslant r$, we define $t_j := \sum_{i\leqslant j} k_i$ (including $t_0 = 0$). Then the set $b_j(\mathbf{k}) := \{t_{j-1}+1,\ldots,t_j\}$ is referred to as the *j-th block* of $\mathbf{k}$, and k_i is called the *size* of the i-th block. The preimage $\mathtt{G}_{\mathbf{k}} := \mathfrak{s}_k^{-1}(\mathfrak{S}_{\mathbf{k}})$ (where $\mathfrak{S}_{\mathbf{k}} := \mathfrak{S}_{k_1}\times\cdots\times\mathfrak{S}_{k_r}$) is called the **k**-*partitioned version* of $\mathtt{G}_k$ and fits into a short exact sequence

$$1 \longrightarrow \mathtt{G}_{\mathbf{k}} \xrightarrow{\mathfrak{g}_{\mathbf{k}}} \mathtt{G}_k \xrightarrow{\mathfrak{s}_{\mathbf{k}}} \mathfrak{S}_{\mathbf{k}} \longrightarrow 1. \tag{1.4}$$

The extremal situations are the discrete partition $\mathbf{k} = \{1,\ldots,1\}$, which corresponds to the *pure version* of the group $\mathtt{G}_k$, and the trivial case $\mathbf{k} = (k)$, which is simply the group $\mathtt{G}_k$ itself.

In all the situations addressed in this paper, the parameter k corresponds to the motion of k points, while the surjection corresponds to the permutations of these points. For the remainder of §1.2.1, we consider a partition $\mathbf{k} = \{k_1;\ldots;k_r\} \vdash k$ of an integer $k \geqslant 1$. We consider the configuration space $\left\{(x_1,\ldots,x_k)\in\mathcal{S}_n^{\times k} \mid x_i \neq x_j \text{ if } i\neq j\right\}/\mathfrak{S}_{\mathbf{k}}$ associated to the partition $\mathbf{k}$ of k points in a surface $\mathcal{S}_n$, denoted by $C_{\mathbf{k}}(\mathcal{S}_n)$. The surface $\mathcal{S}_n$ is defined differently depending on the setting:

- When $G_n = \mathbf{B}_n(S)$ ($S = \Sigma_{g,1}$ or $\mathbb{N}_{h,1}$) we set $\mathcal{S}_n := \mathbb{D}_n\natural S$. We also set $G_{\mathbf{k},n} = \mathbf{B}_{\mathbf{k},n}(S)$.
- When $G_n = \mathrm{MCG}(S^{\natural n})$ ($S = \mathbb{T}$ or $\mathbb{M}$) we set $\mathcal{S}_n := S^{\natural n}$. We also set $G_{\mathbf{k},n} = \mathrm{MCG}(\mathcal{S}_n,\mathbf{k})$.

In each case, there is a split short exact sequence

$$1 \longrightarrow \mathbf{B}_{\mathbf{k}}(\mathcal{S}_n) \longrightarrow G_{\mathbf{k},n} \longrightarrow G_n \longrightarrow 1, \tag{1.5}$$

known as the *Fadell-Neuwirth exact sequence* if $G_n = \mathbf{B}_n(S)$ (see for instance [PS21, Prop. 4.15] or [DPS22, Prop. 6.15]) and as the *Birman short exact sequence* if $G_n = \mathrm{MCG}(S^{\natural n})$ (see for instance [FM12, Lem. 4.16] or [PS21, Cor. 4.19, §5.1.3]). This short exact sequence provides an action (by conjugation) of G_n on $\mathbf{B}_{\mathbf{k}}(\mathcal{S}_n)$. We note in passing that, in the case $G_n = \mathbf{B}_n(S)$, the section of (1.5), denoted by $s_{(\mathbf{k},n)}$, is such that its composition with the injection $G_{\mathbf{k},n} \hookrightarrow G_{k+n}$ of (1.4) is equal to $\mathrm{id}_k \natural (-)_n \colon G_n \hookrightarrow G_{k+n}$.

Twisted representations. Another preliminary is the recollection of the notion of *twisted* representations. For simplicity, we use $\mathbb{Z}$ as ground ring, although we could consider all of the following framework over a non-zero associative unital ring $\mathbb{A}$; each mention of "ring" should then be replaced by "$\mathbb{A}$-algebra".

Definition 1.4 (*Category of twisted modules.*) Let Q be a group. The category of *twisted* $\mathbb{Z}[Q]$-modules, denoted by $\mathbb{Z}[Q]\text{-}\mathrm{Mod}^{\mathrm{tw}}$, is defined as follows. An object of $\mathbb{Z}[Q]\text{-}\mathrm{Mod}^{\mathrm{tw}}$ is a left $\mathbb{Z}[Q]$-module V. A morphism from V to V' is an $\mathbb{Z}$-algebra automorphism $\psi \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}[Q])$ preserving units together with a morphism of left $\mathbb{Z}[Q]$-modules $\theta \colon V \to \psi^*(V')$. By definition, the module category $\mathbb{Z}[Q]\text{-}\mathrm{Mod}$ is the subcategory of $\mathbb{Z}[Q]\text{-}\mathrm{Mod}^{\mathrm{tw}}$ on the same objects and those morphisms (ψ, θ) with $\psi = \mathrm{id}_{\mathbb{Z}[Q]}$. In a larger context, one may consider the *category of left modules*, whose objects are pairs of a unital ring together with a left module over that ring and whose morphisms are compatible pairs of ring and module homomorphisms. There is an evident forgetful functor to the category of rings and $\mathbb{Z}[Q]\text{-}\mathrm{Mod}^{\mathrm{tw}}$ is the pre-image of the full subcategory on the ring $\mathbb{Z}[Q]$. There is also a forgetful functor $\mathbb{Z}[Q]\text{-}\mathrm{Mod}^{\mathrm{tw}} \to \mathbb{Z}\text{-}\mathrm{Mod}$, where we forget the $\mathbb{Z}[Q]$-module structure on objects and we forget the ψ component of a morphism (ψ, θ).

A functor $\langle \mathcal{G}, \mathcal{M} \rangle \to \mathbb{Z}[Q]\text{-}\mathrm{Mod}^{\mathrm{tw}}$ encodes *twisted* $\mathbb{Z}[Q]$-representations: at the level of group representations, it means that the action of G_n on the corresponding $\mathbb{Z}[Q]$-module commutes with the $\mathbb{Z}[Q]$-structure only up to a "twist", i.e., an action $G_n \to \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}[Q])$. However, these representations are always *genuine* $\mathbb{Z}$-module representations. Indeed, we may always post-compose:

$$\langle \mathcal{G}, \mathcal{M} \rangle \longrightarrow \mathbb{Z}[Q]\text{-}\mathrm{Mod}^{\mathrm{tw}} \longrightarrow \mathbb{Z}\text{-}\mathrm{Mod}, \tag{1.6}$$

in order to view *twisted* $\mathbb{Z}[Q]$-module representations as *genuine* $\mathbb{Z}$-module representations.

Transformation group functor. We now introduce the key parameter to define a homological representation functor. For the remainder of §1.2.1, we consider an integer $\ell \geqslant 1$ corresponding to a lower central series index. For each n, the short exact sequence (1.5) induces the key defining diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{B}_{\mathbf{k}}(\mathcal{S}_n) & \longrightarrow & G_{\mathbf{k},n} & \longrightarrow & G_n & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Q_{(\mathbf{k},\ell,n)}(\mathcal{S}) & \longrightarrow & G_{\mathbf{k},n}/\Gamma_\ell & \longrightarrow & G_n/\Gamma_\ell & \longrightarrow & 1.
\end{array} \tag{1.7}$$

More precisely, the right-exactness of the quotient $-/\Gamma_\ell$ gives the right-half of the bottom short exact sequence and ensures that the right-hand square of the diagram is commutative; the group $Q_{(\mathbf{k},\ell,n)}(\mathcal{S})$ is defined as the kernel of the surjection $G_{\mathbf{k},n}/\Gamma_\ell \twoheadrightarrow G_n/\Gamma_\ell$; the map $\mathbf{B}_{\mathbf{k}}(\mathcal{S}_n) \twoheadrightarrow Q_{(\mathbf{k},\ell,n)}(\mathcal{S})$ is uniquely defined by the universal property of $\mathbf{B}_{\mathbf{k}}(\mathcal{S}_n)$ as a kernel and its surjectivity follows from the 5-lemma. Furthermore, the universal property of $G_{\mathbf{k},n}/\Gamma_\ell$ and G_n/Γ_ℓ as cokernels ensure there exist unique maps $G_{\mathbf{k},n}/\Gamma_\ell \to G_{\mathbf{k},n+1}/\Gamma_\ell$ and $G_n/\Gamma_\ell \to G_{n+1}/\Gamma_\ell$, induced by $\mathrm{id}_1 \natural (-)$, making the following square commutative:

$$\begin{array}{ccc}
G_{\mathbf{k},n}/\Gamma_\ell & \twoheadrightarrow & G_n/\Gamma_\ell \\
\downarrow & & \downarrow \\
G_{\mathbf{k},n+1}/\Gamma_\ell & \twoheadrightarrow & G_{n+1}/\Gamma_\ell.
\end{array}$$

Hence, by the universal property of a kernel, there exists a canonical map $q_{(\mathbf{k},\ell,n)}\colon Q_{(\mathbf{k},\ell,n)}(\mathcal{S}) \to Q_{(\mathbf{k},\ell,n+1)}(\mathcal{S})$ making the clear diagram commutative. The colimit of the modules $\{(Q_{(\mathbf{k},\ell,n)}(\mathcal{S}))\}_{n\in\mathbb{N}}$ with respect to the maps $q_{(\mathbf{k},\ell,n)}$ is denoted by $Q_{(\mathbf{k},\ell)}(\mathcal{S})$. In many examples of interest, this colimit is isomorphic to $Q_{(\mathbf{k},\ell,n)}(\mathcal{S})$ for all n sufficiently large. This phenomenon is called *Q-stability* in [PS21, Def. 5.14], although we will not need this in the present paper. With or without the above Q-stability property, the surjection $\mathbf{B}_{\mathbf{k}}(\mathcal{S}_n) \twoheadrightarrow Q_{(\mathbf{k},\ell,n)}(\mathcal{S})$ defines a regular covering of $C_{\mathbf{k}}(\mathcal{S}_n)$ by classical covering space theory (see for instance [Hat02, §1.3]), on which the group G_n acts naturally. In particular, G_n acts by conjugation on the image of the map $\mathbf{B}_{\mathbf{k}}(\mathcal{S}_n) \to Q_{(\mathbf{k},\ell)}(\mathcal{S})$. We define below a *transformation group functor* $F_{(\mathbf{k},\ell)}(\mathcal{S})$ encoding all these covering spaces and group actions.

Beforehand, we need to define a category associated to these coverings. Let Q be a group. The category of *twisted (proper) coverings over* Q is denoted by $\mathrm{Cov}_Q^{\mathrm{tw}}$ and defined as follows. An object of $\mathrm{Cov}_Q^{\mathrm{tw}}$ is a path-connected, based, locally-compact space X admitting a universal covering (i.e., locally path-connected and semi-locally simply-connected), equipped with a homomorphism $\phi\colon \pi_1(X) \to Q$. Via the correspondence between path-connected, regular coverings of X and normal subgroups of $\pi_1(X)$, the surjective morphism $\pi_1(X) \twoheadrightarrow \mathrm{Im}(\phi)$ determines a regular covering X^{ϕ}. A morphism in $\mathrm{Cov}_Q^{\mathrm{tw}}$ from (X,ϕ) to (X',ϕ') is a based, proper map $f\colon X \to X'$ such that the induced homomorphism $\pi_1(f)$ sends $\ker(\phi)$ into $\ker(\phi')$. This implies that there is a unique homomorphism $\alpha\colon Q \to Q$ such that $\phi' \circ \pi_1(f) = \alpha \circ \phi$. The category Cov_Q of *(proper) coverings over* Q is the subcategory of $\mathrm{Cov}_Q^{\mathrm{tw}}$ on the same objects and those morphisms f such that the induced homomorphism α is equal to id_Q; see [PS21, Def. 2.3] for further details.

Using the topological lifts of the categories recalled in §1.1.3, it follows from the general theory of [PS21, §5.1.1–§5.1.5] that there is a well-defined functor

$$F_{(\mathbf{k},\ell)}(\mathcal{S})\colon \langle \mathcal{G}^{\mathrm{t}}, \mathcal{M}^{\mathrm{t}} \rangle \longrightarrow \mathrm{Cov}^{\mathrm{tw}}_{Q_{(\mathbf{k},\ell)}(\mathcal{S})} \tag{1.8}$$

defined, for each object X_n of $\mathcal{M}^{\mathrm{t}}$ such that $\mathrm{Aut}_{\langle \mathcal{G},\mathcal{M}\rangle}(X_n) \cong G_n$, by sending X_n to the configuration space $C_{\mathbf{k}}(\mathcal{S}_n)$ equipped with the morphism $\mathbf{B}_{\mathbf{k}}(\mathcal{S}_n) \twoheadrightarrow Q_{(\mathbf{k},\ell,n)}(\mathcal{S}) \to Q_{(\mathbf{k},\ell)}(\mathcal{S})$.

A natural goal is to choose transformation groups Q_n such that the actions of the groups G_n on the colimit group Q are trivial, i.e. so that the functor (1.8) factors through the category Cov_Q. The optimal way to do this consists in taking the coinvariants of the group $Q_{(\mathbf{k},\ell)}(\mathcal{S})$ under the natural action of each G_n. Namely, let $Q^u_{(\mathbf{k},\ell)}(\mathcal{S})$ be the colimit of the modules $\{(Q_{(\mathbf{k},\ell,n)}(\mathcal{S}))_{G_n}\}_{n\in\mathbb{N}}$ with respect to the maps $(Q_{(\mathbf{k},\ell,n)}(\mathcal{S}))_{G_n} \to (Q_{(\mathbf{k},\ell,n+1)}(\mathcal{S}))_{G_{n+1}}$ induced by the canonical morphisms $\mathrm{id}_1 \natural (-)_n\colon G_n \hookrightarrow G_{n+1}$. Hence, there is a canonical surjective morphism

$$Q_{(\mathbf{k},\ell)}(\mathcal{S}) \twoheadrightarrow Q^u_{(\mathbf{k},\ell)}(\mathcal{S}). \tag{1.9}$$

The quotient $Q^u_{(\mathbf{k},\ell)}(\mathcal{S})$ is *optimal* in the sense that any other *untwisted* (i.e. with trivial G_n-actions) quotient Q' of $Q_{(\mathbf{k},\ell)}(\mathcal{S})$ is a quotient of $Q^u_{(\mathbf{k},\ell)}(\mathcal{S})$; the "$u$" in the notation stands for *untwisted*. Repeating the procedure of [PS21, §5.1.1–§5.1.5], there is a well-defined functor

$$F^u_{(\mathbf{k},\ell)}(\mathcal{S})\colon \langle \mathcal{G}^{\mathrm{t}}, \mathcal{M}^{\mathrm{t}} \rangle \longrightarrow \mathrm{Cov}_{Q^u_{(\mathbf{k},\ell)}(\mathcal{S})} \tag{1.10}$$

Homology of covering spaces. We finally describe the remaining steps of Construction 1.7, encoding the idea of taking twisted Borel-Moore homology of covering spaces.

To do this, we first recall the category $\mathrm{Top}_{\bullet}$ of *bundles of (right) modules*. An object of $\mathrm{Top}_{\bullet}$ is a locally-compact space X together with an associative, unital ring S and a bundle of right S-modules over X; equivalently, a functor $\xi\colon \pi_{\leqslant 1}(X) \to S\text{-Mod}$. A morphism from (X,S,ξ) to (X',S',ξ') is a proper map $f\colon X \to X'$, a ring homomorphism $\psi\colon S \to S'$, and an endofunctor $E\colon S\text{-Mod} \to S'\text{-Mod}$ such that $\psi^* \circ \xi' \circ \pi_{\leqslant 1}(f) = E \circ \xi$, where $\psi^*\colon S\text{-Mod} \to S'\text{-Mod}$ is the restriction functor induced by ψ. Denote by $\mathrm{Top}^{\mathrm{tw}}_{\mathbb{Z}[Q]} \subset \mathrm{Top}_{\bullet}$ the full subcategory on those objects (X,S,ξ) where $S = \mathbb{Z}[Q]$, and let $\mathrm{Top}_{\mathbb{Z}[Q]} \subset \mathrm{Top}^{\mathrm{tw}}_{\mathbb{Z}[Q]}$ be the subcategory on the same objects but with only those morphisms where $E = \mathrm{id}_{\mathbb{Z}[Q]\text{-Mod}}$. See [PS21, §2.1] for further details.

As explained in [PS21, Prop. 2.10], there is a continuous functor

$$\mathrm{Lift}\colon \mathrm{Cov}^{\mathrm{tw}}_{Q_{(\mathbf{k},\ell)}(\mathcal{S})} \longrightarrow \mathrm{Top}_{\bullet} \tag{1.11}$$

defined by sending a space X and homomorphism $\phi\colon \pi_1(X) \to Q_{(\mathbf{k},\ell)}(\mathcal{S})$ to the bundle of $\mathbb{Z}[\mathrm{Im}(\phi)]$-modules over X freely generated by the regular covering of X with deck transformation group $\mathrm{Im}(\phi)$ corresponding to the kernel of ϕ. We may then slightly modify this construction by changing the base ring of the bundle via the inclusion $\mathrm{Im}(\phi) \to Q_{(\mathbf{k},\ell)}(\mathcal{S})$ (formally, we are taking a fibrewise tensor product here). We may do the same construction with $Q^u_{(\mathbf{k},\ell)}(\mathcal{S})$ in place of $Q_{(\mathbf{k},\ell)}(\mathcal{S})$, restricting to the "untwisted" subcategories on both sides. We thus obtain functors

$$\mathrm{Cov}^{\mathrm{tw}}_{Q_{(\mathbf{k},\ell)}(\mathcal{S})} \longrightarrow \mathrm{Top}^{\mathrm{tw}}_{\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})]} \qquad \text{and} \qquad \mathrm{Cov}_{Q^u_{(\mathbf{k},\ell)}(\mathcal{S})} \longrightarrow \mathrm{Top}_{\mathbb{Z}[Q^u_{(\mathbf{k},\ell)}(\mathcal{S})]}, \tag{1.12}$$

that we denote by $\mathrm{Lift}(-) \otimes \mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})]$ and $\mathrm{Lift}(-) \otimes \mathbb{Z}[Q^u_{(\mathbf{k},\ell)}(\mathcal{S})]$.

Remark 1.5 The framework detailed above slightly differs from that of [PS21]. First of all, we introduce and use the colimit $Q_{(\mathbf{k},\ell)}(\mathcal{S})$ to define the key functors (1.8) and (1.10). This requires a slight adaptation of the categories of twisted and untwisted (proper) coverings $\mathrm{Cov}^{\mathrm{tw}}_Q$ and Cov_Q as well as the functor (1.11). Here, we consider a not-necessarily-surjective morphism $\phi\colon \pi_1(X) \to Q$, contrary to [PS21, Def. 2.2]. In particular, this avoids the necessity of checking the *Q-stability* property (see the paragraph after diagram (1.7)) for the functors (1.8) and (1.10) in order for their images to lie in a subcategory of the form $\mathrm{Cov}^{\mathrm{tw}}_Q \subset \mathrm{Cov}_\bullet$ or $\mathrm{Cov}_Q \subset \mathrm{Cov}_\bullet$. Also, in the present paper, we define the untwisted transformation group $Q^u_{(\mathbf{k},\ell)}(\mathcal{S})$ and directly define all the topological categories with proper maps in order to apply Borel-Moore homology functors.

Finally, we recall that twisted Borel-Moore homology in degree k is a functor

$$H^{\mathrm{BM}}_k\colon \mathrm{Top}^{\mathrm{tw}}_{\mathbb{Z}[Q]} \longrightarrow \mathbb{Z}[Q]\text{-}\mathrm{Mod}^{\mathrm{tw}}, \tag{1.13}$$

whose restriction to the subcategory $\mathrm{Top}_{\mathbb{Z}[Q]}$ has image in the untwisted subcategory $\mathbb{Z}[Q]$-Mod.

Remark 1.6 There are variants of the functor (1.13) and a fortiori of Construction 1.7 below for *ordinary homology* and for *reduced homology* (where we work with categories of pairs of spaces). However, we prefer to use Borel-Moore homology because we can calculate the representation modules and group actions when considering configuration spaces (see §§2.1–2.3). In contrast, most of the analogous computations using ordinary homology groups are beyond current knowledge, and the very few occurrences for which computations are done lead to representations that are much harder to handle (see for instance [Sta21, Th. 1.4]). Also, anticipating the results of §2, we only consider homology in degree k. This is because the Borel-Moore homology $H^{\mathrm{BM}}_*(\mathbf{B}_{\mathbf{k}}(\mathcal{S}_n); \mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S}_n)])$ is concentrated in that degree for all the situations that we consider; see §§2.2–2.3.

The construction. We now recall the general procedure of [PS21] for defining homological representation functors: it is summarised in Construction 1.7 below. We refer to [PS21, §2.5] for further details.

For the framework of Construction 1.7, we consider the following coupled pairs of assignments: F is either (1.8) or (1.10); Q is either $Q_{(\mathbf{k},\ell)}(\mathcal{S})$ or $Q^u_{(\mathbf{k},\ell)}(\mathcal{S})$; Cov^*_Q is either $\mathrm{Cov}^{\mathrm{tw}}_Q$ or Cov_Q; $\mathrm{Top}^*_{\mathbb{Z}[Q]}$ is either $\mathrm{Top}^{\mathrm{tw}}_{\mathbb{Z}[Q]}$ or $\mathrm{Top}_{\mathbb{Z}[Q]}$; $\mathbb{Z}[Q]\text{-}\mathrm{Mod}^\star$ is either $\mathbb{Z}[Q]\text{-}\mathrm{Mod}^{\mathrm{tw}}$ or $\mathbb{Z}[Q]$-Mod.

Construction 1.7 Let $\mathbf{k} = \{k_1; \ldots; k_r\} \vdash k$ be a partition of an integer $k \geqslant 1$ and let $\ell \geqslant 1$ be an integer. We then define $\mathfrak{L}_{(\mathbf{k},\ell)}(F)\colon \langle \mathcal{G}, \mathcal{M} \rangle \to \mathbb{Z}[Q]\text{-}\mathrm{Mod}^\star$ to be the functor induced on π_0 by the composition $H^{\mathrm{BM}}_k \circ ((\mathrm{Lift} \circ F) \otimes \mathbb{Z}[Q])$. This may be written as:

$$\begin{array}{ccccccc} \langle \mathcal{G}^{\mathrm{t}}, \mathcal{M}^{\mathrm{t}} \rangle & \xrightarrow{\quad F \quad} & \mathrm{Cov}^*_Q & \xrightarrow{\ \mathrm{Lift}(-) \otimes \mathbb{Z}[Q]\ } & \mathrm{Top}^*_{\mathbb{Z}[Q]} & \xrightarrow{H^{\mathrm{BM}}_k} & \mathbb{Z}[Q]\text{-}\mathrm{Mod}^\star. \\ \downarrow{\scriptstyle \pi_0} & & & & & \nearrow & \\ \langle \mathcal{G}, \mathcal{M} \rangle & & & \xrightarrow{\qquad \mathfrak{L}_{(\mathbf{k},\ell)}(F) \qquad} & & & \end{array} \tag{1.14}$$

In Construction 1.7, when $F = (1.10)$, then $Q = Q^u_{(\mathbf{k},\ell)}(\mathcal{S})$, $\mathrm{Cov}^*_Q = \mathrm{Cov}_Q$, $\mathrm{Top}^*_{\mathbb{Z}[Q]} = \mathrm{Top}_{\mathbb{Z}[Q]}$ and $\mathbb{Z}[Q]\text{-}\mathrm{Mod}^\star = \mathbb{Z}[Q]$-Mod. Therefore the encoded representations are *untwisted* and we distinguish the resulting homological representation functor by denoting it by $\mathfrak{L}^u_{(\mathbf{k},\ell)}(F)$. Moreover, it follows from the construction (cf. for instance [PS21, Lem. 5.24, Diag. (5.14)]) that the surjection (1.9) defines a natural transformation $\mathfrak{L}_{(\mathbf{k},\ell)}(F) \to \mathfrak{L}^u_{(\mathbf{k},\ell)}(F)$. In particular, the homological representations obtained by Construction 1.7 with the parameters $\ell \in \{1,2\}$ are always untwisted:

Lemma 1.8 *There are canonical identifications $\mathfrak{L}^u_{(\mathbf{k},1)}(F) = \mathfrak{L}_{(\mathbf{k},1)}(F)$ and $\mathfrak{L}^u_{(\mathbf{k},2)}(F) = \mathfrak{L}_{(\mathbf{k},2)}(F)$.*

Proof. The result for $\ell = 1$ is obvious since $Q_{(\mathbf{k},1)}(\mathcal{S}) = 0$. For $\ell = 2$, the G_n-action on $Q_{(\mathbf{k},2,n)}(\mathcal{S})$ is trivial for each n, since this is induced by conjugation in the abelian group $G_{\mathbf{k},n}/\Gamma_2$. Hence $Q^u_{(\mathbf{k},2)}(\mathcal{S}) = Q_{(\mathbf{k},2)}(\mathcal{S})$, and the result follows by construction. □

Remark 1.9 (*Alternative to [PS21].*) Instead of using the functorial machinery of [PS21], we may introduce all the functors of §1.3 that we study in the present paper as follows. For each fixed n, the quotient $\mathbf{B_k}(\mathcal{S}_n) \twoheadrightarrow Q_{(\mathbf{k},\ell,n)}(\mathcal{S})$ defined by (1.7) formally induces a representation of G_n on the twisted Borel-Moore homology group $H^{\mathrm{BM}}_k(C_{\mathbf{k}}(\mathcal{S}_n); \mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})])$ via covering space theory, working at the level of individual groups. This defines a functor $\mathfrak{L}_{(\mathbf{k},\ell)}(F)\colon \mathcal{G} \to \mathbb{Z}[Q]\text{-Mod}^{\star}$. We may then extend this functor to $\langle \mathcal{G}, \mathcal{M}\rangle$ following Lemma 1.1: namely, we just have to assign images to the morphisms of type $[1, \mathrm{id}_{n+1}]\colon n \to n+1$ and check the compatibility relation (1.2). The only appropriate candidate for these morphisms may then be guessed from the description of the module structure results in §2.

The vertical-type alternatives. Finally, we mention an important general modification that we may make in the parameters of Construction 1.7. We recall that we consider the configuration space $C_{\mathbf{k}}(\mathcal{S}_n)$ of k points in $\mathcal{S}_n$ which is obtained from a compact surface S possibly minus finitely many punctures in the interior. We may alternatively use the "dual" surface $\check{\mathcal{S}}_n$ obtained by blowing up each puncture to a new boundary component and removing the original boundary ∂S; see Definition 2.12. Then we consider the configuration space of points on the surface $\check{\mathcal{S}}_n$ and all the steps of Construction 1.7 repeat verbatim. This alteration has a deep impact on the module structures of the representations; see §2.3. Therefore, we single this variant out by denoting it by $\mathfrak{L}^v_{(\mathbf{k},\ell)}(F)$. Here "$v$" stands for "vertical" and we call this variant the *vertical-type alternative*: this terminology comes from the basis we obtain for the modules for surface braid group representations in §2.3. Also, when $\check{\mathcal{S}}_n$ is orientable and under some mild assumptions on the ground ring, the representations encoded by the functor $\mathfrak{L}^v_{(\mathbf{k},\ell)}(F)$ are the *dual representations* of those encoded by $\mathfrak{L}^v_{(\mathbf{k},\ell)}(F)$; see Corollary 2.16 and the isomorphisms (2.17).

1.2.2. Change of rings operation and transformation groups

For a category $\mathcal{C}$, a (non-zero) ring R and a (non-zero) ring homomorphism $f\colon \mathbb{Z}[Q] \to R$, the *change of rings operation* on a functor $F\colon \mathcal{C} \to \mathbb{Z}[Q]$-Mod consists in composing with the *induced module functor* $f_!\colon \mathbb{Z}[Q]\text{-Mod} \to R\text{-Mod}$, also known as the tensor product functor $R \otimes_f -$. A typical application of such an operation consists in making the local coefficient system $\mathbb{Z}[Q]$ of a homological representation functor *generic* in the sense of §2.4; see Trick 2.23.

Another key use of the change of rings operations is the following natural modification of the ground rings of homological representation functors with respect to partitions. We consider partitions $\mathbf{k} = \{k_1; \ldots; k_r\} \vdash k$ and $\mathbf{k}' = \{k'_1; \ldots; k'_r\} \vdash k'$ such that $1 \leqslant k' \leqslant k$, $k_l \geqslant 1$ and $0 \leqslant k'_l \leqslant k_l$ for all $1 \leqslant l \leqslant r$. For each $k'_i < k_i$, there is an evident analogue of the short exact sequence (1.5) with $G_{k'_i,n}$ as quotient and $G_{\{k_i-k'_i,k'_i\},n}$ as middle term. The section $s_{\{k_i-k'_i,k'_i\},n}$ of that short exact sequence provides an injection $G_{k'_i,n} \hookrightarrow G_{\{k_i-k'_i,k'_i\},n}$. Composing with the clear injection $G_{\{k_i-k'_i,k'_i\},n} \hookrightarrow G_{k_i,n}$ of the type of that of (1.4), we obtain an injection $G_{k'_i,n} \hookrightarrow G_{k_i,n}$. Applying this procedure for each block, we obtain a canonical injection $G_{\mathbf{k}',n} \hookrightarrow G_{\mathbf{k},n}$. Now, for some fixed $\ell \geqslant 1$, we consider the transformation groups $Q_{(\mathbf{k},\ell)}(\mathcal{S})$ and $Q_{(\mathbf{k}',\ell)}(\mathcal{S})$ associated to homological representation functors $\mathfrak{L}_{(\mathbf{k},\ell)}$ and $\mathfrak{L}_{(\mathbf{k}',\ell)}$ respectively.

Lemma 1.10 *There is a canonical group homomorphism*

$$Q_{(\mathbf{k}',\ell)}(\mathcal{S}) \longrightarrow Q_{(\mathbf{k},\ell)}(\mathcal{S}). \tag{1.15}$$

Proof. The injection $G_{\mathbf{k}',n} \hookrightarrow G_{\mathbf{k},n}$ induces the following commutative triangle:

$$\begin{array}{ccc} G_{\mathbf{k}',n}/\varGamma_\ell & \twoheadrightarrow & G_n/\varGamma_\ell. \\ \downarrow & \nearrow & \\ G_{\mathbf{k},n}/\varGamma_\ell. & & \end{array}$$

Taking kernels and the colimit as $n \to \infty$ (noting that this construction commutes with passing from n to $n+1$), we obtain (1.15). □

In many situations, we will apply a change of rings operation using (1.15) in order to identify some quotients of homological representation functors with some other ones; see Convention 3.7 and its applications in §4. In particular, we will use the following fact:

Observation 1.11 A change of rings operation $(1.15)_!\colon \mathbb{Z}[Q_{(\mathbf{k}',\ell)}(\mathcal{S})] \to \mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})]$ gives $\mathfrak{L}_{(\mathbf{k}',\ell)}$ the same ground ring as $\mathfrak{L}_{(\mathbf{k},\ell)}$, as well as the same actions of the groups G_n on $\mathbb{Z}[Q_{(\mathbf{k}',\ell)}]$.

1.3. Applications for surface braid groups and mapping class groups

We now review the application of Construction 1.7 to produce homological representation functors for classical braid groups (see §1.3.1), surface braid groups (see §1.3.2) and mapping class groups of surfaces (see §1.3.3). This is essentially a summary and straightforward generalisation of [PS21, §5.2, §5.4]. In the exposition, we separate the homological functors defined by Construction 1.7 with the parameter $\ell \leqslant 2$ from those for $\ell \geqslant 3$, because they always encode untwisted representations (see Lemma 1.8) and we can always compute their transformation groups. Most properties from [PS21; DPS22; PS22; PS23] (e.g. the transformation group computations or the relevant choices for ℓ with respect to the partition $\mathbf{k}$) are not necessary for our further work on the polynomiality properties in §4, which are thus *independent* of these previous works. We however explain them here for the sake of completeness.

We consider an integer $k \geqslant 1$ and a partition $\mathbf{k} = \{k_1; \ldots ; k_r\} \vdash k$ for all §1.3; we denote by r' the number of indices $i \leqslant r$ in $\mathbf{k}$ such that $k_i \geqslant 2$. When $\mathcal{M} = \mathcal{G}$, we abbreviate $\langle \mathcal{G}, \mathcal{G} \rangle = \mathfrak{U}\mathcal{G}$.

1.3.1. Classical braid groups

We apply Construction 1.7 with the setting $G_n := \mathbf{B}_n$, $\mathcal{S}_n := \mathbb{D}_n$ and $\mathcal{G} = \mathcal{M} = \boldsymbol{\beta}$. Taking quotients by the $\varGamma_1$ and $\varGamma_2$ terms, Construction 1.7 provides functors

$$\mathfrak{LB}_{(\mathbf{k},1)} \colon \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{Z}\text{-Mod} \qquad \text{and} \qquad \mathfrak{LB}_{(\mathbf{k},2)} \colon \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{Z}[Q_{(\mathbf{k},2)}(\mathbb{D})]\text{-Mod}. \tag{1.16}$$

[PS23, Prop. 3.2] ensures that these functors satisfy the Q-stability property, and we know that $Q_{(\mathbf{k},2)}(\mathbb{D}) \cong \mathbb{Z}^{r'} \times \mathbb{Z}^{r(r-1)/2} \times \mathbb{Z}^r$; see [PS23, Lem. 3.1].

Example 1.12 (*The Lawrence-Bigelow representations [Law90; Big04].*) Each functor $\mathfrak{LB}_{(\mathbf{k},2)}$ is called the $(\mathbf{k},2)$-*Lawrence-Bigelow functor*. This terminology comes from the fact that, when $\mathbf{k} = k$, the functor $\mathfrak{LB}_{(k,2)}$ encodes the k-th family of the Lawrence-Bigelow representations. These were originally introduced by Lawrence [Law90] as representations of Hecke algebras and then by Bigelow [Big04] via topological methods. The Burau representations originally introduced in [Bur35] are encoded by the functor $\mathfrak{LB}_{(1,2)}$, while the *Lawrence-Krammer-Bigelow* representations that Bigelow [Big01] and Krammer [Kra02] independently proved to be faithful are encoded by the functor $\mathfrak{LB}_{(2,2)}$; see [PS21, §5.2.1.1]. Also, each functor $\mathfrak{LB}_{(k,1)}$ corresponds to the trivial specialisation $\mathbb{Z}[Q_{(k,2)}(\mathbb{D})] \twoheadrightarrow \mathbb{Z}$ of the functor $\mathfrak{LB}_{(k,2)}$, and Lawrence [Law90, §3.4] proves that it encodes the representations factoring through $\mathbf{B}_n \twoheadrightarrow \mathfrak{S}_n$.

Taking quotients by the $\varGamma_\ell$ terms for each $\ell \geqslant 3$, Construction 1.7 provides functors

$$\mathfrak{LB}_{(\mathbf{k},\ell)} \colon \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]\text{-Mod}^{\mathrm{tw}} \qquad \text{and} \qquad \mathfrak{LB}^u_{(\mathbf{k},\ell)} \colon \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{Z}[Q^u_{(\mathbf{k},\ell)}(\mathbb{D})]\text{-Mod}, \tag{1.17}$$

which we call the twisted and untwisted $(\mathbf{k},\ell)$-*Lawrence-Bigelow functors*. Again, [PS23, Prop. 3.2] ensures that these functors satisfy the Q-stability property. Also, we deduce from [DPS22, Th. 3.6] that $\mathfrak{LB}_{(\mathbf{k},\ell)} = \mathfrak{LB}^u_{(\mathbf{k},\ell)} = \mathfrak{LB}_{(\mathbf{k},2)}$ if $k_i \geqslant 3$ for all $1 \leqslant i \leqslant r$ or $\mathbf{k}$ is either 1 or $\{1;1\}$. In contrast, it follows from [PS22, Table 2] that as soon as $\mathbf{k}$ is of the form $\{2;\mathbf{k}'\}$, $\{1;1;1;\mathbf{k}'\}$, $\{2;2;\mathbf{k}'\}$ or $\{1;2;\mathbf{k}'\}$, then $\mathfrak{LB}_{(\mathbf{k},\ell)} \neq \mathfrak{LB}_{(\mathbf{k},\ell+1)}$ for each $\ell \geqslant 1$. These fit together into a *pro-nilpotent tower* of representations as explained in [PS22, §4]. Furthermore, when $\mathbf{k} = \{2;\mathbf{k}'\}$ for $\mathbf{k}'$ such that each $k'_l \geqslant 3$, the transformation group $Q_{(\mathbf{k},\ell)}(\mathbb{D})$ is recalled in [PS23, Lem. 3.3]. We prove in [PS22, §5] that the representations are untwisted in this case, and a fortiori that $\mathfrak{LB}_{(\mathbf{k},\ell)} = \mathfrak{LB}^u_{(\mathbf{k},\ell)}$. We may also compute the explicit formulas of the $\mathbf{B}_n$-actions; see [PS22, Tab. 1 and Rem. 4.9].

Finally, considering $\mathcal{S}_n := \check{\mathbb{D}}_n$ rather than $\mathbb{D}_n$ defines the *vertical* Lawrence-Bigelow functors $\mathfrak{L}^v_{(\mathbf{k},\ell)} \colon \mathfrak{U}\boldsymbol{\beta} \to \mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]\text{-Mod}^{\mathrm{tw}}$ and $\mathfrak{L}^{u,v}_{(\mathbf{k},\ell)} \colon \mathfrak{U}\boldsymbol{\beta} \to \mathbb{Z}[Q^u_{(\mathbf{k},\ell)}(\mathbb{D})]\text{-Mod}$ for each $\ell \geqslant 1$. The properties mentioned in the previous paragraph for the functors (1.16) and (1.17) are exactly the same for these alternatives.

1.3.2. Surface braid groups

We fix two integers $g \geqslant 1$ and $h \geqslant 1$, and a surface S to be either $\Sigma_{g,1}$ or else $\mathcal{N}_{h,1}$ defined in §1.1.2.2. We apply Construction 1.7 with the setting $G_n := \mathbf{B}_n(S)$, $\mathcal{S}_n := \mathbb{D}_n \natural S$, $\mathcal{G} = \boldsymbol{\beta}$ and $\mathcal{M} = \boldsymbol{\beta}^S$. Taking quotients by the Γ_1 and Γ_2 terms, Construction 1.7 defines functors

$$\mathfrak{L}_{(\mathbf{k},1)}(\Sigma_{g,1}) \colon \langle \boldsymbol{\beta}, \boldsymbol{\beta}^{\Sigma_{g,1}} \rangle \to \mathbb{Z}\text{-Mod} \text{ and } \mathfrak{L}_{(\mathbf{k},2)}(\Sigma_{g,1}) \colon \langle \boldsymbol{\beta}, \boldsymbol{\beta}^{\Sigma_{g,1}} \rangle \to \mathbb{Z}[Q_{(\mathbf{k},2)}(\Sigma_{g,1})]\text{-Mod}; \quad (1.18)$$

$$\mathfrak{L}_{(\mathbf{k},1)}(\mathcal{N}_{h,1}) \colon \langle \boldsymbol{\beta}, \boldsymbol{\beta}^{\mathcal{N}_{h,1}} \rangle \to \mathbb{Z}\text{-Mod} \text{ and } \mathfrak{L}_{(\mathbf{k},2)}(\mathcal{N}_{h,1}) \colon \langle \boldsymbol{\beta}, \boldsymbol{\beta}^{\mathcal{N}_{h,1}} \rangle \to \mathbb{Z}[Q_{(\mathbf{k},2)}(\mathcal{N}_{h,1})]\text{-Mod}. \quad (1.19)$$

[PS23, Prop. 3.2] ensures that these functors satisfy the Q-stability property. In addition, we know from [PS23, Lem. 3.1] that $Q_{(\mathbf{k},2)}(\Sigma_{g,1}) \cong (\mathbb{Z}/2)^{r'} \times H_1(\Sigma_{g,1};\mathbb{Z})^{\times r}$ and that $Q_{(\mathbf{k},2)}(\mathcal{N}_{h,1}) \cong (\mathbb{Z}/2)^{r'} \times H_1(\mathcal{N}_{h,1};\mathbb{Z})^{\times r}$.

Taking quotients by the Γ_ℓ terms for each $\ell \geqslant 3$, Construction 1.7 provides homological representation functors, for $S \in \{\Sigma_{g,1}, \mathcal{N}_{h,1}\}$:

$$\mathfrak{L}_{(\mathbf{k},\ell)}(S) \colon \langle \boldsymbol{\beta}, \boldsymbol{\beta}^S \rangle \longrightarrow \mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]\text{-Mod}^{\mathrm{tw}} \quad \text{and} \quad \mathfrak{L}^u_{(\mathbf{k},\ell)}(S) \colon \langle \boldsymbol{\beta}, \boldsymbol{\beta}^S \rangle \longrightarrow \mathbb{Z}[Q^u_{(\mathbf{k},\ell)}(S)]\text{-Mod}. \quad (1.20)$$

Again, [PS23, Prop. 3.2] ensures that these functors satisfy the Q-stability property. Let us denote by $\mathfrak{L}_{(\mathbf{k},\ell)}$ any one of the functors (1.20). If $k_i \geqslant 3$ for all $1 \leqslant i \leqslant r$, we deduce from [DPS22, Th. 6.52 and Prop. 6.62] that $\mathfrak{L}_{(\mathbf{k},\ell)} = \mathfrak{L}^u_{(\mathbf{k},\ell)} = \mathfrak{L}_{(\mathbf{k},3)}$ for all $\ell \geqslant 4$. Moreover, we may compute explicitly the transformation groups $Q_{(\mathbf{k},3)}(\Sigma_{g,1})$, $Q^u_{(\mathbf{k},3)}(\Sigma_{g,1})$, $Q_{(\mathbf{k},3)}(\mathcal{N}_{h,1})$ and $Q^u_{(\mathbf{k},3)}(\mathcal{N}_{h,1}) = Q_{(\mathbf{k},3)}(\mathcal{N}_{h,1})/\mathbb{Z}^r$; see [PS23, Lem. 3.3]. A fortiori, the untwisted representation functors $\mathfrak{L}^u_{(\mathbf{k},\ell)}$ are really distinct from $\mathfrak{L}_{(\mathbf{k},\ell)}$ if $k_i \geqslant 3$ for all $1 \leqslant i \leqslant r$. In contrast, it follows from [PS22, Table 2] that if $\mathbf{k}$ is of the form $\{2;\mathbf{k}'\}$ or $\{1;\mathbf{k}'\}$ (assuming that $S \neq \mathbb{M}$ for the latter), then $\mathfrak{L}_{(\mathbf{k},\ell)} \neq \mathfrak{L}_{(\mathbf{k},\ell+1)}$ for each $\ell \geqslant 3$. In particular, these functors form a pro-nilpotent tower of representations as explained in [PS22, §5]. Our current knowledge is not sufficient to decide whether or not $\mathfrak{L}_{(\mathbf{k},\ell)} \neq \mathfrak{L}^u_{(\mathbf{k},\ell)}$ in this situation; see [PS23, Rem. 3.4].

Taking $\mathcal{S}_n$ to be the "dual" surface $(\mathbb{D}_n \natural S)^\vee$ rather than $\mathbb{D}_n \natural S$, the construction defines the vertical homological representation functors $\mathfrak{L}^{u,v}_{(\mathbf{k},\ell)}(\Sigma_{g,1})$, $\mathfrak{L}^v_{(\mathbf{k},\ell)}(\Sigma_{g,1})$, $\mathfrak{L}^{u,v}_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ and $\mathfrak{L}^v_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ for each $\ell \geqslant 1$. Their source and target categories are the same as their non-vertical counterparts, and the properties mentioned in the paragraph above for the functors (1.18)–(1.17) are exactly the same for these alternatives.

Example 1.13 (*The An-Ko representations [AK10].*) For orientable surfaces, the trivial partition $\mathbf{k} = k$ and $\ell = 3$, the $\mathbf{B}_n(\Sigma_{g,1})$-representation $\mathfrak{L}_{(k,3)}(\Sigma_{g,1})(n) \otimes_{Q_{(k,3)}(\Sigma_{g,1})} \mathbf{B}_{k,n}(\Sigma_{g,1})/\Gamma_3$ is isomorphic to the one introduced by An and Ko in [AK10, Th. 3.2]; see [PS21, §5.2.2.2.]. The group $Q_{(k,3)}(\Sigma_{g,1})$ is abstractly defined in [AK10] in terms of group presentations to satisfy certain technical homological constraints, while [BGG17, §4] gives all the connections to the third lower central series quotient. In contrast, the untwisted representations encoded by the functor $\mathfrak{L}^u_{(k,3)}(\Sigma_{g,1})$ are specific to [PS21, §5.2.2.2].

1.3.3. Mapping class groups of surfaces

We denote by $I' \subsetneq I$ the image of the open subinterval $[-1/2, 1/2]$ in the boundary ∂S. We apply Construction 1.7 with the setting $G_n := \boldsymbol{\Gamma}_{n,1}$ or $\boldsymbol{\mathcal{N}}_{n,1}$, $\mathcal{S}_n := S_n \smallsetminus I'$ where $S_n := \mathbb{T}^{\natural n}$ or $\mathbb{M}^{\natural n}$ and $\mathcal{G} = \mathcal{M} = \mathcal{M}_2^+$ or $\mathcal{M}_2^-$.

Remark 1.14 A more natural assignment for applying Construction 1.7 would be to take $\mathcal{S}_n := S_n$, i.e. not to remove the subinterval I'. We do however choose $S_n \smallsetminus I'$ instead because it is necessary for applying Lemma 2.1 in order to compute the underlying modules of the representations; see §2.3. Otherwise, the calculations of the representations using $\mathcal{S}_n := S_n$ are much more complicated; see for instance the work of Stavrou [Sta21, Th. 1.4], who computes the $\mathbf{\Gamma}_{n,1}$-representation equivalent to that obtained from Construction 1.7 with $\mathcal{S}_n := \mathbb{T}^{\natural n}$, $\ell = 1$, taking $\mathbb{Q}$ as ground ring and using classical homology (see Remark 1.6).

Taking quotients by the Γ_1 and Γ_2 terms, Construction 1.7 defines homological representation functors

$$\mathfrak{L}_{(\mathbf{k},1)}(\mathbf{\Gamma})\colon \mathfrak{U}\mathcal{M}_2^+ \to \mathbb{Z}\text{-Mod} \quad \text{and} \quad \mathfrak{L}_{(\mathbf{k},2)}(\mathbf{\Gamma})\colon \mathfrak{U}\mathcal{M}_2^+ \to \mathbb{Z}[Q_{(\mathbf{k},2)}(\mathbb{T})]\text{-Mod} \tag{1.21}$$

$$\mathfrak{L}_{(\mathbf{k},1)}(\boldsymbol{\mathcal{N}})\colon \mathfrak{U}\mathcal{M}_2^- \to \mathbb{Z}\text{-Mod} \quad \text{and} \quad \mathfrak{L}_{(\mathbf{k},2)}(\boldsymbol{\mathcal{N}})\colon \mathfrak{U}\mathcal{M}_2^- \to \mathbb{Z}[Q_{(\mathbf{k},2)}(\mathbb{M})]\text{-Mod} \tag{1.22}$$

where $Q_{(1,2)}(\mathbb{T}) = 0$, $Q_{(\mathbf{k},2)}(\mathbb{T}) \cong (\mathbb{Z}/2)^{r'}$ for $k \geqslant 2$ and $Q_{(\mathbf{k},2)}(\mathbb{M}) \cong (\mathbb{Z}/2)^{r'} \times (\mathbb{Z}/2)^r$; see [PS23, Cor. 3.6]. These homological representation functors also satisfy the Q-stability property.

Example 1.15 (*The Moriyama representations [Mor07].*) For orientable surfaces, the discrete partition $\mathbf{k} = \{1, ..., 1\}$ and $\ell = 1$, the functor $\mathfrak{L}_{(\{1,\ldots,1\},1)}(\mathbf{\Gamma})$ encodes the mapping class group representations introduced by Moriyama [Mor07]; see Remark 2.18. It is thus called the k-th *Moriyama functor*. In particular, the representations encoded by the functor $\mathfrak{L}_{(1,1)}(\mathbf{\Gamma})$ are equivalent to the well-known ones on $H_1(\Sigma_{g,1};\mathbb{Z})$, which factor through the symplectic groups $\mathrm{Sp}_{2g}(\mathbb{Z})$.

For orientable surfaces, it follows from [PS23, Cor. 3.6 and 3.7] that, for all $\ell \geqslant 3$, we have $Q_{(\mathbf{k},\ell)}(\mathbb{T}) = Q_{(\mathbf{k},2)}(\mathbb{T})$. A fortiori $\mathfrak{L}_{(\mathbf{k},\ell)} = \mathfrak{L}_{(\mathbf{k},2)}$ by construction, and thus it is not relevant to consider Construction 1.7 with parameters $\ell \geqslant 3$. On the other hand, for non-orientable surfaces, our current knowledge of the lower central series of the mapping class groups $\mathrm{MCG}(\mathbb{M}^{\natural h}, \mathbf{k})$ is not sufficient to decide whether or not $Q_{(\mathbf{k},\ell)}(\mathbb{M}) \neq Q_{(\mathbf{k},2)}(\mathbb{M})$ for $\ell \geqslant 3$; see [PS23, Rem. 3.8]. Therefore, it may be relevant to consider Construction 1.7 with quotients by the Γ_ℓ terms, defining homological representation functors for each $\ell \geqslant 3$:

$$\mathfrak{L}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})\colon \mathfrak{U}\mathcal{M}_2^- \to \mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{M})]\text{-Mod}^{\mathrm{tw}} \quad \text{and} \quad \mathfrak{L}^u_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})\colon \mathfrak{U}\mathcal{M}_2^- \to \mathbb{Z}[Q^u_{(\mathbf{k},\ell)}(\mathbb{M})]\text{-Mod}. \tag{1.23}$$

Due to our lack of detailed knowledge of $Q_{(\mathbf{k},\ell)}(\mathbb{M})$ and $Q^u_{(\mathbf{k},\ell)}(\mathbb{M})$, it is not clear whether $\mathfrak{L}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}}) \neq \mathfrak{L}^u_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})$ (see [PS23, Rem. 3.8]) and whether the Q-stability property is satisfied by these functors.

Finally, we may consider the "dual" surface $\mathcal{S}_n := (S_n \smallsetminus I')^{\check{}}$ instead of $S_n \smallsetminus I'$ (as before, S_n is either $\mathbb{T}^{\natural n}$ or $\mathbb{M}^{\natural n}$). In other words, instead of removing the interval I' from the (circle) boundary of S_n, we remove the complementary interval, i.e. the closure of $\partial S_n \smallsetminus I'$. However, in this case, we also change our convention on the braiding for the groupoid $\mathcal{M}_2$ by choosing its opposite:

Convention 1.16 In this setting, we apply Construction 1.7 with the groupoids $(\mathcal{M}_2^+)^\dagger$ and $(\mathcal{M}_2^-)^\dagger$ for $\mathcal{G} = \mathcal{M}$, instead of $\mathcal{M}_2^+$ and $\mathcal{M}_2^-$ respectively. This purely arbitrary choice is motivated by the polynomiality properties we aim at proving in Theorem 4.17. These rely on computations explained in §3.3.3.2 that would not be satisfied defining these functors over $\mathcal{M}_2^+$ and $\mathcal{M}_2^-$; see Remarks 3.17 and 4.18.

Therefore we define from Construction 1.7 the vertical-type alternative homological representation functors $\mathfrak{L}^v_{(\mathbf{k},\ell')}(\mathbf{\Gamma})\colon \mathfrak{U}(\mathcal{M}_2^+)^\dagger \to \mathbb{Z}[Q_{(\mathbf{k},\ell')}(\mathbb{T})]\text{-Mod}$ for each $\ell' \leqslant 2$, and $\mathfrak{L}^v_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})\colon \mathfrak{U}(\mathcal{M}_2^-)^\dagger \to \mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{M})]\text{-Mod}^{\mathrm{tw}}$ and its untwisted version $\mathfrak{L}^{u,v}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})$ for each $\ell \geqslant 1$. All of the properties mentioned above for the functors (1.21)–(1.23) are exactly the same for these alternatives.

2. Structure of the representations

The homological representations described in the previous sections are induced from actions on the twisted homology of configuration spaces on manifolds. Recall (see Remark 1.6) that we will always work with *Borel-Moore homology*, since in all of our examples it has the useful property that its underlying module is *free*, with an explicitly describable free generating set.

In §2.1 we prove a general criterion implying that the (possibly twisted) Borel-Moore homology of configuration spaces on a given underlying space is isomorphic to the Borel-Moore homology of configuration spaces on a subspace. Roughly, this works when the underlying space has a metric and the subspace is a "skeleton" onto which it deformation retracts in a controlled, non-expanding way; see Lemma 2.1 for the precise statement.

In §2.2 we describe several examples to which this applies, which are the underlying modules of representations of surface braid groups, mapping class groups, loop braid groups and related groups. In §2.3 we study several applications of Lemma 2.1 in more detail, describing explicit free generating sets for certain Borel-Moore homology modules. We also describe their "dual bases" with respect to certain non-degenerate pairings, which span the "vertical-type" alternative representations described in the last paragraph of §1.2.1. These dual bases, together with some diagrammatic reasoning, will be used in §3.3 to prove some key lemmas needed in our polynomiality arguments in §4.

The discussion so far applies only to Borel-Moore homology, but we recall in §2.4 a "genericity" criterion on local systems that implies that ordinary and Borel-Moore homology coincide. This criterion is due to Kohno [Koh17, Th. 3.1] and has been mildly generalised in [AP20, §5].

Formulas. In total, this gives us a detailed understanding of the underlying module structure of the surface braid group and mapping class group representations that we consider. One may then fruitfully attempt to derive explicit formulas for the group action in these models. We shall not pursue this here (beyond some qualitative diagrammatic statements that we prove in §3.3), since such explicit formulas are not needed to prove our polynomiality results. On the other hand, explicit formulas are derived in our forthcoming work [PS], where they are used to prove irreducibility results for surface braid group and mapping class group representations.

2.1. An isomorphism criterion for twisted Borel-Moore homology

In this section, we give a criterion for an inclusion of metric spaces to induce isomorphisms on the (possibly twisted) Borel-Moore homology of their associated configuration spaces. It abstracts the essential idea of [Big04, Lemma 3.1], where the underlying space is a surface of genus zero (see also [Mar22, Lemma 3.7] for a slight variation of this). Similar results for more general surfaces appear in [AK10, Lemma 3.3], [AP20, Theorem 6.6], [BPS21, Theorem A(a)] and (for thickenings of ribbon graphs) in [Bla23, Theorem 2]. The general criterion below (Lemma 2.1) recovers all of these examples, as well as many interesting examples in higher dimensions.

The advantage of this general criterion is that it applies very naturally to a wide variety of settings in all dimensions (see §2.2 for many examples), whereas the special cases covered in the literature quoted above only consider unordered configurations on surfaces. We also note that the criterion works equally well for partitioned configuration spaces; not just unordered ones.

Lemma 2.1 *Let M be a compact metric space with closed subspaces $A \subseteq B \subseteq M$, where M and B are locally compact. Suppose that there exists a strong deformation retraction h of M onto B, in other words a map $h\colon [0,1] \times M \to M$ satisfying the following two conditions:*

- *$h(t,x) = x$ whenever $t = 0$ or $x \in B$,*
- *$h(1,x) \in B$ for all $x \in M$,*

such that moreover the following two additional conditions hold:

- *$h(t,-)$ is non-expanding for all t, i.e. $d(x,y) \geqslant d(h(t,x),h(t,y))$ for all $x,y \in M$,*
- *$h(t,-)$ is a topological self-embedding of M for all $t < 1$.*

Then, for all $k \in \mathbb{N}$ and partitions $\mathbf{k} \vdash k$, the inclusion of configuration spaces

$$C_{\mathbf{k}}(B \smallsetminus A) \longhookrightarrow C_{\mathbf{k}}(M \smallsetminus A)$$

induces isomorphisms on Borel-Moore homology in all degrees and for all local coefficient systems that extend to $C_{\mathbf{k}}(M)$.

Remark 2.2 The condition that the local coefficient systems under consideration must extend to the larger space $C_{\mathbf{k}}(M)$ is automatically satisfied in all of the examples that we shall consider, since in these examples the inclusion $C_{\mathbf{k}}(M \smallsetminus A) \hookrightarrow C_{\mathbf{k}}(M)$ is a homotopy equivalence. Indeed, this holds whenever M is a manifold and $A \subseteq M$ is a subset of its boundary.

Notice also that the hypotheses on A are rather weak in Lemma 2.1: it is simply any closed subset of B. (For example, it may be empty, although the result is typically most useful when it is not.) The non-trivial hypothesis is the existence of a controlled deformation retraction of M onto B, and does not refer to A.

Typical examples. Typically, we will apply this in situations when M is a manifold that deformation retracts onto a "skeleton" $B \subset M$. For example, if $M = S$ is a compact, connected surface with non-empty boundary, we may take $B = \Gamma$ to be an embedded graph in S with its vertices in ∂M and A to be a subset of $\Gamma \cap \partial M$. This example, and others, are detailed in §2.2.

Calculational utility. The point of Lemma 2.1 is that, in many situations, the Borel-Moore homology of $C_{\mathbf{k}}(M \smallsetminus A)$ is interesting (it is the underlying module of a representation that we are studying), whereas the Borel-Moore homology of its subspace $C_{\mathbf{k}}(B \smallsetminus A)$ is easily computable. In the first example mentioned in the previous remark, $B \smallsetminus A$ is a disjoint union of open intervals, and the configuration space $C_{\mathbf{k}}(B \smallsetminus A)$ is therefore a disjoint union of open k-balls, whose Borel-Moore homology (either untwisted or twisted) is concentrated in degree k and free in that degree. In the second example, $B \smallsetminus A$ is a disjoint union of $n-1$ open annuli and one open 2-disc, so the Borel-Moore homology of the configuration spaces $C_{\mathbf{k}}(B \smallsetminus A)$ is a little more complicated, but still well-understood.

Proof of Lemma 2.1. We follow the outline of [AP20, Th. 6.6], which in turn is inspired by the idea of [Big04, Lem. 3.1]. For $t \in [0,1]$, write $h_t = h(t,-)\colon M \to M$ and recall that $h_0 = \mathrm{id}$ and $h_1(M) = B$. For $\epsilon > 0$, define

$$C_\epsilon := \bigl\{[c_1,\ldots,c_k] \in C_{\mathbf{k}}(M) \mid d(c_i,c_j) < \epsilon \text{ for some } i \neq j \text{ or } d(c_i,a) < \epsilon \text{ for some } a \in A\bigr\}.$$

For each $t \in [0,1]$, every compact subspace of $C_{\mathbf{k}}(h_t(M) \smallsetminus A)$ is disjoint from C_ϵ for some $\epsilon > 0$, so we may write its Borel-Moore homology as the inverse limit

$$H_*^{\mathrm{BM}}\bigl(C_{\mathbf{k}}(h_t(M) \smallsetminus A);\mathcal{L}\bigr) \cong \lim_{\epsilon \to 0} H_*\bigl(C_{\mathbf{k}}(h_t(M) \smallsetminus A), C_{\mathbf{k}}(h_t(M) \smallsetminus A) \cap C_\epsilon;\mathcal{L}\bigr)$$

for any local system $\mathcal{L}$. Thus it suffices to show that the inclusion of pairs

$$(C_{\mathbf{k}}(B \smallsetminus A), C_{\mathbf{k}}(B \smallsetminus A) \cap C_\epsilon) \hookrightarrow (C_{\mathbf{k}}(M \smallsetminus A), C_{\mathbf{k}}(M \smallsetminus A) \cap C_\epsilon) \tag{2.1}$$

induces isomorphisms on twisted homology in all degrees for all local systems extending to $C_{\mathbf{k}}(M)$, for all $\epsilon > 0$. This fits into a diagram of inclusions of pairs of spaces

$$\begin{array}{ccc} (C_{\mathbf{k}}(B \smallsetminus A), C_{\mathbf{k}}(B \smallsetminus A) \cap C_\epsilon) & \hookrightarrow & (C_{\mathbf{k}}(M \smallsetminus A), C_{\mathbf{k}}(M \smallsetminus A) \cap C_\epsilon) \\ \downarrow & & \downarrow \\ (C_{\mathbf{k}}(B), C_{\mathbf{k}}(B) \cap C_\epsilon) & \hookrightarrow & (C_{\mathbf{k}}(M), C_{\mathbf{k}}(M) \cap C_\epsilon). \end{array} \tag{2.2}$$

The vertical inclusions in (2.2) induce isomorphisms on twisted homology in all degrees by excision. Hence, abbreviating $C^t := C_{\mathbf{k}}(h_t(M))$ and $C := C^0$, it suffices to show that the inclusion of pairs $(C^1, C^1 \cap C_\epsilon) \hookrightarrow (C, C_\epsilon)$ induces isomorphisms on twisted homology in all degrees,for all $\epsilon > 0$.

Let us fix $\epsilon > 0$. The hypothesis that $h_t\colon M \to M$ is a topological self-embedding for $t < 1$ implies that it induces well-defined maps of configuration spaces that define a strong deformation retraction of C onto C^t for any $t < 1$. Moreover, the hypothesis that h_t is non-expanding means that these maps of configuration spaces preserve the subspace C_ϵ, so we in fact have a strong deformation retraction of the pair (C, C_ϵ) onto the pair $(C^t, C^t \cap C_\epsilon)$ for any $t < 1$. On the other hand, we cannot conclude the same statement for $t = 1$, since $h_1\colon M \to M$ is not assumed to be an embedding (and in our key examples it will not be). In order to continue the deformation retraction of configuration spaces, we first pass to a subspace: for any $t < 1$ we define

$$\check{C}^t := \bigl\{[c_1,\ldots,c_k] \in C^t \mid h_1(h_t^{-1}(c_i)) \neq h_1(h_t^{-1}(c_j)) \text{ for each } i \neq j\bigr\}.$$

This additional condition precisely ensures that points do not collide if we continue applying the deformation retraction h_t to configurations until time $t = 1$. Thus there is a strong deformation retraction of the pair $(\check{C}^t, \check{C}^t \cap C_\epsilon)$ onto the pair $(C^1, C^1 \cap C_\epsilon)$ for any $t < 1$. It therefore remains to show that there exists some $t < 1$ (depending on ϵ) such that the inclusion

$$(\check{C}^t, \check{C}^t \cap C_\epsilon) \longhookrightarrow (C^t, C^t \cap C_\epsilon)$$

induces isomorphisms on twisted homology in all degrees. By excision, it suffices to show that $\check{C}^t$ and $C^t \cap C_\epsilon$ form an open covering of C^t. It is clear that these are both open subspaces, so we just have to show that there exists some $t < 1$ such that $\check{C}^t \cup (C^t \cap C_\epsilon) = C^t$, or equivalently such that $C^t \smallsetminus \check{C}^t \subseteq C_\epsilon$.

By continuity of h and compactness of M, there exists $\delta < 1$ such that $d(h_\delta(x), h_1(x)) < \epsilon/2$ for all $x \in M$. By above, it suffices to show that $C^\delta \smallsetminus \check{C}^\delta \subseteq C_\epsilon$. Let $c = [c_1, \ldots, c_k]$ be a configuration in $C^\delta \smallsetminus \check{C}^\delta$, in other words we have $c_i = h_\delta(x_i)$ for some configuration $[x_1, \ldots, x_k]$ in $C = C_{\mathbf{k}}(M)$ and $h_1(x_i) = h_1(x_j)$ for some $i \neq j$. The distance from c_i to c_j is therefore at most the sum of the distances from $c_i = h_\delta(x_i)$ to $h_1(x_i)$ and from $h_1(x_i) = h_1(x_j)$ to $h_\delta(x_j) = c_j$. These latter distances are both less than $\epsilon/2$ by our choice of δ, so we have $d(c_i, c_j) < \epsilon$ and hence $c \in C_\epsilon$. Thus we complete the excision argument in the previous paragraph with $t = \delta$.

In summary, we have proved Lemma 2.1 by showing that, in the diagram

$$\begin{array}{ccccccc}
 & (C_{\mathbf{k}}(B \smallsetminus A), C_{\mathbf{k}}(B \smallsetminus A) \cap C_\epsilon) & & \longhookrightarrow & & (C_{\mathbf{k}}(M \smallsetminus A), C_{\mathbf{k}}(M \smallsetminus A) \cap C_\epsilon) & \\
 \swarrow^{(*)} & & & & & & \searrow^{(*)} \\
(C^1, C^1 \cap C_\epsilon) & \xhookrightarrow{(\ddagger)} & (\check{C}^t, \check{C}^t \cap C_\epsilon) & \xhookrightarrow{(**)} & (C^t, C^t \cap C_\epsilon) & \xhookrightarrow{(\ddagger)} & (C, C_\epsilon),
\end{array}$$

the arrows $(*)$ induce isomorphisms on twisted homology in all degrees (by excision), the arrows $(\ddagger)$ are homotopy equivalences and for each $\epsilon > 0$ there exists $t \in (0, 1)$ such that the arrow $(**)$ induces isomorphisms on twisted homology in all degrees (again by excision). ☐

2.2. Examples

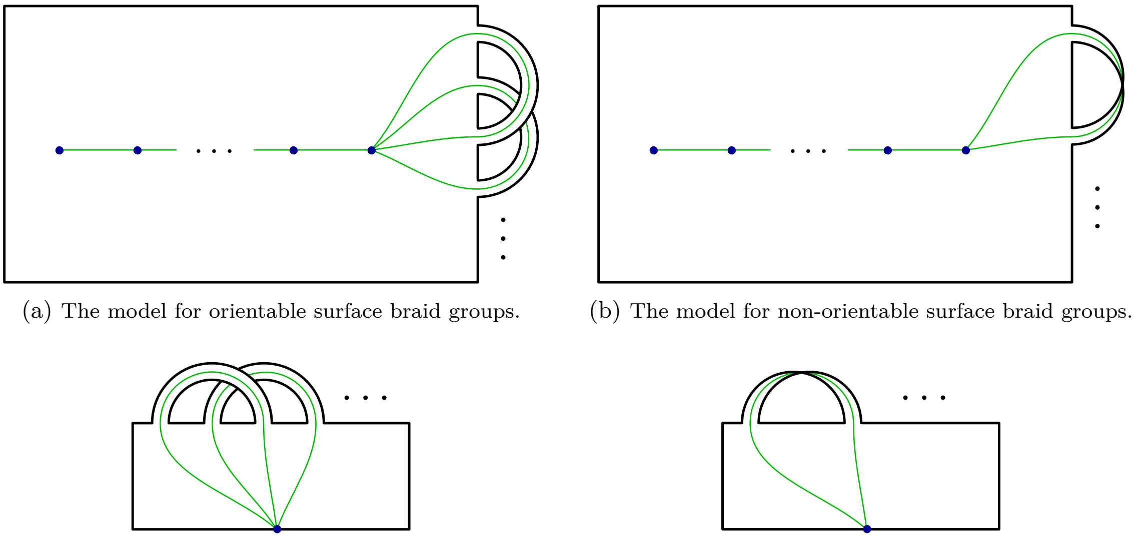
We describe various examples of $A \subseteq B \subseteq M$ and the inclusion of configuration spaces

$$C_{\mathbf{k}}(B \smallsetminus A) \longhookrightarrow C_{\mathbf{k}}(M \smallsetminus A). \tag{2.3}$$

Example 2.3 (*Configurations on punctured discs.*) First, let $M = \Sigma_{0,n+1}$ be the connected, compact surface of genus 0 with $n+1$ boundary components; this may be viewed as a closed 2-disc with n holes. Let A be the union of the n inner boundary components of M and let B be the union of A with $n-1$ arcs connecting consecutive components of A. Taking a little care about the metric, it is easy to see that there is a strong deformation retraction of M onto B satisfying the hypotheses of Lemma 2.1. Also, since A is part of the boundary of M, all local coefficient systems on $C_{\mathbf{k}}(M \smallsetminus A)$ extend to $C_{\mathbf{k}}(M)$. Thus Lemma 2.1 tells us that (2.3) induces isomorphisms on twisted Borel-Moore homology in all degrees. This special case recovers [Big04, Lem. 3.1]. In this case, $M \smallsetminus A$ is the closed 2-disc minus n punctures in its interior and $B \smallsetminus A$ is a disjoint union of $n-1$ open arcs.

One could instead take A to be the union of the n inner boundary components together with a point p on the outer boundary component. We may then take B to be the union of A with n arcs, each connecting a different component of A to the point p. There is again a strong deformation retraction of M onto B satisfying the hypotheses of Lemma 2.1, so in this case we also deduce that (2.3) induces isomorphisms on twisted Borel-Moore homology in all degrees. This special case recovers [Mar22, Lem. 3.7]. In this case, $M \smallsetminus A$ is the closed 2-disc minus n punctures in its interior and one puncture on its boundary and $B \smallsetminus A$ is a disjoint union of n open arcs.

Example 2.4 (*Configurations on non-closed surfaces.*) As a slight generalisation, one may take $M = S$ to be any compact surface with non-empty boundary and $B = \Gamma \subseteq S$ to be an embedded finite graph onto which it deformation retracts. With a little care about the metric, one may ensure that there is a controlled deformation retraction of S onto Γ satisfying the hypotheses of Lemma 2.1. If we then take A to be any closed subset of Γ, we conclude that the inclusion (2.3) induces isomorphisms on twisted Borel-Moore homology in all degrees (for local systems on $C_{\mathbf{k}}(S \smallsetminus A)$ that extend to $C_{\mathbf{k}}(S)$; this is automatic if $A \subseteq \Gamma \cap \partial S$). In the case $S = \Sigma_{g,1}$, this recovers [AK10, Lem. 3.3], [AP20, Th. 6.6] and [BPS21, Th. A(a)].



(a) The model for orientable surface braid groups.
(b) The model for non-orientable surface braid groups.
(c) The model for orientable mapping class groups.
(d) The model for non-orientable mapping class groups.

Figure 2.1 Four examples of the setting of Lemma 2.1 where S is a compact, connected surface with one boundary component, Γ is the embedded graph drawn in blue/green and A is its set of vertices.

Example 2.5 (*Configurations in higher-dimensional analogues of surfaces.*) Generalising the example of connected, orientable surfaces, one may take M to be the manifold $W_{g,1} = (S^n \times S^n)^{\sharp g} \smallsetminus \mathring{D}^{2n}$, where $\sharp$ denotes the connected sum. It is homotopy equivalent to a wedge sum of $2g$ copies of S^n, and with a little care about the metric one may find a controlled deformation retraction as in Lemma 2.1 onto a subspace $B \subseteq W_{g,1}$ that is homeomorphic to $\vee^{2g} S^n$, with the basepoint of the wedge sum corresponding to a point p in the boundary of $W_{g,1}$. Take $A = \{p\}$. Lemma 2.1 then implies that (2.3) induces isomorphisms on twisted Borel-Moore homology in all degrees. Thus we have isomorphisms of (twisted) Borel-Moore homology:

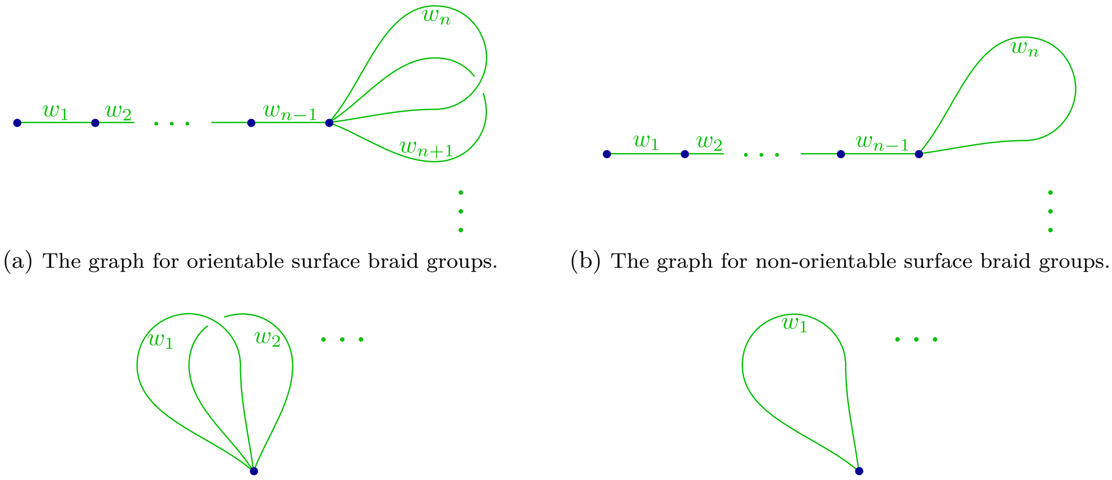
$$H_*^{\mathrm{BM}}\big(C_{\mathbf{k}}(W_{g,1} \smallsetminus \{p\}); \mathcal{L}\big) \;\cong\; H_*^{\mathrm{BM}}\big(C_{\mathbf{k}}(\textstyle\coprod^{2g} \mathbb{R}^n); \mathcal{L}\big).$$

Example 2.6 (*Configurations in the complement of a link.*) For an example of a different flavour, we may consider configurations in the complement of a link L in $\mathbb{D}^3$. More precisely, let U be an open tubular neighbourhood of L, set $M = \mathbb{D}^3 \smallsetminus U$ and let A be the union of the n torus boundary components of M. How to describe $B \supset A$, the subspace of $M = \mathbb{D}^3 \smallsetminus U$ onto which it deformation retracts in a controlled way as in Lemma 2.1, depends on properties of the link L. If L is an unlink, we may assume that it consists of n concentric circles contained in an embedded plane in $\mathbb{D}^3$. We may then take B to be the union of A with $n-1$ annuli (connecting consecutive components of A) and one 2-disc (filling the component of A corresponding to the innermost circle). If L is not an unlink but each of its components L_i is an unknot, we may look for transversely-intersecting Seifert discs D_i cobounding the L_i whose union is contractible (this is possible for example for iterated Hopf links, "necklace links" (as studied in [BB16]) or the Borromean rings); in this case we take $B = A \cup \bigcup_i (D_i \smallsetminus (U \cap D_i))$. The twisted Borel-Moore homology of some of these examples is studied in more detail in forthcoming work.

2.3. Free bases and dual bases

The key setting for the rest of the paper will be Example 2.4, which we now consider in more detail. Specifically, let S be a connected, compact surface with one boundary component, let Γ be the embedded graph pictured in Figure 2.1 and let A be the set of vertices of Γ.

To apply Lemma 2.1, it will be convenient to modify these spaces a little in cases (a) and (b), where the vertices lie in the interior of S. In these cases, let $\overline{S}$ be the result of blowing up each vertex of Γ to a boundary component (so that the total number of boundary components of $\overline{S}$ is $|A|+1$), let $\overline{\Gamma}$ be the result of replacing each vertex v of Γ with a circle (coinciding with the corresponding new boundary component of $\overline{S}$) subdivided into $\nu(v)$ vertices and $\nu(v)$ edges,



(a) The graph for orientable surface braid groups.

(b) The graph for non-orientable surface braid groups.

(c) The graph for orientable mapping class groups.

(d) The graph for non-orientable mapping class groups.

Figure 2.2 The graphs from Figure 2.1, considered as abstract graphs and equipped with labels, viewed as generators of the Borel-Moore homology of $C_k(\Gamma \smallsetminus A)$, equivalently (by Lemma 2.1), the Borel-Moore homology of $C_k(S \smallsetminus A)$. In each case, a cycle representing the homology class under consideration is given by the subspace of configurations where every configuration point lies on the interior of a green edge and as we move along a given edge (according to an orientation fixed in advance) the ordered list of blocks of the partition $\mathbf{k}$ that the configuration points that we meet belong to agrees with the list (word on the alphabet of blocks of $\mathbf{k}$) that decorates that edge in the diagram.

where $\nu(v)$ is the valence of v, and finally let $\overline{A} \subset \overline{\Gamma}$ be the union of these circles (equivalently, the new boundary components of $\overline{S}$). We clearly have homeomorphisms $C_{\mathbf{k}}(\overline{S} \smallsetminus \overline{A}) \cong C_{\mathbf{k}}(S \smallsetminus A)$ and $C_{\mathbf{k}}(\overline{\Gamma} \smallsetminus \overline{A}) \cong C_{\mathbf{k}}(\Gamma \smallsetminus A)$. In cases (c) and (d) we simply take $\overline{S} = S$, $\overline{\Gamma} = \Gamma$ and $\overline{A} = A$.

By Lemma 2.1, the inclusion $C_{\mathbf{k}}(\overline{\Gamma} \smallsetminus \overline{A}) \hookrightarrow C_{\mathbf{k}}(\overline{S} \smallsetminus \overline{A})$ induces isomorphisms on Borel-Moore homology for all local coefficient systems on $C_{\mathbf{k}}(\overline{S} \smallsetminus \overline{A})$ that extend to $C_{\mathbf{k}}(\overline{S})$. Note that $\overline{A} \subset \partial \overline{S}$ (the purpose of replacing S, Γ, A with $\overline{S}, \overline{\Gamma}, \overline{A}$ was precisely to ensure this), so the inclusion $\overline{S} \smallsetminus \overline{A} \hookrightarrow \overline{S}$ is an isotopy equivalence and thus $C_{\mathbf{k}}(\overline{S} \smallsetminus \overline{A}) \hookrightarrow C_{\mathbf{k}}(\overline{S})$ is a homotopy equivalence. This means in particular that all local coefficient systems on $C_{\mathbf{k}}(\overline{S} \smallsetminus \overline{A})$ extend to $C_{\mathbf{k}}(\overline{S})$, so the result of Lemma 2.1 is that the inclusion $C_{\mathbf{k}}(\overline{\Gamma} \smallsetminus \overline{A}) \hookrightarrow C_{\mathbf{k}}(\overline{S} \smallsetminus \overline{A})$ induces isomorphisms on Borel-Moore homology with *all* local coefficient systems.

The twisted Borel-Moore homology of $C_{\mathbf{k}}(S \smallsetminus A)$ may therefore be computed from the twisted Borel-Moore homology of $C_{\mathbf{k}}(\Gamma \smallsetminus A)$, where we may now consider Γ as an abstract graph (forgetting its embedding into S) with vertex set A, as depicted in Figure 2.2. Since $\Gamma \smallsetminus A$ is simply the disjoint union of the (open) edges of the graph Γ, its configuration space $C_{\mathbf{k}}(\Gamma \smallsetminus A)$ is a disjoint union of open k-dimensional simplices, one for each choice of:

- the number of points that lie on each edge of Γ;
- for each edge of Γ, an ordered list of blocks of the partition $\mathbf{k}$, prescribing which blocks of the partition the configuration points that lie on this edge must belong to, as we pass from left to right along the edge (with respect to an arbitrary orientation of the edge, chosen once and for all).

This combinatorial information may be summarised succinctly as a choice, for each edge e of Γ, of a word w_e on the alphabet of blocks of the partition $\mathbf{k}$. This choice must have the property that the total number of times that a block of $\mathbf{k}$ appears in w_e, as e runs over all edges of Γ, is equal to the size of the block. See Figure 2.2.

We summarise this discussion in the following corollary.

Corollary 2.7 *Let S be a connected, compact surface with one boundary component, let A be either a finite subset of its interior or a single point on its boundary and let $\mathbf{k}$ be a partition of a positive integer k. Choose any local system $\mathcal{L}$ on $C_{\mathbf{k}}(S \smallsetminus A)$ defined over a ring R. As an R-module, the twisted Borel-Moore homology*

$$H_k^{\mathrm{BM}}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L}) \tag{2.4}$$

decomposes as a direct sum of copies of the fibre of $\mathcal{L}$, indexed by functions

$$w\colon \{\textit{edges of } \Gamma\} \longrightarrow \{\textit{blocks of } \mathbf{k}\}^*, \tag{2.5}$$

where Γ is the embedded graph in S depicted in Figure 2.1, the notation X^* means the monoid of words on a set X and the total number of times that a block of $\mathbf{k}$ appears in the word w_e, as e runs over all edges of Γ, is equal to the size of the block.

Notation 2.8 It will be convenient later to fix some standard notation for the different parts of the graphs Γ appearing in Corollary 2.7 and depicted in Figure 2.2. In cases (a) and (b), assuming that there are n punctures, i.e. $|A| = n$, let us write $\mathbb{I}_n$ for the linear (or "tail") part of the graph, which is a linear graph with n vertices and $n-1$ edges. When the surface S is orientable (cases (a) and (c)), we write $\mathbb{W}_g^{\Sigma}$ for the "wedge" part of the graph in Figure 2.2a, which is a graph with one vertex and $2g$ edges, where g is the genus of S. When the surface S is non-orientable (cases (b) and (d)), we write instead $\mathbb{W}_h^{\mathcal{N}}$ for the "wedge" part of the graph in Figure 2.2b, which is a graph with one vertex and h edges, where h is the non-orientable genus of S. The elements (2.5) of the set indexing the decomposition of (2.4) will typically be denoted by

$$(w_1, \ldots, w_{n-1}, [w_n, w_{n+1}], \ldots, [w_{n+2g-2}, w_{n+2g-1}]) \tag{2.6}$$

when $S = \Sigma_{g,1}$ and by

$$(w_1, \ldots, w_{n-1}, [w_n], \ldots, [w_{n+h-1}]) \tag{2.7}$$

when $S = \mathcal{N}_{h,1}$. The first $n-1$ terms are the values of w on $\mathbb{I}_n$ and the remaining $2g$ respectively h terms in square brackets are the values of w on $\mathbb{W}_g^{\Sigma}$ respectively $\mathbb{W}_h^{\mathcal{N}}$.

Dual bases. We now describe, using Poincaré-Lefschetz duality, a perfect pairing between (2.4) and another naturally-defined homology R-module, for which we describe a "dual" basis. In order to apply Poincaré-Lefschetz duality, **we assume for the remainder of §2.3 that the surface S is orientable** By tensoring appropriately with the orientation local system, one could generalise this discussion to allow also non-orientable surfaces; this is explained briefly in Remark 2.11 below.

Let us now consider the relative homology group $H_k(C_{\mathbf{k}}(S \smallsetminus A), \partial; \mathcal{L})$, where ∂ is an abbreviation of $\partial C_{\mathbf{k}}(S \smallsetminus A)$, the boundary of the topological manifold $C_{\mathbf{k}}(S \smallsetminus A)$, which consists of all configurations that non-trivially intersect the boundary of $S \smallsetminus A$.

In case (c), we implicitly make a small modification here: we replace A, which is a single point in ∂S, with a small closed interval in ∂S; we also replace $S \smallsetminus A$ with the closure in S of the complement of this small closed interval. This is analogous to the modification that we made earlier in this section in case (a): now, in case (c), we are essentially *blowing up* the (unique) vertex of the graph Γ on ∂S.

From now on, we assume that $\mathcal{L}$ is a *rank-one* local system; i.e. its fibre over each point is a free module of rank one over the ground ring R. Under this assumption, Corollary 2.7 describes a *free basis* for $H_k^{\mathrm{BM}}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L})$ over R.

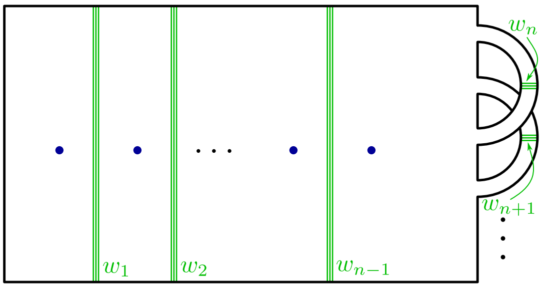
There is a naturally corresponding set of elements of $H_k(C_{\mathbf{k}}(S \smallsetminus A), \partial; \mathcal{L})$, depicted in Figure 2.3 and indexed by the same combinatorial data as described in Corollary 2.7. As explained in [AP20, Thm. A], Poincaré-Lefschetz duality and the relative cap product induce a pairing

$$H_k^{\mathrm{BM}}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L}) \otimes_R H_k(C_{\mathbf{k}}(S \smallsetminus A), \partial; \mathcal{L}) \longrightarrow R \tag{2.8}$$

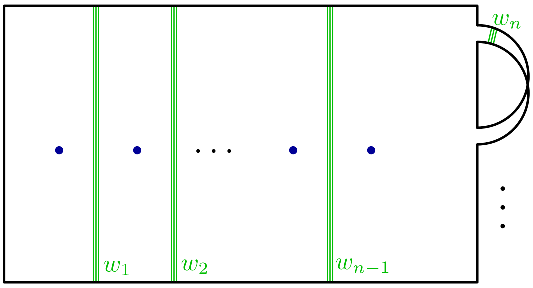
whose evaluation on a basis element of $H_k^{\mathrm{BM}}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L})$ together with one of the elements depicted in Figure 2.3 is equal to 1 if the two elements are indexed by the same function w and equal to 0 otherwise. It follows that the submodule spanned by the elements in Figure 2.3 is *freely* spanned by them, and the pairing (2.8) restricts to a *perfect pairing* when we restrict to this submodule on the right-hand factor.

Notation 2.9 We write $H_k^{\partial}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L})$ for the R-submodule of $H_k(C_{\mathbf{k}}(S \smallsetminus A), \partial; \mathcal{L})$ (freely) spanned by the elements depicted in Figure 2.3. As another piece of general notation, for module W over a ring R, we denote by $W^{\vee}$ the dual module $\mathrm{Hom}_R(W, R)$.

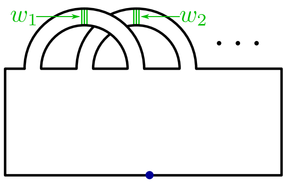
With this notation, the discussion above may be summarised as follows.



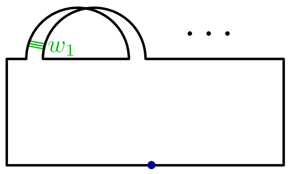
(a) Dual basis for orientable surface braid groups.



(b) Dual basis for non-orientable surface braid groups.



(c) Dual basis for orientable mapping class groups.



(d) Dual basis for non-orientable mapping class groups.

Figure 2.3 A linearly independent collection of elements of $H_k(C_{\mathbf{k}}(S \smallsetminus A), \partial; \mathcal{L})$ whose span is isomorphic to the dual of $H_k^{\mathrm{BM}}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L})$ via the perfect pairing given by the restriction of (2.8). In each of the four cases, the figure consists of a collection of arcs beginning and ending on the boundary of the surface, each labelled by a word w_i on the alphabet of blocks of the partition $\mathbf{k} = \{k_1; \ldots; k_r\} \vdash k$. Each arc is in fact to be thought of as $|w_i|$ parallel arcs, where $|w_i|$ is the length of the word w_i; these arcs are then labelled by letters of w_i (which are blocks of the partition $\mathbf{k}$). The homology class depicted by the figure is the one represented by the cycle given by the subspace of configurations where exactly one point lies on each arc and this point belongs to the block of the partition specified by the label of that arc.
The same diagrams, interpreted differently, also describe a basis for the module $H_k^{\mathrm{BM}}(C_{\mathbf{k}}(\check{S}); \mathcal{L})$ (see Definition 2.12). More precisely, if we now interpret each arc labelled by w_i as a single arc (instead of a collection of parallel arcs) specifying that there must be precisely $|w_i|$ points lying on it and belonging to the blocks of the partition $\mathbf{k}$ given by the letters of w_i (in other words, interpreted exactly as in Figure 2.2), then the corresponding homology classes form a free basis of $H_k^{\mathrm{BM}}(C_{\mathbf{k}}(\check{S}); \mathcal{L})$.

Corollary 2.10 *Let S be a connected, compact surface with one boundary component, let A be either a finite subset of its interior or a closed interval in its boundary and let $\mathbf{k}$ be a partition of a positive integer k. Choose any rank-one local system $\mathcal{L}$ on $C_{\mathbf{k}}(S \smallsetminus A)$ defined over a ring R. Then the R-module $H_k^{\partial}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L})$ is freely generated over R by the same combinatorial data as described in Corollary 2.7. Moreover, there is a perfect pairing*

$$H_k^{\mathrm{BM}}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L}) \otimes_R H_k^{\partial}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L}) \longrightarrow R \tag{2.9}$$

given by Poincaré-Lefschetz duality and the relative cap product, whose matrix with respect to the two bases that we have described is the identity matrix. In particular, we therefore have

$$H_k^{\partial}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L}) \;\cong\; \left(H_k^{\mathrm{BM}}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L})\right)^{\vee}. \tag{2.10}$$

Remark 2.11 For non-orientable surfaces S, an analogue of Corollary 2.10 also holds, the only difference being that, on the left-hand side of (2.10), we must tensor $\mathcal{L}$ with the orientation local system of the non-orientable manifold $C_{\mathbf{k}}(S \smallsetminus A)$.

Embeddings. We now recall from [AP20] certain embeddings of mapping class group representations, which become isomorphisms after localising the coefficients appropriately.

Definition 2.12 Consider a finite-type surface $S \smallsetminus \mathcal{P}$, namely a compact surface S minus a finite subset $\mathcal{P} \subset S$. Its *blow-up* $\overline{S}$ is then obtained from S by blowing up each $p \in \mathcal{P}$ to a new boundary component (if $p \in S \smallsetminus \partial S$) or an interval (if $p \in \partial S$). Furthermore, its *dual surface* $\check{S}$ is obtained by removing from $\overline{S}$ the original boundary $\partial(S \smallsetminus \mathcal{P})$. Note that $(\overline{S}; S \smallsetminus \mathcal{P}, \check{S})$ is a manifold triad.

For example, in cases (a) and (b) under consideration, $\overline{S}$ is obtained from $S \smallsetminus A$ by blowing up each (interior) puncture in A to a new boundary component and $\check{S}$ is given by removing the original boundary component ∂S from $\overline{S}$ but keeping the $|A|$ new boundary components. In cases (c) and (d), $\overline{S}$ simply replaces the single boundary puncture A in ∂S with a closed interval and $\check{S}$ is the union of the interior of S with the complement of this closed interval in the boundary of $\overline{S}$.

The twisted Borel-Moore homology module $H_k^{BM}(C_{\mathbf{k}}(\check{S}); \mathcal{L})$ has an explicit description similar to the description of $H_k^{BM}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L})$ in Corollary 2.7. This is another direct application of Lemma 2.1, this time with $M = \overline{S}$, $A = \partial S$ (so that $M \smallsetminus A = \check{S}$) and B equal to the union of A with the green arcs drawn in Figure 2.3. We record this here:

Lemma 2.13 *Let S and A be as above and define $\check{S}$ as in Definition 2.12. Let $\mathbf{k}$ be a partition of a positive integer k and choose any local system $\mathcal{L}$ on $C_{\mathbf{k}}(\check{S})$ defined over a ring R. As an R-module, the twisted Borel-Moore homology module $H_k^{BM}(C_{\mathbf{k}}(\check{S}); \mathcal{L})$ decomposes as a direct sum of copies of the fibre of $\mathcal{L}$, indexed by the same combinatorial data as in Corollary 2.7, but using the embedded graph in Figure 2.3 instead of Figure 2.1.*

Definition 2.14 ([AP20, Def. 2.14]) Let γ be a loop in $C_{\mathbf{k}}(\overline{S})$ where all points remain fixed apart from two, which exchange places anticlockwise within a subdisc of $\overline{S}$ that is disjoint from the other points. If the monodromy of the rank-one local system $\mathcal{L}$ around every such loop γ is the *same* element $u \in R^{\times}$ then $\mathcal{L}$ is called *u-homogeneous*.

Under the assumption that the local system $\mathcal{L}$ is homogeneous, we have the following maps of representations.

Proposition 2.15 ([AP20, Thm. B]) *Let S be a connected, compact surface with one boundary component, let A be either a finite subset of its interior or a closed interval in its boundary and let $\mathbf{k}$ be a partition of a positive integer k. Let $\mathcal{L}$ be a rank-one local system on $C_{\mathbf{k}}(S \smallsetminus A)$ defined over a ring R that is u-homogeneous for some $u \in R^{\times}$. Then there are R-linear maps*

$$\begin{aligned} H_k^{\partial}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L}) &\longrightarrow H_k^{\mathrm{BM}}(C_{\mathbf{k}}(\check{S}); \mathcal{L}) \\ H_k^{\partial}(C_{\mathbf{k}}(\check{S}); \mathcal{L}) &\longrightarrow H_k^{\mathrm{BM}}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L}) \end{aligned} \tag{2.11}$$

that are equivariant with respect to the natural actions of the mapping class group of $(S, \partial S)$. With respect to appropriate free bases over R, these two maps are given by diagonal matrices all of whose diagonal entries are u-quantum factorials $[i]_u! = [1]_u[2]_u \cdots [i]_u$, where $[i]_u = 1 + u + \cdots + u^{i-1}$.

Corollary 2.16 *If $u \in R^{\times}$ is such that $[i]_u! \in R$ does not divide zero for all $i \geqslant 1$, then the maps (2.11) are embeddings of representations of the mapping class group of $(S, \partial S)$.*

Applying Corollary 2.10 and its analogue when $S \smallsetminus A$ is replaced by $\check{S}$ (which is also part of [AP20, Thm. A]), it follows that in this setting we have embeddings of representations

$$V^{\vee} \longhookrightarrow W \qquad \text{and} \qquad W^{\vee} \longhookrightarrow V \tag{2.12}$$

where $V = H_k^{\mathrm{BM}}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L})$ and $W = H_k^{\mathrm{BM}}(C_{\mathbf{k}}(\check{S}); \mathcal{L})$. This holds in particular if R is an integral domain and $u \in R^{\times}$ is such that $[i]_u \neq 0$ for all $i \geqslant 1$. In this setting, after tensoring (2.12) with the field of fractions $\mathrm{F}(R)$ we have isomorphisms

$$V^{\vee} \otimes \mathrm{F}(R) \cong W \otimes \mathrm{F}(R) \qquad \text{and} \qquad W^{\vee} \otimes \mathrm{F}(R) \cong V \otimes \mathrm{F}(R). \tag{2.13}$$

Example 2.17 This applies in particular to the representations $V = \mathfrak{LB}_{\mathbf{k}}(n)$ and their "vertical-type" alternatives $W = \mathfrak{LB}_{\mathbf{k}}^{v}(n)$ described in §1.2.1. In this case the ground ring R is $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ and the local coefficient system $\mathcal{L}$ is t-homogeneous. We therefore have embeddings

$$\mathfrak{LB}_{\mathbf{k}}(n)^{\vee} \longhookrightarrow \mathfrak{LB}_{\mathbf{k}}^{v}(n) \qquad \text{and} \qquad \mathfrak{LB}_{\mathbf{k}}^{v}(n)^{\vee} \longhookrightarrow \mathfrak{LB}_{\mathbf{k}}(n)$$

that become isomorphisms after tensoring over $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ with the field of rational functions $\mathbb{Q}(t, q)$.

The representations $\mathfrak{LB}_{\mathbf{k}}(n)$ are part of a functor $\mathfrak{LB}_{\mathbf{k}}$ defined on $\mathfrak{U}\boldsymbol{\beta}$, described in §1.2.1. Their duals $\mathfrak{LB}_{\mathbf{k}}(n)^{\vee}$ may similarly be extended to a functor defined on $\mathfrak{U}\boldsymbol{\beta}$, so the above may be thought of as embeddings (and isomorphisms) of representations of the category $\mathfrak{U}\boldsymbol{\beta}$.

The Moriyama representations. In the case $S = \Sigma_{g,1}$, Moriyama [Mor07] considered the $\mathbf{\Gamma}_{g,1}$-representation given by its action on the relative homology group $H_n(S^n, \Delta \cup A)$, where Δ denotes the "fat diagonal" of S^n where at least two points coincide and A denotes the subspace of S^n where at least one point is equal to p_0, a chosen basepoint on ∂S.

Let us write $S' = S \smallsetminus \{p_0\}$. Since S^n is a compactification of $F_n(S') = S^n \smallsetminus (\Delta \cup A)$, the Borel-Moore homology of $F_n(S')$ is isomorphic to the relative homology group $H_*(S^n, \Delta \cup A)$. Thus Moriyama's representation may be viewed as an action on Borel-Moore homology. Denoting by $F_n(X, Y) \subseteq F_n(X)$ the subspace of configurations that intersect $Y \subseteq X$ non-trivially, by Poincaré duality we have $H_*^{\mathrm{BM}}(F_n(S')) \cong H^n(F_n(S'), F_n(S', \partial S'))$ since $F_n(S')$ is a connected, orientable manifold. Using the fact that $S' \subset S$ and $\{p_1\} \subset \partial S'$ are isotopy equivalences, where $p_1 \in \partial S'$ is another (different) point on the boundary of S, this is naturally isomorphic to $H^n(F_n(S), F_n(S, \{p_1\}))$.

A special case of Lemma 2.1 implies that $H_*^{\mathrm{BM}}(F_n(S'))$ is concentrated in degree $* = n$, so by the universal coefficient theorem, Moriyama's representation $H_n(S^n, \Delta \cup A)$ is dual to the relative cohomology group $H^n(S^n, \Delta \cup A)$. Analogous identifications to those above, replacing H_*^{BM} with compactly-supported cohomology H_c^*, etc., apply to these dual representations.

In summary, we have identifications:

$$\begin{array}{ccccccc} H_n(S^n, \Delta \cup A) & \cong & H_n^{\mathrm{BM}}(F_n(S')) & \cong & H^n(F_n(S'), F_n(S', \partial S')) & \cong & H^n(F_n(S), F_n(S, \{p_1\})) \\ {\scriptstyle dual}\updownarrow & & & & & & \\ H^n(S^n, \Delta \cup A) & \cong & H_c^n(F_n(S')) & \cong & H_n(F_n(S'), F_n(S', \partial S')) & \cong & H_n(F_n(S), F_n(S, \{p_1\})), \end{array} \tag{2.14}$$

where the top row are models for Moriyama's representation and the bottom row are models for its dual.

Remark 2.18 Recall from §1.3.3 that the functor $\mathfrak{L}_{(\{1,\ldots,1\},1)}(\mathbf{\Gamma})$, restricted to the n-th automorphism group $G_n = \mathbf{\Gamma}_{n,1}$, is given by the natural action of the mapping class group on $H_n^{\mathrm{BM}}(F_n(S'))$. In fact in §1.3.3 we remove a closed interval from the boundary of S, instead of just a point, but the resulting configuration spaces are homotopy equivalent. By the discussion above, this is the n-th Moriyama representation [Mor07].

2.4. Generic local systems

We finally recall the notion of *genericity* for a local coefficient system, allowing us to equivalently consider classical homology and Borel-Moore homology for the homological representation functors (see Proposition 2.21).

Definition 2.19 Let $\mathcal{L}$ be a rank-one local system on a space X defined over a ring R. It is called *generic* if it satisfies the following property. Let γ be an unbased loop in X that may be homotoped to be disjoint from any given compact subset. Then the monodromy m_γ of $\mathcal{L}$ around γ, which is an element of $R^\times$, has the property that $1_R - m_\gamma \in R$ also lies in $R^\times \subset R$.

Remark 2.20 Genericity of $\mathcal{L}$ is in a sense a strong opposite of the property that $\mathcal{L}$ extends to the one-point compactification of X: the latter occurs if and only if $1_R - m_\gamma = 0$ for all loops γ as above. In particular, $\mathcal{L}$ is generic *and* extends to the one-point compactification of X if and only if either X is compact or R is the zero ring.

Proposition 2.21 *Suppose that $X = C_{\mathbf{k}}(M \smallsetminus A)$ for a triple (M, B, A) satisfying the hypotheses of Lemma 2.1, where $M \smallsetminus A$ is a connected, non-compact, orientable surface and $B \smallsetminus A$ is a disjoint union of open intervals. Let $\mathcal{L}$ be a local system on X defined over an Ore domain R, denote by* $\mathrm{F}(R)$ *the division ring of fractions of R and assume that $\mathcal{L} \otimes_R \mathrm{F}(R)$ is generic. Then the natural map*

$$H_k(X; \mathcal{L}) \longrightarrow H_k^{\mathrm{BM}}(X; \mathcal{L}) \tag{2.15}$$

becomes an isomorphism after tensoring $- \otimes_R \mathrm{F}(R)$.

Beforehand, we need the following version of the Künneth theorem:

Lemma 2.22 *Let G be a group, R an Ore domain whose division ring of fractions is denoted by $\mathrm{F}(R)$ and M an $R[G]$-module. There is a natural isomorphism $H_*(G; M \otimes_R \mathrm{F}(R)) \cong H_*(G; M) \otimes_R \mathrm{F}(R)$.*

Proof. Let $P_\bullet \to R$ be a projective right $R[G]$-module resolution. Then $P_\bullet \otimes_{R[G]} M$ and $P_\bullet \otimes_{R[G]} (M \otimes_R \mathrm{F}(R))$ are chain complexes computing $H_*(G; M)$ and $H_*(G; M \otimes_R \mathrm{F}(R))$ respectively. We recall that there is a natural isomorphism $P_\bullet \otimes_{R[G]} (M \otimes_R \mathrm{F}(R)) \cong (P_\bullet \otimes_{R[G]} M) \otimes_R \mathrm{F}(R)$ and that the division ring of fractions $\mathrm{F}(R)$ is a flat R-module; see [GW04, Cor. 10.13] for instance. Then the result follows from applying the Künneth spectral sequence for chain complexes; see [Wei94, Th. 5.6.4] for example. □

Proof of Proposition 2.21. The result is a consequence of the following three natural isomorphisms:

$$H_k(X; \mathcal{L}) \otimes_R \mathrm{F}(R) \cong H_k(X; \mathcal{L} \otimes_R \mathrm{F}(R)) \cong H_k^{\mathrm{BM}}(X; \mathcal{L} \otimes_R \mathrm{F}(R)) \cong H_k^{\mathrm{BM}}(X; \mathcal{L}) \otimes_R \mathrm{F}(R). \quad (2.16)$$

The first isomorphism follows from Lemma 2.22, since X is an aspherical space (because $M \smallsetminus A$ is an aspherical surface and hence so are its configuration spaces by the Fadell-Neuwirth fibration sequences) so its homology coincides with the group homology of its fundamental group.

The second isomorphism follows from the fact that we have assumed that $\mathcal{L} \otimes_R \mathrm{F}(R)$ is a generic local system on X. In general, ordinary and Borel-Moore homology are isomorphic, via the natural transformation $H_k \to H_k^{\mathrm{BM}}$, whenever taking coefficients in a generic local system on configuration spaces on orientable surfaces. This fact is originally due to Kohno [Koh17, Thm. 3.1], who proved this statement for local systems defined over $\mathbb{C}$, and was mildly generalised to other ground rings in [AP20, Prop. D].

Finally, the third isomorphism follows from Lemma 2.1, since that lemma implies that both sides are free $\mathrm{F}(R)$-modules with the same basis. □

The utility of Proposition 2.21 is that, under its hypotheses, we understand the Borel-Moore homology $H_k^{\mathrm{BM}}(X; \mathcal{L})$ completely by Lemma 2.1, whereas we do not have any structural result about the ordinary homology $H_k(X; \mathcal{L})$. The proposition tells us that they become isomorphic after tensoring with the field of fractions.

Trick 2.23 (*Making the coefficients generic.*) Only the first of the three isomorphisms (2.16) depends on the fact that we are tensoring with the division ring of fractions $\mathrm{F}(R)$. The third isomorphism works for any change of ring operation $- \otimes_R S$ for a ring morphism $R \to S$ and the second isomorphism works for any such change of ring operation making $\mathcal{L} \otimes_R S$ generic. It is therefore useful to note the minimal change of ring required to force genericity: this is given by considering all monodromies $m_\gamma \in R^\times$ for free loops γ that may be homotoped outside of any compact subset and then setting S to be the localisation of R at the multiplicative subset generated by $1_R - m_\gamma$ for all such γ. Note also that the local system $\mathcal{L} \otimes_R \mathrm{F}(R)$ will be generic if and only if $m_\gamma \neq 1_R$ for all such γ, since in this case the elements $1_R - m_\gamma$ will all be non-zero in R and hence invertible in its division ring of fractions.

3. Polynomial functors: background and preliminaries

This section recollects the general theory on polynomial functors (see §3.1) and prepares the study of polynomiality of homological representation functors of §4 (see §3.2).

3.1. Notions of polynomiality

In this section, we review the notions and basic properties of *strong*, *very strong*, *split* and *weak* polynomial functors. The definitions and results extend verbatim to the present slightly larger framework from the previous literature on that topic (see [DV19] and [Sou22, §4] for instance), the various proofs being mutatis mutandis generalisations of these previous works. For the remainder of §3.1, we fix a Grothendieck category $\mathcal{A}$, a left-module $(\mathcal{M}, \natural)$ over strict monoidal small groupoid $(\mathcal{G}, \natural, 0)$, where $\mathcal{M}$ is small groupoid, $(\mathcal{G}, \natural, 0)$ has *no zero divisors* and $\mathrm{Aut}_{\mathcal{G}}(0) = \{\mathrm{id}_0\}$.

Strong, very strong and split polynomial functors. Let X be an object of $\mathcal{G}$. Let τ_X be the endofunctor of $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ defined by $\tau_X(F) = F(X \natural -)$, called the *translation* functor. Let $i_X : Id \to \tau_X$ be the natural transformation of $\mathbf{Fct}(\mathcal{M}, \mathcal{A})$ induced by precomposition with the morphisms $[X, \mathrm{id}_{X \natural A}] : 0 \natural Y \to X \natural A$ for all $A \in \mathrm{Obj}(\mathcal{M})$. We define $\delta_X = \mathrm{coker}(i_X)$, called the *difference* functor, and $\kappa_X = \ker(i_X)$, called the *evanescence* functor. We denote by τ_X^m and δ_X^m the m-fold iterations $\tau_X \cdots \tau_X \tau_X$ and $\delta_X \cdots \delta_X \delta_X$ respectively. The translation functor τ_X is exact and induces the following exact sequence of endofunctors of $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$:

$$0 \longrightarrow \kappa_X \xrightarrow{\Omega_X} Id \xrightarrow{i_X} \tau_X \xrightarrow{\Delta_X} \delta_X \longrightarrow 0. \tag{3.1}$$

Moreover, for a short exact sequence $0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$ in the category $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$, there is a natural exact sequence following from the snake lemma:

$$0 \longrightarrow \kappa_X F \longrightarrow \kappa_X G \longrightarrow \kappa_X H \longrightarrow \delta_X F \longrightarrow \delta_X G \longrightarrow \delta_X H \longrightarrow 0. \tag{3.2}$$

In addition, for Y another object of $\mathcal{G}$, τ_X and τ_Y commute up to natural isomorphism and they commute with limits and colimits, δ_X and δ_Y commute up to natural isomorphism and they commute with colimits, κ_X and κ_Y commute up to natural isomorphism and they commute with limits, and τ_X commute with the functors δ_X and κ_X up to natural isomorphism.

The category of *strong* polynomial functors of degree less than or equal to $d \in \mathbb{N}$, denoted by $\mathcal{P}ol_d^{str}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$, is the full subcategory of $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ defined by $\mathcal{P}ol_d^{str}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}) = \{0\}$ if $d < 0$ and the objects of $\mathcal{P}ol_d^{str}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ for $d \in \mathbb{N}$ are the functors F such that the functor $\delta_X(F)$ is an object of $\mathcal{P}ol_{d-1}^{str}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$. The smallest integer $d \in \mathbb{N}$ for which an object F of $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ is an object of $\mathcal{P}ol_d^{str}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ is called the strong degree of F. The category of *very strong* polynomial functors of degree less than or equal to $d \in \mathbb{N}$, denoted by $\mathcal{VP}ol_d(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$, is the full subcategory of $\mathcal{P}ol_d^{str}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ of the objects F such that $\kappa_1(F) = 0$ and the functor $\delta_1(F)$ is an object of $\mathcal{VP}ol_{d-1}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$. The category of *split* polynomial functors of degree less than or equal to $d \in \mathbb{N}$, denoted by $\mathcal{SP}ol_d(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$, is the full subcategory of $\mathcal{VP}ol_d(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ of the objects F such that the translation map $i_X F : F \to \tau_X F$ is split injective in $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ for each object X of $\mathcal{G}$.

Weak polynomial functors. Let F be an object of $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$. We denote by $\kappa(F)$ the subfunctor $\sum_{X \in \mathrm{Obj}(\mathcal{G})} \kappa_X F$ of F. Let $K(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ be the full subcategory of $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ of the objects F such that $\kappa(F) = F$. The category $K(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ is a thick subcategory of $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ and it is closed under colimits; see [Sou22, Prop. 4.6]. Since the functor category $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ is a Grothendieck category, the subcategory $K(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ is *localising* and we may define the quotient category of $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ by $K(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$; see [Gab62, Chapitre III]. We denote by $\mathbf{St}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ this quotient category and by $\pi_{\langle \mathcal{G}, \mathcal{M} \rangle}$ the associated quotient functor.

For an object X of $\mathcal{G}$, the translation functor τ_X and the difference functor δ_X in the category $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ respectively induce an exact endofunctor of $\mathbf{St}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ which commutes with colimits, respectively again called the translation functor τ_X and the difference functor δ_X. In addition, we have the commutation relations $\delta_X \circ \pi_{\langle \mathcal{G}, \mathcal{M} \rangle} = \pi_{\langle \mathcal{G}, \mathcal{M} \rangle} \circ \delta_X$ and $\tau_X \circ \pi_{\langle \mathcal{G}, \mathcal{M} \rangle} = \pi_{\langle \mathcal{G}, \mathcal{M} \rangle} \circ \tau_X$. Therefore, the exact sequence (3.1) induces a short exact sequence $Id \hookrightarrow \tau_X \twoheadrightarrow \delta_X$ for the induced endofunctors of $\mathbf{St}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$. Finally for another object X' of $\mathcal{M}$, the endofunctors δ_X, $\delta_{X'}$, τ_X and $\tau_{X'}$ of $\mathbf{St}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ pairwise commute up to natural isomorphism.

We then define, inductively on $d \in \mathbb{N}$, the category of polynomial functors of degree less than or equal to d, denoted by $\mathcal{P}ol_d(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$, to be the full subcategory of $\mathbf{St}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ as follows. If $d < 0$, $\mathcal{P}ol_d(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}) = K(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$; if $d \geqslant 0$, the objects of $\mathcal{P}ol_d(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ are the functors F such that the functor $\delta_1(F)$ is an object of $\mathcal{P}ol_{d-1}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$. For an object F of $\mathbf{St}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ which is polynomial of degree less than or equal to $d \in \mathbb{N}$, the smallest integer $n \leqslant d$ for which F is an object of $\mathcal{P}ol_d(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ is called the degree of F. An object F of $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$ is weak polynomial of degree at most d if its image $\pi_{\langle \mathcal{G}, \mathcal{M} \rangle}(F)$ is an object of $\mathcal{P}ol_d(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A})$. The degree of polynomiality of $\pi_{\langle \mathcal{G}, \mathcal{M} \rangle}(F)$ is called the weak degree of F.

Finally, we recall useful properties of the categories associated with the different types of polynomial functors. We recall that the category $\langle \mathcal{G}, \mathcal{M} \rangle$ is said to be *finitely generated by the monoidal structure* $\natural$ if there exists a finite set $E_{\mathcal{G}}$ of objects of $\mathcal{G}$, such that each object m of $\langle \mathcal{G}, \mathcal{M} \rangle$ is isomorphic to a monoidal product of type $e^{\natural n} \natural m_0$ where $e \in E_{\mathcal{G}}$, $m_0 \in \mathcal{M}$ and $n \in \mathbb{N}$.

The following properties are proven in [Sou22, Props. 4.4, 4.10] (split polynomial functors are not considered there, but their study follows repeating mutatis mutandis this reference).

Proposition 3.1 *Let $d \geqslant 0$ be an integer. The category $\mathcal{P}ol_d^{str}(\langle \mathcal{G}, \mathcal{M}\rangle, \mathcal{A})$ is closed under the translation functor, under quotients, under extensions and under colimits. The category $\mathcal{VP}ol_d(\langle \mathcal{G}, \mathcal{M}\rangle, \mathcal{A})$ is closed under the translation functor, under normal subobjects and under extensions. The category $\mathcal{SP}ol_d(\langle \mathcal{G}, \mathcal{M}\rangle, \mathcal{A})$ is closed under the translation functor and under normal subobjects. As a subcategory of $\mathbf{St}(\langle \mathcal{G}, \mathcal{M}\rangle, \mathcal{A})$, the category $\mathcal{P}ol_d(\langle \mathcal{G}, \mathcal{M}\rangle, \mathcal{A})$ is thick and closed under limits and colimits.*

Furthermore, we assume that $\langle \mathcal{G}, \mathcal{M}\rangle$ is finitely generated by the monoidal structure $\natural$ with generating set $E_{\mathcal{G}}$. Let F be an object of $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M}\rangle, \mathcal{A})$. Then, it is enough to check the criteria of the above definitions of strong, very strong, split and weak polynomiality assuming that X is an object of $E_{\mathcal{G}}$.

3.2. Framework preliminaries for polynomiality

We now briefly discuss some preliminaries for the work on polynomiality of §4. First of all, each source category of the homological representation functors of §1.3 is of type $\langle \mathcal{G}, \mathcal{M}\rangle$ and finitely generated by the monoidal structure $\natural$ using exactly two objects $X \in \mathrm{Obj}(\mathcal{G})$ and $O \in \mathrm{Obj}(\mathcal{M})$; see §1.3. Hence by Proposition 3.1, it is enough to study the natural transformation i_X to prove the polynomiality results. We consider any one of the homological representation functors of §1.3 that we denote by $\mathfrak{L}_{(\mathbf{k},\ell)}$ (with $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})]$ the ground ring of the target module category).

Twisted representations and target category. We assume that $\mathfrak{L}_{(\mathbf{k},\ell)}$ is *twisted*, i.e. a functor of the form $\langle \mathcal{G}, \mathcal{M}\rangle \to \mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})]\text{-Mod}^{\mathrm{tw}}$. The notions of polynomiality defined in §3.1 require the target category of a functor to be a Grothendieck category. This subtlety does not impact the core points to deal with polynomiality, and we solve this issue by adopting the following convention:

Convention 3.2 In §4, when considering a *twisted* homological representation functor $\mathfrak{L}_{(\mathbf{k},\ell)}$, we always postcompose it by the forgetful functor $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})]\text{-Mod}^{\mathrm{tw}} \to \mathbb{Z}\text{-Mod}$, as done in (1.6). We generically denote by $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})]\text{-Mod}^{\star}$ the target category of $\mathfrak{L}_{(\mathbf{k},\ell)}$, which is either $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})]$-Mod or $\mathbb{Z}$-Mod.

Change of rings operation. The ground ring of most of the homological representation functors is a group ring of type $\mathbb{Z}[Q]$ that we may modify by some change of rings operation, as explained in §1.2.2. The following lemma is a key point to prove that such an alteration does not impact our polynomiality results in §4.

Lemma 3.3 *Let Q be a group, R be a (non-zero) ring and $f\colon \mathbb{Z}[Q] \to R$ be a (non-zero) ring morphism. We consider a functor $F\colon \langle \mathcal{G}, \mathcal{M}\rangle \to \mathbb{Z}[Q]\text{-Mod}$ such that $F(Y)$, $\tau_X F(Y)$ and $\delta_X F(Y)$ are free $\mathbb{Z}[Q]$-modules for all $Y \in \mathrm{Obj}(\langle \mathcal{G}, \mathcal{M}\rangle)$. If $\delta_X F = 0$, then $\kappa_X f_! F = 0$.*

Proof. We note that the functor $f_!$ clearly commutes with the translation functor τ_1, and also with the difference functor δ_X because it is right-exact. Also, by our assumption on F, the R-modules $f_! F(Y)$ and $\delta_X f_! F(Y)$ are free for all $Y \in \mathrm{Obj}(\langle \mathcal{G}, \mathcal{M}\rangle)$. Hence the kernel of $\tau_X f_! F(Y) \twoheadrightarrow \delta_X f_! F(Y)$ is a free R-module whose free generating set is in bijection with that of $f_! F(Y)$. Therefore, the map $f_! F \to \tau_X f_! F$ is injective, which ends the proof. □

Borel-Moore homology vs. classical homology. All of our reasoning in order to prove polynomiality results in §4 decisively relies on the module structures for the homological representations using Borel-Moore homology exhibited in §2. However, these results may be extended to functors defined with classical homology thanks to the notion of *genericity* explained in §2.4 as follows.

Lemma 3.4 (*Genericity.*) *Assume that the surfaces $\mathcal{S}_n$ defining $\mathfrak{L}_{(\mathbf{k},\ell)}$ are **orientable** and consider a ring homomorphism $f\colon \mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})] \to \mathrm{F}(R)$, where $\mathrm{F}(R)$ is the division ring of fractions of an Ore domain R, such that the local system $f_! \mathfrak{L}_{(\mathbf{k},\ell)}$ is **generic**. Then, the results for $\mathfrak{L}_{(\mathbf{k},\ell)}$ repeat verbatim for the analogue of $f_! \mathfrak{L}_{(\mathbf{k},\ell)}$ using **ordinary** homology.*

Proof. This is a consequence of an application of Proposition 2.21 along with the naturality of the isomorphisms of Lemma 2.1, of (2.15) and of Lemma 2.22. □

Example 3.5 Let $\mathbf{k} \vdash k$ be any partition of $k \geqslant 1$ and $\mathbf{k}' = \{k'_1; \ldots; k'_r\} \vdash k'$ be a partition of $k' \geqslant 3$ such that $k'_l \geqslant 3$ for all $1 \leqslant l \leqslant r$. We may apply Lemma 3.4 to the functors $\mathfrak{LB}_{(\mathbf{k},2)}$, $\mathfrak{L}_{(\mathbf{k}',3)}(\Sigma_{g,1})$ and $\mathfrak{L}_{(k',3)}(\mathcal{N}_{h,1})$ for f the canonical inclusion $\mathtt{i}\colon \mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})] \to \mathrm{F}(\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})])$ in each case. Indeed, in each case, the group ring $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})]$ is an Ore domain by [PS23, Cor. 3.10], and we then easily check that the local system $\mathtt{i}_!\mathfrak{L}_{(\mathbf{k},\ell)}$ is generic using Trick 2.23.

Partitions for the difference functors. Considering any homological representation functor of §1.3 with a partition $\mathbf{k} \vdash k$ as parameter, the description of its difference functor in §4 makes key use of some appropriate partitions of $k-1$ obtained from $\mathbf{k} \vdash k$. We denote these partitions and their sets as follows:

Notation 3.6 Considering integers $k \geqslant k' \geqslant 1$ and a partition $\mathbf{k} = \{k_1; \ldots; k_r\} \vdash k$, we denote by $\{\mathbf{k} - k'\}$ the set $\left\{\{k_1 - k'_1; \ldots; k_r - k'_r\};\, \sum_{1\leqslant l\leqslant r} k'_l = k'\right\}$.

For $k' = 1$ and each $1 \leqslant i \leqslant r$, we denote by $\mathbf{k}_i$ the element $\{k_1; \ldots; k_i - 1; \ldots; k_r\}$ of $\{\mathbf{k}-1\}$. In particular, we have $\{\mathbf{k}-1\} = \{\mathbf{k}_i\,;\, 1 \leqslant i \leqslant r\}$.

For $k' = 2$ and each $1 \leqslant i \leqslant j \leqslant r$, we denote by $\mathbf{k}_{i,j}$ the element $\{k_1; \ldots; k_i - 1; \ldots; k_j - 1; \ldots; k_r\}$ of $\{\mathbf{k}-2\}$. Then, we have $\{\mathbf{k}-2\} = \{\mathbf{k}_{i,j}\,;\, 1 \leqslant i \leqslant j \leqslant r\}$.

Transformation groups of the difference functors. Let $\mathbf{k}' \in \{\mathbf{k} - k'\}$, and let $\mathfrak{L}_{(\mathbf{k}',\ell)}$ and $\mathfrak{L}_{(\mathbf{k},\ell)}$ be the homological representation functors associated to the partitions $\mathbf{k}'$ and $\mathbf{k}$. We recall from Lemma 1.10 that there is a canonical map for the associated transformation groups (1.15): $Q_{(\mathbf{k}',\ell)}(\mathcal{S}) \to Q_{(\mathbf{k},\ell)}(\mathcal{S})$. The change of rings operation $(1.15)_!\mathfrak{L}_{(\mathbf{k}',\ell)}$ allows us to canonically switch the module structure of $\mathfrak{L}_{(\mathbf{k}',\ell)}$ from $\mathbb{Z}[Q_{(\mathbf{k}',\ell)}(\mathcal{S})]$ to $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathcal{S})]$, as well as the potential twisted actions of the groups on these modules; see Observation 1.11. This change of ground ring map is just the identity in many situations, and it anyway does not impact the key underlying structures of $\mathfrak{L}_{(\mathbf{k}',\ell)}$ for polynomiality. Broadly speaking, its purpose is to allow us to identify $\mathfrak{L}_{(\mathbf{k}',\ell)}$ as a summand of the difference functor $\delta_1\mathfrak{L}_{(\mathbf{k},\ell)}$. Also, the partition $\mathbf{k}'$ may have null blocks, so we denote by $\overline{\mathbf{k}'}$ the partition of $k - k'$ obtained from $\mathbf{k}'$ by removing the 0-blocks.

Because the above subtleties are minor points and do not affect the key points of the reasoning, we choose to use the following conventions on the simplification of the notations:

Convention 3.7 In §4, the change of ground ring operations of type $(1.15)_!\mathfrak{L}_{(\mathbf{k}',\ell)}$ on the functors are always clear from the context and implicitly applied at several points. For the sake of simplicity and to not overload the notation, we keep the notation $\mathfrak{L}_{(\mathbf{k}',\ell)}$ for the functor modified in this way. We also always implicitly identify $\mathfrak{L}_{(\mathbf{k}',\ell)}$ with $\mathfrak{L}_{(\overline{\mathbf{k}'},\ell)}$ because they are obviously isomorphic. For consistency, we generically denote by $\mathfrak{L}_{(0,\ell)}\colon \langle \mathcal{G}, \mathcal{M}\rangle \to \mathbb{Z}[Q_{(1,\ell)}]$-Mod the constant functor at $\mathbb{Z}[Q_{(1,\ell)}]$, except for Lawrence-Bigelow functors (see Notation 4.1).

3.3. Diagrammatic arguments

In §2.3 we gave explicit descriptions of the underlying (free) modules of the representations that we consider. For the purposes of the present paper, we do not need explicit formulas for the group actions on these modules, but we will need to use several qualitative properties of the group actions, which we establish in this subsection.

We take up the notations of §2.3 and consider any one of the homological representation functors of §1.3 in the classical (i.e. non-vertical) setting, that we denote by $\mathfrak{L}_{(\mathbf{k},\ell)}$ with associated transformation groups $Q_{(\mathbf{k},\ell)}$. Before restricting our attention to automorphism groups, we first note that the action of the canonical morphism $n \to 1\natural n$ of the category on which our representations are defined has a very simple description: in each case, it is induced by the evident inclusion of configuration spaces. For example, its action under the representation $\mathfrak{L}_{(\mathbf{k},\ell)}$ is the map

$$\mathfrak{L}_{(\mathbf{k},\ell)}(n) = H^{\mathrm{BM}}_k(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L}) \longrightarrow H^{\mathrm{BM}}_k(C_{\mathbf{k}}(S \smallsetminus A^+); \mathcal{L}) = \mathfrak{L}_{(\mathbf{k},\ell)}(1\natural n), \tag{3.3}$$

where $A^+ = \{*\} \sqcup A$, induced by the closed inclusion $S \smallsetminus A \hookrightarrow S \smallsetminus A^+$ given by regarding $S \smallsetminus A^+$ as the boundary connected sum of a punctured disc with $S \smallsetminus A$. The local systems, which we have denoted simply by $\mathcal{L}$ here for simplicity, are explained in the general construction of §1.2.1.

3.3.1. The cloud lemma

We consider the representation $\mathfrak{L}_{(\mathbf{k},\ell)}(n) = H_k^{\mathrm{BM}}(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L})$ of the surface braid group $\mathbf{B}_n(S)$, where we are in the setting of Figure 2.1a of §2.3 and $\mathcal{L}$ is a rank-one local system; i.e. this representation is part of a homological representation functor from §1.3.1 or §1.3.2. Corollary 2.7 describes a free basis for the underlying module of $\mathfrak{L}_{(\mathbf{k},\ell)}(n)$ indexed by labellings of the embedded graph $\Gamma \subset S$ by words in the blocks of the partition $\mathbf{k}$. Choosing an ordering of the edges of Γ, we write this as $(w_1, \ldots, w_{2g+n-1}) \vdash \mathbf{k}$. (This is a slight simplification of Notation 2.8 just for this subsection.)

Remark 3.8 The diagrammatic arguments in this subsection work both for orientable surfaces $S = \Sigma_{g,1}$ and non-orientable surfaces $S = \mathcal{N}_{h,1}$. However, for convenience, we will write everything for the case $S = \Sigma_{g,1}$. To obtain the case $S = \mathcal{N}_{h,1}$, one simply has to replace "$2g$" with "h" everywhere (such as in the tuple of words indexing a basis element above) and modify the right-hand sides of Figures 3.1–3.3 (only the planar parts of these figures are important).

The representation $\tau_1\mathfrak{L}_{(\mathbf{k},\ell)}(n)$ has a very similar description as a free module: the only difference being that there is one extra edge of the embedded graph Γ^+ whose edge-labellings index the free generating set for $\tau_1\mathfrak{L}_{(\mathbf{k},\ell)}(n)$. We write this as $(w_0, w_1, \ldots, w_{2g+n-1}) \vdash \mathbf{k}$, where w_0 is the label of the extra edge. The cokernel $\delta_1\mathfrak{L}_{(\mathbf{k},\ell)}(n)$ may therefore be described as the free module generated by all edge-labellings of Γ^+ such that w_0 is not the empty word. As one more piece of notation, we write A^+ for the vertices of Γ^+, so it has one more element than A.

Recalling Notation 3.6, the direct sum $\bigoplus_{i=1}^r \tau_1\mathfrak{L}_{(\mathbf{k}_i,\ell)}(n)$ therefore has a basis indexed by pairs $(i, (w_0, w_1, \ldots, w_{2g+n-1}))$, where $1 \leqslant i \leqslant r$ and $(w_0, w_1, \ldots, w_{2g+n-1}) \vdash \mathbf{k}_i$. There is an evident bijection between the basis for $\delta_1\mathfrak{L}_{(\mathbf{k},\ell)}(n)$ and this basis for $\bigoplus_{i=1}^r \tau_1\mathfrak{L}_{(\mathbf{k}_i,\ell)}(n)$ given by

$$(w_0, w_1, \ldots, w_{2g+n-1}) \longmapsto (i, (w_0', w_1, \ldots, w_{2g+n-1})),$$

where $w_0 = iw_0'$. Extending by linearity, we obtain an isomorphism of free modules

$$\delta_1\mathfrak{L}_{(\mathbf{k},\ell)}(n) \longrightarrow \bigoplus_{i=1}^r \tau_1\mathfrak{L}_{(\mathbf{k}_i,\ell)}(n). \tag{3.4}$$

The following lemma is the key ingredient to prove that this is an isomorphism of *representations*. (In fact, we will have an isomorphism of functors as n varies.)

Lemma 3.9 ("*Cloud lemma*".) *The module isomorphism* (3.4) *acts as indicated in Figure 3.1. Namely, given any homology class of the form depicted on the left-hand side of the figure, it is sent under* (3.4) *to the homology class depicted on the right-hand side. A precise version of this statement is given in the caption of Figure 3.1.*

Proof. Let us write $\mathbf{w} = (w_0, w_1, \ldots, w_{2g+n-1}) \vdash \mathbf{k}$ and set $\mathbf{w}'' = (w_0'', w_1, \ldots, w_{2g+n-1})$, where $w_0 = jw_0''$; in other words the operation $(-)''$ removes the first letter of the first word of $\mathbf{w}$. Note that $\mathbf{w}'' \vdash \mathbf{k}_j$. Denote by $e_{\mathbf{w}}$ the standard basis element, depicted in Figure 2.2, indexed by $\mathbf{w} \vdash \mathbf{k}$ and denote by $e'_{\mathbf{w}}$ the corresponding dual basis element depicted in Figure 2.3. By definition, the isomorphism (3.4) takes $e_{\mathbf{w}}$ to the element $e_{\mathbf{w}''}$ in the j-th summand of the right-hand side.

Let us first decompose the left-hand side of Figure 3.1 as

$$\sum_{\mathbf{w} \vdash \mathbf{k}} e_{\mathbf{w}}.\lambda_{\mathbf{w}} = \sum_{j=1}^r \sum_{\substack{\mathbf{w} \vdash \mathbf{k} \\ w_0 = jw_0''}} e_{\mathbf{w}}.\lambda_{\mathbf{w}}, \tag{3.5}$$

where $\lambda_{\mathbf{w}} = \langle \mathrm{LHS}, e'_{\mathbf{w}} \rangle$ is the value of the intersection pairing (2.8) evaluated on the left-hand side of Figure 3.1 and the dual basis element $e'_{\mathbf{w}}$. This is illustrated on the left-hand side of Figure 3.2.

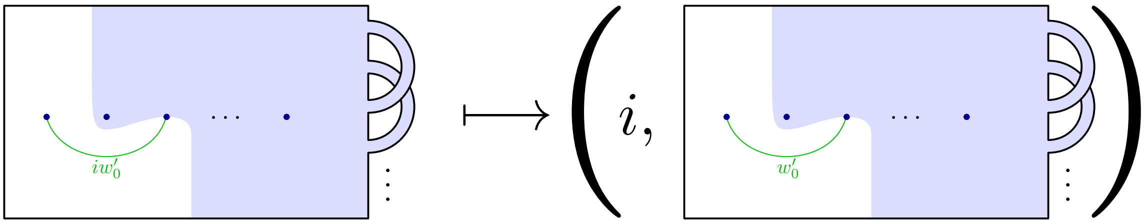


Figure 3.1 The left-hand side of the figure depicts a Borel-Moore cycle on the partitioned configuration space $C_{\mathbf{k}}(S \smallsetminus A^+)$, representing an element of $\tau_1\mathfrak{L}_{(\mathbf{k},\ell)}(n) = H_k^{\mathrm{BM}}(C_{\mathbf{k}}(S \smallsetminus A^+);\mathbb{Z}[Q_{\mathbf{k}}])$ and thus determining an element of the quotient $\delta_1\mathfrak{L}_{(\mathbf{k},\ell)}(n)$ of $\tau_1\mathfrak{L}_{(\mathbf{k},\ell)}(n)$. This cycle is assumed to be of the following form. Choose a non-empty word w_0 on the alphabet $\{1,\ldots,r\}$ and write $\mathbf{l} = \{l_1;\ldots;l_r\}$ where l_i denotes the number of copies of the letter i in w_0. Choose any cycle α on $C_{\mathbf{k}-\mathbf{l}}(S \smallsetminus A^+)$ supported in the blue shaded region (the "*cloud*") and let β denote the cycle on $C_{\mathbf{l}}(S \smallsetminus A^+)$ given by the singular simplex consisting of all configurations lying on the open green arc labelled according to the word w_0. Then $\alpha \times \beta$ is a cycle on $C_{\mathbf{k}}(S \smallsetminus A^+)$; this is the cycle that we consider on the left-hand side.

The right-hand side has a similar description, where we decompose the non-empty word w_0 as iw_0'. The "i" component simply says that the element lies in the i-th summand on the right-hand side of (3.4). The second, pictorial component then describes an explicit Borel-Moore cycle on $C_{\mathbf{k}_i}(S \smallsetminus A^+)$ representing an element of $\tau_1\mathfrak{L}_{(\mathbf{k}_i,\ell)}(n)$. Precisely, it is $\alpha \times \beta'$, where α is the previously-chosen cycle and β' is the cycle on $C_{\mathbf{l}-i}(S \smallsetminus A^+)$ given by the singular simplex consisting of all configurations lying on the open green arc labelled according to the word w_0'.

The precise statement of Lemma 3.9 is that, for any choices of $w_0 = iw_0'$ and α as above, the module isomorphism (3.4) sends the element $[\alpha \times \beta]$ to the element $(i, [\alpha \times \beta'])$, where β and β' are determined by w_0 as described earlier.

From that figure, it is clear that $\lambda_{\mathbf{w}} = 0$ unless $j = i$, so we may remove the outer sum and set $j = i$ in the formula (3.5). Its image under the map (3.4) is

$$\sum_{\substack{\mathbf{w} \vdash \mathbf{k} \\ w_0 = iw_0''}} e_{\mathbf{w}''}.\lambda_{\mathbf{w}}. \tag{3.6}$$

On the other hand, the right-hand side of Figure 3.1 decomposes as

$$\sum_{\mathbf{v} \vdash \mathbf{k}_i} e_{\mathbf{v}}.\mu_{\mathbf{v}}, \tag{3.7}$$

where $\mu_{\mathbf{v}} = \langle \mathrm{RHS}, e'_{\mathbf{v}} \rangle$ is the value of the intersection pairing (2.8) evaluated on the right-hand side of Figure 3.1 and the dual basis element $e'_{\mathbf{v}}$. This is illustrated on the right-hand side of Figure 3.2. There is clearly a bijection between the two indexing sets of the sums above given by sending $\mathbf{w}$ to $\mathbf{v} = \mathbf{w}''$. Thus, in order to prove that (3.6) = (3.7), as desired, it remains to show that we have an equality of coefficients

$$\langle \mathrm{LHS}, e'_{\mathbf{w}} \rangle = \lambda_{\mathbf{w}} = \mu_{\mathbf{w}''} = \langle \mathrm{RHS}, e'_{\mathbf{w}''} \rangle. \tag{3.8}$$

To explain this, we briefly recall some of the details of how the intersection pairings $\langle \mathrm{LHS}, e'_{\mathbf{w}} \rangle$ and $\langle \mathrm{RHS}, e'_{\mathbf{w}''} \rangle$ may be computed; for more precise details, see [Big01, §2.1] or [PS22, §4.3] (when the surface is a disc) or [BPS21, §7] (for more general orientable surfaces).

We first consider $\langle \mathrm{LHS}, e'_{\mathbf{w}} \rangle$ and assume that the Borel-Moore homology class denoted by LHS (the left-hand side of Figure 3.1) is represented by configuration spaces on a collection of pairwise disjoint properly-embedded arcs, one of these being the arc depicted and the others being contained in the shaded "cloud". (It is always possible to represent a Borel-Moore homology class as a formal linear combination of such classes, due to the basis that we have described in §2. We may therefore make this assumption without loss of generality.) The dual basis element $e'_{\mathbf{w}}$ is represented by the cycle given by the red vertical (and horizontal, in the handles) arcs on the left-hand side of Figure 3.2. We assume that these intersect the arcs representing LHS transversely, in particular in finitely many points.

The value of the pairing $\langle \mathrm{LHS}, e'_{\mathbf{w}} \rangle \in \mathbb{Z}[Q_{\mathbf{k}}]$ is then a sum of terms $\epsilon_p \phi(\ell_p)$ indexed by these intersection points p, where $\epsilon_p \in \{\pm 1\}$ is a sign, ℓ_p is a based loop in the configuration space

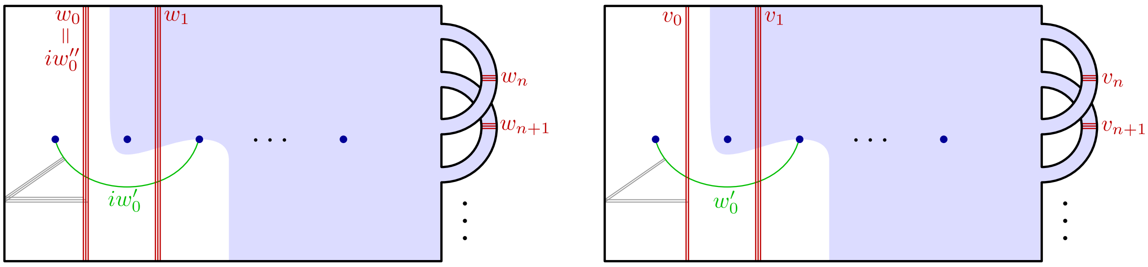


Figure 3.2 An illustration of the intersection pairings $\lambda_{\mathbf{w}} = \langle \mathrm{LHS}, e'_{\mathbf{w}} \rangle$ (left) and $\mu_{\mathbf{w}''} = \langle \mathrm{RHS}, e'_{\mathbf{w}''} \rangle$ (right). Here, LHS and RHS refer to the Borel-Moore homology classes depicted on the left-hand side and right-hand side of Figure 3.1 respectively.

$C_{\mathbf{k}}(S \smallsetminus A^+)$ and $\phi\colon \pi_1(C_{\mathbf{k}}(S \smallsetminus A^+)) \twoheadrightarrow Q_{\mathbf{k}}$ is the quotient determining the local system in the definition of $\mathfrak{L}_{(\mathbf{k},\ell)}$. More precisely, $\phi(\ell_p)$ is a tuple of integers counting various winding numbers of configuration points around the punctures A^+ and around each other during the loop ℓ_p. To determine ℓ_p, one must first of all choose a path from the base configuration to some point x on (the cycle representing the homology class) LHS and another path from the base configuration to some point y on (the cycle representing the homology class) $e'_{\mathbf{w}}$. Such choices of "tethers" are in fact necessary to fully describe the homology classes that we are considering. However, different choices correspond to homology classes that differ only by unit scalars in the ground ring, so the choice does not matter for us, since (3.4) is a module morphism. We will assume, for convenience, that the parts of the tethers attached to the left-most arcs are as depicted in Figure 3.2. Given these choices, the loop ℓ_p is then a concatenation of four paths

$$* \rightsquigarrow x \rightsquigarrow p \rightsquigarrow y \rightsquigarrow *$$

where the first and last are the tethers, the second is a path in LHS from x to the intersection point p and the third is a path in $e'_{\mathbf{w}}$ from p to y.

The intersection pairing $\langle \mathrm{RHS}, e'_{\mathbf{w}''} \rangle$ has an almost identical description, the only difference being that one red vertical arc (containing a single point in the ith block of the partition $\mathbf{k}$) has been removed and the green arc labelled by iw'_0 is now labelled by w'_0, so its left-most point (in the ith block of the partition $\mathbf{k}$) has also been removed.

To compare these two elements of $\mathbb{Z}[Q_{\mathbf{k}}]$, first notice that there is a bijection of intersection points $\mathrm{RHS} \cap e'_{\mathbf{w}''} \to \mathrm{LHS} \cap e'_{\mathbf{w}}$ given by $p \mapsto \bar{p} = p \cup \{p_0\}$, where $p_0 \in S \smallsetminus A^+$ is the unique intersection point between the left-most vertical arc and the curved (green) arc on the left-hand side of Figure 3.2. It therefore suffices to check that we have $\epsilon_{\bar{p}} = \epsilon_p$ and $\phi(\ell_{\bar{p}}) = \phi(\ell_p)$, where the values with a subscript $\bar{p}$ are computed using the left-hand side of Figure 3.2 and those with a subscript p are computed using the right-hand side of Figure 3.2.

The loop $\ell_{\bar{p}}$ lies in a configuration space with one more point than the configuration space containing the loop ℓ_p. The key observation is that – up to basepoint-preserving homotopy – $\ell_{\bar{p}}$ is obtained from ℓ_p by simply adjoining a *stationary* point in the boundary of the surface. This fact may be read off directly, using the description above of how the loops are constructed, by comparing the two sides of Figure 3.2. In particular, no winding numbers are changed by adjoining this additional stationary point, so $\phi(\ell_{\bar{p}}) = \phi(\ell_p)$.

Finally, we recall that the sign ϵ_p is the product of the local signs of the intersections of arcs in the surface at each point of $p = \{p_1, \ldots, p_r\}$ together with an additional sign recording the parity of the permutation of the base configuration induced by ℓ_p. The local sign of the intersection at p_0 is $+1$, so adjoining p_0 does not change the product of the local signs. In addition, as a consequence of the paragraph above, the permutation induced by $\ell_{\bar{p}}$ is obtained from the permutation induced by ℓ_p by adjoining a fixed point; in particular they both have the same parity. Thus $\epsilon_{\bar{p}} = \epsilon_p$. $\square$

3.3.2. Other diagrammatic arguments for surface braid groups

We take up the notations from §3.3.1. We will need two other (easier) diagrammatic facts in the setting of surface braid groups. The first is an identity taking place in the Borel-Moore homology group $\mathfrak{L}_{(\mathbf{k},\ell)}(n) = H^{\mathrm{BM}}_k(C_{\mathbf{k}}(S \smallsetminus A); \mathcal{L})$.

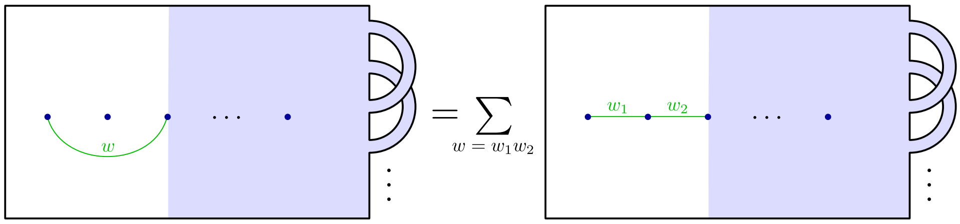


Figure 3.3 The identity of Lemma 3.10 in the orientable case. The identity in the non-orientable case is the obvious analogue; only the left-hand side of each diagram, which is planar, is important.

Lemma 3.10 *In $\mathfrak{L}_{(\mathbf{k},\ell)}(n)$, we have the identity depicted in Figure 3.3.*

Proof. This follows immediately by verifying that each side of the equation evaluates to the same element of the ground ring when applying the intersection pairing $\langle -, e'_{\mathbf{v}} \rangle$ with the dual basis element $e'_{\mathbf{v}}$ for each $\mathbf{v} = (v_1, v_2, v_3 \ldots) \vdash \mathbf{k}$. (Details of how these intersection pairings are computed are explained in the proof of Lemma 3.9 above.) $\square$

The second fact concerns the behaviour of the vertical-type alternative functors $\mathfrak{L}^v_{(\mathbf{k},\ell)}$ after applying the operation δ_1.

Lemma 3.11 *The functor $\delta_1 \mathfrak{L}^v_{(\mathbf{k},\ell)}$ sends every morphism that is not an endomorphism to zero.*

Proof. As mentioned at the beginning of this subsection, the canonical morphism $n \to 1 \natural n$ of the domain category is sent, under each of our functors $\mathfrak{L}^v_{(\mathbf{k},\ell)}$, to the map on Borel-Moore homology induced by the evident inclusion of configuration spaces. (In the non-vertical setting this is (3.3); in the vertical setting it is the obvious analogue.) Since every morphism of the domain category that is not an endomorphism factors through one of these canonical morphisms, it suffices to show that all of these are sent to zero under $\delta_1 \mathfrak{L}^v_{(\mathbf{k},\ell)}$. In other words, we wish to show that the map labelled by $(*)$ in the following diagram is zero, where the rows are exact:

$$
\begin{array}{ccccccc}
\mathfrak{L}^v_{(\mathbf{k},\ell)}(n) & \longrightarrow & \tau_1 \mathfrak{L}^v_{(\mathbf{k},\ell)}(n) & \longrightarrow & \delta_1 \mathfrak{L}^v_{(\mathbf{k},\ell)}(n) & \longrightarrow & 0 \\
\downarrow & \swarrow & \downarrow (\dagger) & & \downarrow (*) & & \\
\mathfrak{L}^v_{(\mathbf{k},\ell)}(1 \natural n) & \xrightarrow[(\ddagger)]{} & \tau_1 \mathfrak{L}^v_{(\mathbf{k},\ell)}(1 \natural n) & \longrightarrow & \delta_1 \mathfrak{L}^v_{(\mathbf{k},\ell)}(1 \natural n) & \longrightarrow & 0
\end{array}
\tag{3.9}
$$

To do this, it suffices to show that there is a diagonal morphism making the two triangles commute. Recalling that $\tau_1 F(n) = F(1 \natural n)$ in general, we will be able to take the diagonal morphism to be the identity as long as the two maps labelled $(\dagger)$ and $(\ddagger)$ are equal (the top horizontal and left vertical maps in that square are always equal by definition of the natural transformation $Id \to \tau_1$).

By definition of $\tau_1 \mathfrak{L}^v_{(\mathbf{k},\ell)}$, its action on the canonical morphism $n \to 1 \natural n$ is given by the action of $\mathfrak{L}^v_{(\mathbf{k},\ell)}$ on the canonical morphism $1 \natural n \to 2 \natural n$ composed with $(b_{1,1})^{-1} \natural \mathrm{id}_n$, where $b_{1,1}$ denotes the braiding $b^{\boldsymbol{\beta}}_{1,1} \colon 1 \natural 1 \cong 1 \natural 1$ of the groupoid $\boldsymbol{\beta}$. This describes the map $(\dagger)$; on the other hand, the map $(\ddagger)$ is given simply by the action of $\mathfrak{L}^v_{(\mathbf{k},\ell)}$ on the canonical morphism $1 \natural n \to 2 \natural n$. It is therefore enough to prove that the automorphism $b_{1,1} \natural \mathrm{id}_n$ acts by the identity on the image of $(\ddagger)$. This is immediate from Figure 3.4, where the image of an arbitrary basis element under $(\ddagger)$ is depicted in green (supported on the vertical arcs) and the support of a diffeomorphism representing the mapping class $b_{1,1} \natural \mathrm{id}_n$ is shaded in grey. Since these supports are disjoint, the action of $b_{1,1} \natural \mathrm{id}_n$ on the image of $(\ddagger)$ is trivial. $\square$

Remark 3.12 It is instructive to consider why the same argument does *not* also show that the functor $\delta_1 \mathfrak{L}_{(\mathbf{k},\ell)}$ sends every canonical morphism $n \to 1 \natural n$ to the zero morphism. This boils down to the fact that, in the analogue of Figure 3.4 for the non-vertical version $\mathfrak{L}_{(\mathbf{k},\ell)}$ of the functor, the supports are *not* disjoint.

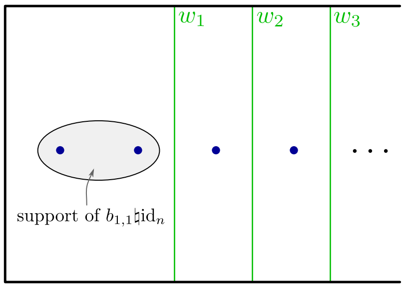


Figure 3.4 The support of a diffeomorphism representing the mapping class $b_{1,1}\natural\mathrm{id}_n$ and the image of an arbitrary basis element under the map (‡) of (3.9).

3.3.3. Other diagrammatic arguments for mapping class groups

In the setting of mapping class groups, we will need two other general diagrammatic arguments: a disjoint support argument in the case of boundary connected sums of two surfaces (§3.3.3.1) and some calculations of the actions of various braiding actions (§3.3.3.2). We now assume that $\mathfrak{L}_{(\mathbf{k},\ell)}$ is one of the (non-vertical) homological representation functors defined in §1.3.3.

3.3.3.1. Boundary connected sums

We will sometimes use the following general principle, for representations $\mathfrak{L}(S)$ (where $\mathfrak{L}$ is either $\mathfrak{L}_{(\mathbf{k},\ell)}$ or its vertical-type alternative $\mathfrak{L}^{v}_{(\mathbf{k},\ell)}$) of the mapping class group of a surface S that splits as a boundary connected sum $S = S'\natural S''$.

Lemma 3.13 *Suppose that $S = S'\natural S''$ and let $g \in \mathrm{MCG}(S')$. Let $e_{\mathbf{w}}$ be a basis element of $\mathfrak{L}(S)$, using the bases described in §2 and write the tuple $\mathbf{w}$ as $\mathbf{w} = (\mathbf{w}', \mathbf{w}'')$, where the entries of $\mathbf{w}'$ correspond to arcs supported in S' and the entries of $\mathbf{w}''$ correspond to arcs supported in S''. Then $\mathfrak{L}(g\natural\mathrm{id}_{S''})(e_{\mathbf{w}})$ is a linear combination of basis elements of the form $e_{(\mathbf{v}',\mathbf{w}'')}$, where $\mathbf{v}'$ runs over possible labellings of arcs supported in S'.*

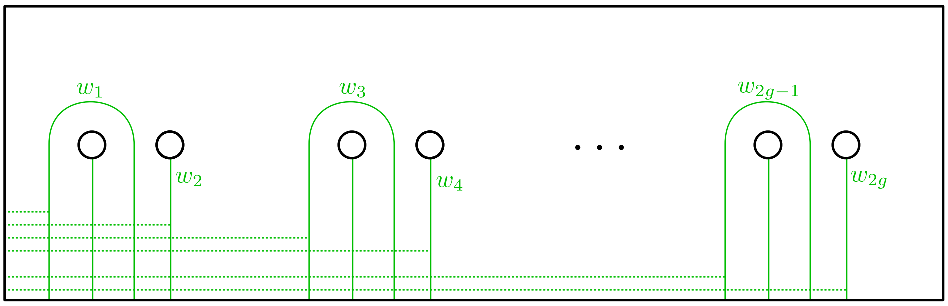
Proof. Let $e'_{\mathbf{v}}$ be an arbitrary dual basis element and write $\mathbf{v} = (\mathbf{v}', \mathbf{v}'')$ similarly to the decomposition $\mathbf{w} = (\mathbf{w}', \mathbf{w}'')$. It suffices to show that $\langle \mathfrak{L}(g\natural\mathrm{id}_{S''})(e_{\mathbf{w}}), e'_{\mathbf{v}}\rangle = 0$ unless $\mathbf{v}'' = \mathbf{w}''$. To see this, recall that the homology class $e_{\mathbf{w}}$ is represented by certain configuration spaces on embedded arcs in S. Since $g\natural\mathrm{id}_{S''}$, by construction, is supported in S', the homology class $\mathfrak{L}(g\natural\mathrm{id}_{S''})(e_{\mathbf{w}})$ may be represented by certain configuration spaces on embedded arcs, which are *identical*, on the boundary connected summand S'', to those representing $e_{\mathbf{w}}$. The intersection pairing with $e'_{\mathbf{v}}$ must therefore be zero unless $\mathbf{v}'' = \mathbf{w}''$. □

3.3.3.2. Interaction with the braiding

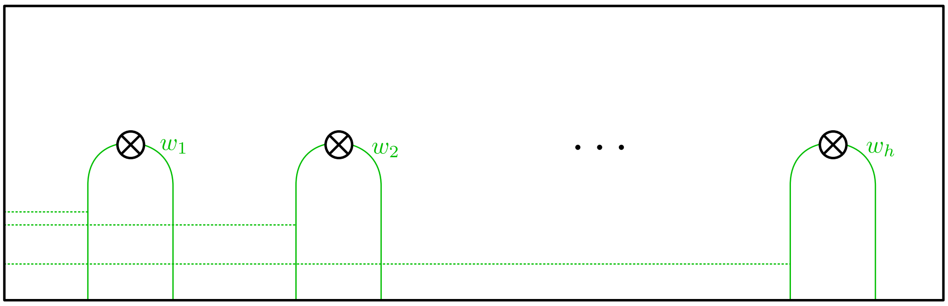
To discuss elements of mapping class groups that act by "braiding" handles or crosscaps of a surface S, it is convenient to pass, in this section, to a different way of representing S diagrammatically. Instead of a rectangle to which we have glued a finite number of strips (as in, for example, Figure 2.1), we will represent S as a rectangle from which we have either erased the interiors of $2g$ discs and glued their boundaries in pairs (when $S = \Sigma_{g,1}$) or erased the interiors of h discs and glued each resulting boundary component to itself by a degree-2 map (when $S = \mathcal{N}_{h,1}$). The basis elements of the representations $\mathfrak{L}_{(\mathbf{k},\ell)}(S)$ (see Figures 2.2c and 2.2d) look as illustrated in Figure 3.5 in this picture, where we have also included explicit choices of "tethers", i.e. paths from a point on the (cycle representing the) homology class to the base configuration. Similarly, the basis elements of the "vertical-type alternative" representations $\mathfrak{L}^{v}_{(\mathbf{k},\ell)}(S)$ (see Figures 2.3c and 2.3d) look as illustrated in Figure 3.6 in this picture.

Notation 3.14 Denote by $\sigma_1 \in \mathrm{MCG}(S)$ the mapping class illustrated in Figure 3.7: it braids the left-most two handles if $S = \Sigma_{g,1}$ and it braids the left-most two crosscaps if $S = \mathcal{N}_{h,1}$.

Denote by $e_{\mathbf{w}}$ the basis elements illustrated in Figure 3.5, where $\mathbf{w} = ([w_1, w_2], [w_3, w_4], \ldots)$ or $\mathbf{w} = ([w_1], [w_2], \ldots)$ in the orientable and non-orientable settings respectively. Similarly, denote by $f_{\mathbf{w}}$ the basis elements illustrated in Figure 3.6.

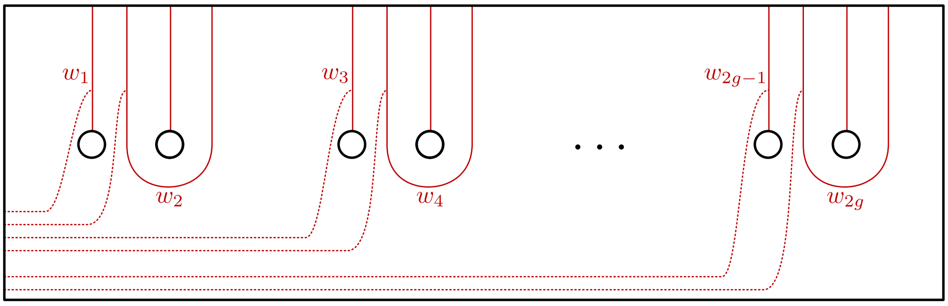


(a) The orientable case.

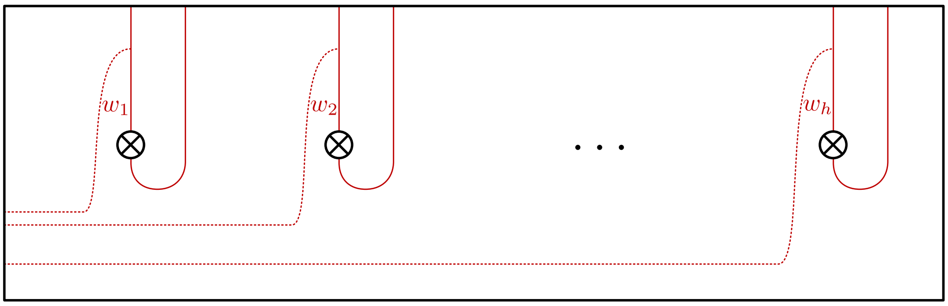


(b) The non-orientable case.

Figure 3.5 Another perspective on the basis elements depicted in Figures 2.2c and 2.2d.



(a) The orientable case.



(b) The non-orientable case.

Figure 3.6 Another perspective on the basis elements depicted in Figures 2.3c and 2.3d.

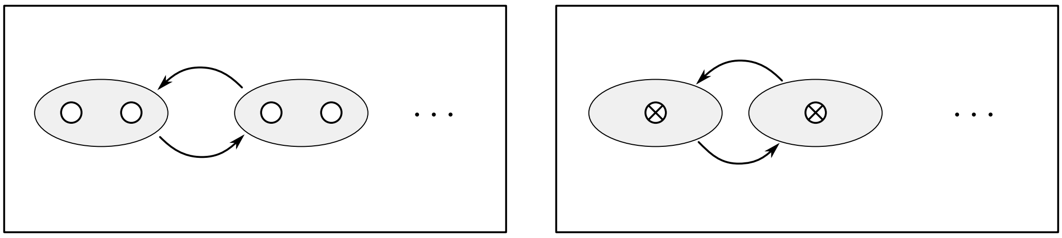


Figure 3.7 The braiding element $\sigma_1 \in \mathrm{MCG}(S)$ when $S = \Sigma_{g,1}$ (left) and $S = \mathbb{N}_{h,1}$ (right).

Lemma 3.15 *The following identities hold in $\mathfrak{L}_{(\mathbf{k},\ell')}(S)$ and $\mathfrak{L}^v_{(\mathbf{k},\ell')}(S)$ where $\ell' \leqslant 2$:*

$$\sigma_1^{-1}\left(e_{([\varnothing,\varnothing],[w_3,w_4],\ldots)}\right) = e_{([w_3,w_4],[\varnothing,\varnothing],\ldots)} \tag{3.10}$$

$$\sigma_1\left(f_{([\varnothing,\varnothing],[w_3,w_4],\ldots)}\right) = f_{([w_3,w_4],[\varnothing,\varnothing],\ldots)} \tag{3.11}$$

$$\sigma_1^{-1}\left(e_{([\varnothing],[w_2],\ldots)}\right) = e_{([w_2],[\varnothing],\ldots)} \tag{3.12}$$

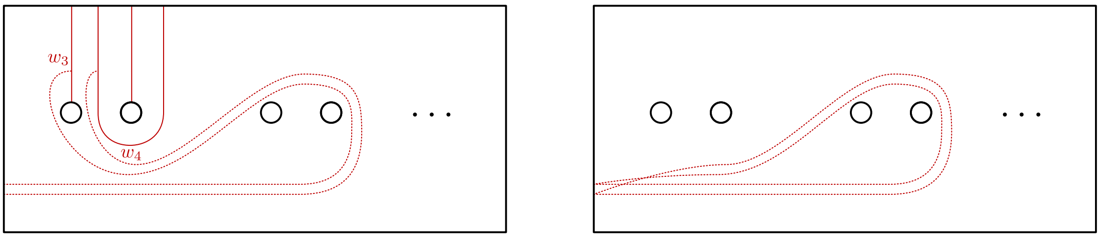
$$\sigma_1\left(f_{([\varnothing],[w_2],\ldots)}\right) = f_{([w_2],[\varnothing],\ldots)} \tag{3.13}$$

Proof. Equations (3.10) and (3.12) are clear from the diagrams, using the fact that the label(s) corresponding to the left-most handle or crosscap are empty. On the other hand, we see from the diagrams that the left-hand side of equation (3.11) is equal to the element illustrated in Figure 3.8a. This differs from the right-hand side of equation (3.11) only in the choice of tether. We therefore need to show that the difference between the two tethers, which is the based loop of configurations illustrated in Figure 3.8b, projects to the trivial element of the group $Q_{(\mathbf{k},\ell')}$. This is obvious for $\ell' = 1$ since the group is trivial in this case. For $\ell' = 2$, we recall from §1.3.3 that $Q_{(\mathbf{k},2)}$ is simply a product of copies of $\mathbb{Z}/2$, one for each block of the partition $\mathbf{k} = \{k_1; \ldots; k_r\}$ with $k_i \geqslant 2$. The projection onto $Q_{(\mathbf{k},2)}$ records the writhe (modulo 2) of each block of strands (in a surface of positive genus the writhe is only well-defined modulo 2). It is clear that the writhe of the loop of configurations illustrated in Figure 3.8b is trivial for each block. This establishes equation (3.11).

We argue similarly for equation (3.13). (Again, the case $\ell' = 1$ being obvious, we just consider $\ell' = 2$.) The left-hand side is equal to the element illustrated in Figure 3.9a, which differs from the right-hand side only by its choice of tether; the difference between the two tethers forms the based loop of configurations illustrated in Figure 3.9b. We therefore just have to show that this projects to the trivial element of the group $Q_{(\mathbf{k},2)}$. This time the group $Q_{(\mathbf{k},2)}$ is a product of $r' + r$ copies of $\mathbb{Z}/2$, where r' denotes the number of blocks of the partition $\mathbf{k}$ with $k_i \geqslant 2$; see §1.3.3. The first r' copies of $\mathbb{Z}/2$, in the projection to $Q_{(\mathbf{k},2)}$ of a loop of configurations, record the writhe of each block of strands; the remaining r copies of $\mathbb{Z}/2$ record, for each block of strands, the number of times modulo 2 that a strand from that block passes through a crosscap. As before, it is clear that the writhe of the loop of configurations illustrated in Figure 3.9b is trivial for each block; thus the first r' coordinates of its projection to $Q_{(\mathbf{k},2)}$ are zero. Moreover, each strand in this loop of configurations passes *around* a crosscap an integer number of times, which corresponds to passing *through* a crosscap an even number of times; thus the last r coordinates of its projection to $Q_{(\mathbf{k},2)}$ are also zero. This establishes equation (3.13). □

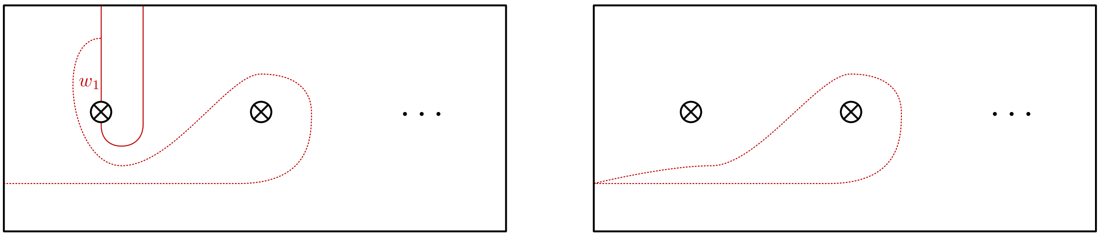
Remark 3.16 Equations (3.10) and (3.12) of Lemma 3.15 hold in fact for all $\ell' \geqslant 1$. On the other hand, we used the explicit structure of the quotient group $Q_{(\mathbf{k},2)}$ (and the fact that $Q_{(\mathbf{k},1)}$ is trivial) to prove equations (3.11) and (3.13). For $\ell' \geqslant 3$ the proof shows that these equations hold up to a unit scalar, which is the image in $Q_{(\mathbf{k},\ell')}$ of the loops in Figures 3.8b and 3.9b respectively. We do not know whether this scalar is trivial in these cases.

Remark 3.17 We will view equations (3.11) and (3.13) as being statements about the action of $(\sigma_1^{-1})^\dagger = \sigma_1$, where $(-)^\dagger$ is the operation that inverts the braiding of a braided monoidal category; see the beginning of §1.1.



(a) The left-hand side of equation (3.11). (b) A loop given by the difference of two tethers.

Figure 3.8 The left-hand side of equation (3.11) differs from the right-hand side of equation (3.11) by the scalar given by the image in $Q_{(\mathbf{k},2)}$ of the loop illustrated on the right.



(a) The left-hand side of equation (3.13). (b) A loop given by the difference of two tethers.

Figure 3.9 The left-hand side of equation (3.13) differs from the right-hand side of equation (3.13) by the scalar given by the image in $Q_{(\mathbf{k},2)}$ of the loop illustrated on the right.

4. Polynomiality of homological representation functors

In this section, we prove our polynomiality results (Theorems B, C and D) for the homological representation functors defined in §1.3. Throughout §4, we consider homological representation functors indexed by a partition $\mathbf{k} = \{k_1; \ldots ; k_r\} \vdash k$ of an integer $k \geqslant 1$ (corresponding to some configuration space of points) and by the stage $\ell \geqslant 1$ of the lower central series defining it. We recall that we use Notation 2.8 for the bases of the representation modules, Notation 3.6 for certain sets of partitions and Convention 3.7 about modifying these functors by change of ground ring.

4.1. For classical braid group functors

4.1.1. For the classical homological representation functors

The aim of this section is to prove the polynomiality results of Theorem B for each $(\mathbf{k}, \ell)$-Lawrence-Bigelow functor $\mathfrak{LB}_{(\mathbf{k},\ell)}$ and its untwisted version $\mathfrak{LB}^u_{(\mathbf{k},\ell)}$ of §1.3.1. The crucial step consists in constructing the key short exact sequences (4.1) and (4.2) of Theorem 4.2 below. The arguments for all of this work are exactly the same for both $\mathfrak{LB}_{(\mathbf{k},\ell)}$ and $\mathfrak{LB}^u_{(\mathbf{k},\ell)}$: for the sake of simplicity, we therefore use the notation $\mathfrak{LB}^\star_{(\mathbf{k},\ell)}$ with $\star \in \{\ , u\}$ and make all further reasoning with this notation.

Notation 4.1 Let us denote by $\mathfrak{LB}_{(0,\ell)} \colon \mathfrak{U}\boldsymbol{\beta} \to \mathbb{Z}[Q_{(1,\ell)}(\mathbb{D})]$ the subobject of the constant functor at $\mathbb{Z}[Q_{(1,\ell)}(\mathbb{D})]$ with $\mathfrak{LB}_{(0,\ell)}(0) = \mathfrak{LB}_{(0,\ell)}(1) = 0$ and $\mathfrak{LB}_{(0,\ell)}(n) = \mathbb{Z}[Q_{(1,\ell)}(\mathbb{D})]$ for $n \geqslant 2$. This choice is driven by consistency with the definition of the Lawrence-Bigelow functors, in that $\mathfrak{LB}_{(\mathbf{k},\ell)}(0) = \mathfrak{LB}_{(\mathbf{k},\ell)}(1) = 0$.

We recall from Corollary 2.7 that, for each $n \in \mathbb{N}$, the $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]$-module $\mathfrak{LB}^\star_{(\mathbf{k},\ell)}(n)$ has a free generating set over $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]$ indexed by the set of tuples $(w_1, \ldots , w_{n-1})$ of words in the alphabet $\{1, 2, \ldots , r\}$ such that each letter $i \in \{1, 2, \ldots , r\}$ appears precisely k_i times; this condition is indicated by the notation $(w_1, \ldots , w_{n-1}) \vdash \mathbf{k}$.

Therefore, we deduce from the definitions of the evanescence and difference functors that $\kappa_1\mathfrak{LB}^\star_{(\mathbf{k},\ell)}(n) = 0$ and that $\delta_1\mathfrak{LB}^\star_{(\mathbf{k},\ell)}(n)$ is the free $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]$-module with generating set indexed the tuples such that $|w_1| \geqslant 1$. Following the preliminary part of §3.3.1, for each $n \geqslant 1$, we denote by $(\mathfrak{p}_{(\mathbf{k},\ell)})_n$ the $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]$-module isomorphism (3.4):

$$\delta_1\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}(n) \xrightarrow{\cong} \bigoplus_{1\leqslant j\leqslant r} \tau_1\mathfrak{LB}^{\star}_{(\mathbf{k}_j,\ell)}(n).$$

More precisely, it is explicitly defined on the generators by $(jw'_0,\ldots,w_n)\mapsto(j,(w'_0,\ldots,w_n))$, where $(j,(w'_0,\ldots,w_n))$ denotes the generator $(w'_0,\ldots,w_n)$ of the summand $\tau_1\mathfrak{LB}^{\star}_{(\mathbf{k}_j,\ell)}(n)$. We also set $(\mathfrak{p}_{(\mathbf{k},\ell)})_0$ to be the trivial morphism. Then we prove that:

Theorem 4.2 *The exact sequence* (3.1) *induces the short exact sequences:*

$$0 \longrightarrow \mathfrak{LB}_{(\mathbf{k},\ell)} \longrightarrow \tau_1\mathfrak{LB}_{(\mathbf{k},\ell)} \longrightarrow \bigoplus_{1\leqslant j\leqslant r} \tau_1\mathfrak{LB}_{(\mathbf{k}_j,\ell)} \longrightarrow 0 \tag{4.1}$$

$$0 \longrightarrow \mathfrak{LB}^u_{(\mathbf{k},\ell)} \longrightarrow \tau_1\mathfrak{LB}^u_{(\mathbf{k},\ell)} \longrightarrow \bigoplus_{1\leqslant j\leqslant r} \tau_1\mathfrak{LB}^u_{(\mathbf{k}_j,\ell)} \longrightarrow 0 \tag{4.2}$$

in $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]\text{-Mod}^{\star})$ *and* $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{Z}[Q^u_{(\mathbf{k},\ell)}(\mathbb{D})]\text{-Mod})$ *respectively. Furthermore, the short exact sequences* (4.1) *and* (4.2) *still hold after any (non-zero) change of rings operation.*

Proof. The strategy consists in showing that the isomorphisms $\{(\mathfrak{p}_{(\mathbf{k},\ell)})_n\}_{n\in\mathbb{N}}$ define an isomorphism $\mathfrak{p}_{(\mathbf{k},\ell)}\colon \delta_1\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)} \xrightarrow{\sim} \bigoplus_{1\leqslant j\leqslant r}\tau_1\mathfrak{LB}^{\star}_{(\mathbf{k}_j,\ell)}$ in $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]\text{-Mod}^{\star})$.

We start by proving that assembling these isomorphisms defines an isomorphism in the category $\mathbf{Fct}(\boldsymbol{\beta},\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]\text{-Mod}^{\star})$, in other words that each $(\mathfrak{p}_{(\mathbf{k},\ell)})_n$ is a $\mathbf{B}_n$-module morphism. The braid group $\mathbf{B}_n$ being trivial for $n=0$ and $n=1$, we assume that $n\geqslant 2$ and prove the commutation of $(\mathfrak{p}_{(\mathbf{k},\ell)})_n$ with respect to the action of any Artin generator σ_i of $\mathbf{B}_n$ with $1\leqslant i\leqslant n-1$. First, we note from Corollary 2.7 that the morphisms $\tau_1\mathfrak{LB}^{\star}_{(\mathbf{k}_j,\ell)}(\sigma_i)$ for all $1\leqslant j\leqslant r$ and $\delta_1\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}(\sigma_i)$ are defined by the action of the generator σ_{i+1} on the Borel-Moore homology classes on the graph $\mathbb{I}_{1+n}$ (see Figure 2.1a with $g=0$). If $i\geqslant 2$, the generator σ_{i+1} acts trivially on the first edge $(1,2)$ of $\mathbb{I}_{1+n}$ and thus does not affect the first entries of the tuples $(w'_0,\ldots,w_{n-1})$ and $(jw'_0,\ldots,w_{n-1})$, because its action is concentrated in the subsurface containing $\mathbb{I}_n$ and cutting the edge $(1,2)$ out. A fortiori, by definition of $(\mathfrak{p}_{(\mathbf{k},\ell)})_n$, we have $(\mathfrak{p}_{(\mathbf{k},\ell)})_n\circ\delta_1\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}(n)(\sigma_i)=(\bigoplus_{1\leqslant j\leqslant r}\tau_1\mathfrak{LB}^{\star}_{(\mathbf{k}_j,\ell)}(\sigma_i))\circ(\mathfrak{p}_{(\mathbf{k},\ell)})_n$ for $i\geqslant 2$. So the remaining point is to check the action of σ_1. We note that the respective actions of $\tau_1\mathfrak{LB}^{\star}_{(\mathbf{k}_j,\ell)}(\sigma_1)$ and $\delta_1\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}(\sigma_1)$ on the generators $(w'_0,\ldots,w_n)$ and $(jw'_0,\ldots,w_n)$ are precisely those illustrated by the right-hand side and left-hand side of Figure 3.1 (with $g=0$): the labelled green arcs are the images of the first edge $(1,2)$ under the action of σ_2 while the other edges are concentrated in the blue shaded region. It thus follows from Lemma 3.9 and the definition of $(\mathfrak{p}_{(\mathbf{k},\ell)})_n$ that $(\mathfrak{p}_{(\mathbf{k},\ell)})_n\circ\delta_1\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}(n)(\sigma_1)=(\bigoplus_{1\leqslant j\leqslant r}\tau_1\mathfrak{LB}^{\star}_{(\mathbf{k}_j,\ell)}(\sigma_1))\circ(\mathfrak{p}_{(\mathbf{k},\ell)})_n$. Therefore, $\mathfrak{p}_{(\mathbf{k},\ell)}$ is a natural transformation in $\mathbf{Fct}(\boldsymbol{\beta},\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]\text{-Mod}^{\star})$.

We now prove that $\mathfrak{p}_{(\mathbf{k},\ell)}$ is a natural transformation in $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]\text{-Mod}^{\star})$ by using the strategy of Lemma 1.2. We fix an integer $n\geqslant 1$, the proof being trivial for $n=0$. We compute from (1.1) that $\tau_1\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}([1,\mathrm{id}_{1+n}])=\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}(\sigma_1^{-1})\circ\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}([1,\mathrm{id}_{2+n}])$. (Here, we canonically identify the Artin generator σ_1 with the braiding $b^{\boldsymbol{\beta}}_{1,1}\colon 1\natural 1\cong 1\natural 1$.) We recall from the description of (3.3) that the morphism $\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}([1,\mathrm{id}_{2+n}])$ is the map induced by the embedding of $\mathbb{I}_{1+n}$ into $\mathbb{I}_{2+n}$ defined by sending each edge $(i,i+1)$ for $1\leqslant i\leqslant n$ to the edge $(i+1,i+2)$. Hence there is no configuration point on the first edge $(1,2)$ of $\mathbb{I}_{2+n}$ in the image of $\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}([1,\mathrm{id}_{2+n}])$, while the morphism $\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}(\sigma_1^{-1})$ is defined by the action of σ_1^{-1} on $\mathbb{I}_{2+n}$ as illustrated on the left-hand side of Figure 3.3. Therefore, it follows from Lemma 3.10 that $\delta_1\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}([1,\mathrm{id}_{1+n}])(jw'_0,\ldots,w_n)$ and $\tau_1\mathfrak{LB}^{\star}_{(\mathbf{k}_j,\ell)}(n)([1,\mathrm{id}_{1+n}])(w_0,\ldots,w_n)$ for each $1\leqslant j\leqslant r$ are both equal to $\sum_{u_1v_1=w_1}(ju_1,v_1,w_2,\ldots,w_n)$. Then, it follows from the definition of $(\mathfrak{p}_{(\mathbf{k},\ell)})_n$ and a clear induction on the integers $m\geqslant 1$ that $(\bigoplus_{1\leqslant j\leqslant r}\tau_1\mathfrak{LB}^{\star}_{(\mathbf{k}_j,\ell)}([m,\mathrm{id}_{m+n}]))\circ(\mathfrak{p}_{(\mathbf{k},\ell)})_n$ is equal to $(\mathfrak{p}_{(\mathbf{k},\ell)})_{m+n}\circ\delta_1\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}([m,\mathrm{id}_{m+n}])$. Hence Relation (1.3) is satisfied for all $n\geqslant 1$, which provides, by Lemma 1.2, the short exact sequences (4.1) and (4.2) of functors on $\mathfrak{U}\boldsymbol{\beta}$.

For all the above work, whenever we consider a *twisted* homological representation functor, we stress that the associated action of $\delta_1\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}$ on the ground ring $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]$ does not affect the reasoning, since $\bigoplus_{1\leqslant j\leqslant r}\tau_1\mathfrak{LB}^{\star}_{(\mathbf{k}_j,\ell)}$ is automatically equipped with the same action (via the implicit change of rings of Convention 3.7 for each summand; see Observation 1.11).

Finally, the results for the change of rings operations follow from the right-exactness of a change of rings operation and Lemma 3.3. □

Remark 4.3 There is no clear splitting for the short exact sequences (4.1) and (4.2). For instance, let us assume that $\Delta_1\mathfrak{LB}_{(\mathbf{k},\ell)}$ admits the obvious splitting Ψ induced by sending each generator $(w_0,\ldots,w_n)\in\tau_1\mathfrak{LB}_{(\mathbf{k}_j,\ell)}(n)$ where $1\leqslant j\leqslant r$ to $(jw_0,\ldots,w_n)$. Then, it clearly follows from the module structures and actions (see §2.3) that some generators of type $(0,w_1,\ldots,w_n)\in\mathfrak{LB}_{(\mathbf{k},\ell)}(n)$ appear in the decomposition of $\tau_1\mathfrak{LB}_{(\mathbf{k},\ell)}(\sigma_1)(jw_0,\ldots,w_n)$ while $(jw_0,\ldots,w_n)\in\delta_1\mathfrak{LB}_{(\mathbf{k},\ell)}(n)$. Therefore the action of $\tau_1\mathfrak{LB}_{(\mathbf{k},\ell)}(\sigma_1)$ mixes generators of $\mathfrak{LB}_{(\mathbf{k},\ell)}(n)$ and of $\delta_1\mathfrak{LB}_{(\mathbf{k},\ell)}(n)$, contradicting the assumption that Ψ is a splitting.

Proof of Theorem B. Using the commutation property of the difference functor δ_1 and translation functor τ_1, we deduce from Theorem 4.2 that, for all $0\leqslant m\leqslant k$, $\delta_1^m\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}$ is a direct sum of functors of type $\tau_1^m\mathfrak{LB}^{\star}_{\mathbf{k}_{m'}}$ where $\mathbf{k}_{m'}\in\{\mathbf{k}-m'\}$ for $0\leqslant m'\leqslant m$, while $\kappa_1\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}=0$.

For $k=1$, we note that $\tau_1\mathfrak{LB}^{\star}_{(0,\ell)}$ is the subobject of the constant functor at $\mathbb{Z}[Q_{(1,2)}(\mathbb{D})]$ with $\tau_1\mathfrak{LB}^{\star}_{(0,\ell)}(0)=0$ and $\tau_1\mathfrak{LB}^{\star}_{(0,\ell)}(n)=\mathbb{Z}[Q_{(1,2)}(\mathbb{D})]$ for $n\geqslant 1$. Hence $\mathfrak{LB}^{\star}_{(1,\ell)}$ is weak polynomial of degree 1 and $\delta_1^2\mathfrak{LB}^{\star}_{(1,\ell)}$ is the functor whose unique non-null value is $\delta_1^2\mathfrak{LB}^{\star}_{(1,\ell)}(0)=\mathbb{Z}[Q_{(1,2)}(\mathbb{D})]$. A fortiori $\mathfrak{LB}^{\star}_{(1,\ell)}$ is strong polynomial of degree 2. For $k\geqslant 2$, we first note that for all integers $m\geqslant 2$, $\tau_1^2\mathfrak{LB}^{\star}_{(0,\ell)}=\tau_1^m\mathfrak{LB}^{\star}_{(0,\ell)}$ is a constant functor and thus it is both very strong and weak polynomial of degree 0. Then, it follows from the above description of $\delta_1^m(\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)})$ by a clear induction on k that $\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}$ is both strong and weak polynomial of degree k. Then, noting from Theorem 4.2 that $\kappa_1(\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)})=0$ and again viewing $\delta_1^m(\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)})$ as a direct sum of functors of type $\tau_1^m\mathfrak{LB}^{\star}_{\mathbf{k}_{m'}}$, the commutation property of the evanescence functor κ_1 and the translation functor τ_1 implies that $\kappa_1(\delta_1^m(\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}))=0$ for all integers $1\leqslant m\leqslant k$: this proves that $\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}$ is *very* strong polynomial of degree k. □

Remark 4.4 The functors $\mathfrak{LB}_{(1,2)}\otimes\mathbb{C}[\mathbb{Z}]$ and $\mathfrak{LB}_{(2,2)}\otimes\mathbb{C}[\mathbb{Z}^2]$ correspond to the *reduced Burau functor* $\overline{\mathfrak{Bur}}$ and the *Lawrence-Krammer functor* $\mathfrak{LK}$ defined in [Sou19, §1.2]. The (very) strong polynomiality results of Theorems B and 4.2 recover those of [Sou19, Props. 3.25 and 3.33] for the functors $\mathfrak{LB}_{(1,2)}\otimes\mathbb{C}[\mathbb{Z}]$ and $\mathfrak{LB}_{(2,2)}\otimes\mathbb{C}[\mathbb{Z}^2]$.

4.1.2. For the vertical-type alternatives

We consider the vertical $(\mathbf{k},\ell)$-Lawrence-Bigelow functor $\mathfrak{LB}^{\star,v}_{(\mathbf{k},\ell)}\colon\mathfrak{U}\boldsymbol{\beta}\to\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]\text{-Mod}^{\star}$, with $\star\in\{\ ,u\}$, as introduced in §1.3.1. We recall from Lemma 2.13 that, for each $n\in\mathbb{N}$, the $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]$-module $\mathfrak{LB}^{\star,v}_{(\mathbf{k},\ell)}(n)$ has a free generating set over $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]$ indexed by the set of ("vertical") tuples $(w_1,\ldots,w_{n-1})^v$, which has the same dimension as the (original) $(\mathbf{k},\ell)$-Lawrence-Bigelow functor $\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}(n)$.

The following result shows that this alternative $\mathfrak{LB}^{\star,v}_{(\mathbf{k},\ell)}$ exhibits unexpected interesting behaviour if we study its polynomiality, which thoroughly differs from the functor $\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}$ addressed in §4.1.1. Indeed, it is not strong polynomial, which is a counterintuitive property since the dimensions of the representations it encodes grow in the same polynomial way as those of $\mathfrak{LB}^{\star}_{(\mathbf{k},\ell)}$.

Theorem 4.5 *The vertical $(\mathbf{k},\ell)$-Lawrence-Bigelow functor $\mathfrak{LB}^{\star,v}_{(\mathbf{k},\ell)}\colon\mathfrak{U}\boldsymbol{\beta}\to\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]\text{-Mod}^{\star}$ is not strong polynomial, but it is weak polynomial of degree 0.*

Proof. It follows from Figure 2.3a and Lemma 3.11 that $\delta_1\mathfrak{LB}^{\star,v}_{(\mathbf{k},\ell)}(n)$ is a non-trivial $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]$-module for all $n\geqslant 1$, while $\delta_1\mathfrak{LB}^{\star,v}_{(\mathbf{k},\ell)}$ assigns the trivial map for all morphisms of $\mathfrak{U}\boldsymbol{\beta}(n,m)$ with $n\neq m$. So $\delta_1\mathfrak{LB}^{\star,v}_{(\mathbf{k},\ell)}$ is isomorphic to a direct sum of infinitely many *atomic* functors (i.e. functors that are non-trivial just in one entry). Hence $\delta_1^m\mathfrak{LB}^{\star,v}_{(\mathbf{k},\ell)}\neq 0$ for any $m\in\mathbb{N}$ while $\pi_{\mathfrak{U}\boldsymbol{\beta}}(\delta_1\mathfrak{LB}^{\star,v}_{(\mathbf{k},\ell)})=0$ in the stable category $\mathbf{St}(\mathfrak{U}\boldsymbol{\beta},\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]\text{-Mod}^{\star})$, whence the result. □

Remark 4.6 On the other hand, the first step of the proof of Theorem 4.2 does go through in the vertical setting, and shows that the analogue of the short exact sequences (4.1) and (4.2) for $\mathfrak{LB}^{\star,v}_{(\mathbf{k},\ell)}$ is true at the level of automorphism groups, i.e. as functors defined on $\boldsymbol{\beta}$.

4.2. For surface braid group functors

4.2.1. For the classical homological representation functors

We deal here with the functors $\mathfrak{L}_{(\mathbf{k},\ell)}(\Sigma_{g,1})$, $\mathfrak{L}^{u}_{(\mathbf{k},\ell)}(\Sigma_{g,1})$, $\mathfrak{L}_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ and $\mathfrak{L}^{u}_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ of §1.3.2 for any $\mathbf{k} \vdash k \geqslant 1$, $\ell \geqslant 1$, $g \geqslant 1$ and $h \geqslant 1$. We start by proving decomposition results for the translations of these homological representation functors; see Theorem 4.10. The arguments for our work in this section are analogous regardless of which of the homological representation functors among the above list we consider. For the sake of simplicity, and to avoid repetition, we pool the key steps and common arguments for all of them for the remainder of §4.2, only emphasising the (minor) differences when necessary. We use the standard notation $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$ with $\star \in \{\ \ ,u\}$ for any of the homological representation functors under consideration, $Q_{(\mathbf{k},\ell)}(S)$ for the associated transformation group, S for either $\Sigma_{g,1}$ or $\mathcal{N}_{h,1}$ and $\boldsymbol{\beta}^{S}$ for the associated groupoid.

We recall from Corollary 2.7 that, for each integer $n \geqslant 1$, the $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]$-module $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(n)$ has a free generating set over the group ring $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]$ indexed by the set of tuples of the form (2.6) if $S = \Sigma_{g,1}$ and (2.7) if $S = \mathcal{N}_{h,1}$. Therefore, we deduce from the definition of the difference functor that $\delta_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(n)$ is the free $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]$-module with generating set indexed the tuples such that $|w_0| \geqslant 1$ for $n \geqslant 1$. Also, it follows from the definition of the evanescence functor and these module structures that $\kappa_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)} = 0$.

It will be convenient to consider various "cut verions" of functors defined on $\langle \boldsymbol{\beta}, \boldsymbol{\beta}^{S} \rangle$. In fact this definition makes sense more generally:

Definition 4.7 Let $\mathcal{C}$ be a category whose objects form a totally-ordered set and there are no morphisms $a \to b$ in $\mathcal{C}$ if $a > b$. Given such a category $\mathcal{C}$, a functor $F \colon \mathcal{C} \to R\text{-Mod}^{\star}$ and an object c of $\mathcal{C}$, define $F_{|\geqslant c} \colon \mathcal{C} \to R\text{-Mod}^{\star}$ on objects by $F_{|\geqslant c}(a) = F(a)$ for $a \geqslant c$ and $F_{|\geqslant c} = 0$ for $a < c$ and on morphisms by $F_{|\geqslant c}(\phi) = F(\phi)$ if the domain of ϕ is $\geqslant c$ and $F_{|\geqslant c}(\phi) = 0$ otherwise.

In the case of $\mathcal{C} = \langle \boldsymbol{\beta}, \boldsymbol{\beta}^{S} \rangle$, the objects form the totally-ordered set $\mathbb{N}$. These alterations are negligible for our study of polynomiality (see the proof of Theorem C), while these subfunctors are much more convenient to deal with (see Remark 4.12).

For each $n \geqslant 2$, following the preliminary part of §3.3.1, we denote by $(\mathfrak{p}_{(\mathbf{k},\ell)})_n$ the $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]$-module isomorphism (3.4), which may be written as

$$\delta_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(n) \xrightarrow{\cong} \bigoplus_{1 \leqslant j \leqslant r} (\tau_1 \mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)})_{|\geqslant 2}(n) \tag{4.3}$$

since the truncations do not make any difference when $n \geqslant 2$. Recal that this isomorphism is defined on the generators by $(jw'_0, \ldots) \mapsto (j, (w'_0, \ldots))$, where $(j, (w'_0, \ldots))$ denotes the generator $(w'_0, \ldots)$ of the j-th summand on the right-hand side. We also set $(\mathfrak{p}_{(\mathbf{k},\ell)})_0$ to be the trivial morphism; this gives an isomorphism of the form (4.3) for $n = 0$. However, for $n = 1$ there is no isomorphism of the form (4.3), since the right-hand side is zero, whereas we have $\delta_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(1) \cong \tau_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(1)$ as representations of $\mathbf{B}_1(S)$ over $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{D})]$.

Our first goal in this section is to promote (4.3) to a natural isomorphism of functors on $\langle \boldsymbol{\beta}, \boldsymbol{\beta}^{S} \rangle$, so we first need to correct the right-hand side on the object $n = 1$. To do this, we will choose a certain extension of functors. We will choose this via the following lemma:

Lemma 4.8 *Let $\mathcal{M}$ be a module over a braided monoidal groupoid $\mathcal{G}$ that on objects is given by the monoid $\mathbb{N}$ as a module over itself. Let $F, G \colon \langle \mathcal{G}, \mathcal{M} \rangle \to R\text{-Mod}^{\star}$ be two functors with $F(n) = 0$ for $n \leqslant c-1$ and $G(n) = 0$ for $n \geqslant c$ for an integer $c \geqslant 1$. Then there is a one-to-one correspondence between extensions of G by F, i.e. short exact sequences $0 \to F \to ? \to G \to 0$, and morphisms $G(c-1) \to F(c)$ in $R\text{-Mod}^{\star}$, given by evaluating the extension functor at $[1, \mathrm{id}_c]$.*

Proof. Since F and G have disjoint support, there is no choice about the action of any such extension on objects and on automorphisms; in other words, there is a unique extension of G by F when restricting the domain to the subgroupoid $\mathcal{G}$. Lemma 1.1 describes the data and conditions required to extend a functor on $\mathcal{G}$ to a functor on $\langle \mathcal{G}, \mathcal{M} \rangle$. In light of the requirement that this is an extension of G by F, the only remaining choice is the value assigned to the morphism $[1, \mathrm{id}_c]$; conversely, any such choice determines an extension. □

Definition 4.9 Denote by $\widetilde{\bigoplus}_{1\leqslant j\leqslant r}(\tau_1\mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)})_{|\geqslant 2}$ the extension of the atomic functor $\tau_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(1)$ by the functor $\bigoplus_{1\leqslant j\leqslant r}(\tau_1\mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)})_{|\geqslant 2}$ whose value on $[1,\mathrm{id}_2]$ is:

$$\begin{array}{ccccc} \tau_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(1) & \xrightarrow{\tau_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}([1,\mathrm{id}_2])} & \tau_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(2) & \xrightarrow{\Delta_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(2)} & \delta_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(2) \\ & & & & \big\downarrow (4.3) \\ & & & & \bigoplus_{1\leqslant j\leqslant r}(\tau_1\mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)})_{|\geqslant 2}(2). \end{array}$$

We also denote by $(\mathfrak{p}_{(\mathbf{k},\ell)})_1$ the isomorphism

$$\delta_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(1) \xrightarrow{(\Delta_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(1))^{-1}} \tau_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(1) = \widetilde{\bigoplus_{1\leqslant j\leqslant r}}(\tau_1\mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)})_{|\geqslant 2}(1). \tag{4.4}$$

Using the extension of Definition 4.9, we may now upgrade (4.3) to an isomorphism of functors:

Theorem 4.10 *For $S=\Sigma_{g,1}$ or $\mathcal{N}_{h,1}$, $\star\in\{\ \ ,u\}$ and $\ell\geqslant 1$, the exact sequence (3.1) induces a short exact sequence*

$$0 \longrightarrow \mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(S) \longrightarrow \tau_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(S) \longrightarrow \widetilde{\bigoplus_{1\leqslant j\leqslant r}}(\tau_1\mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)})_{|\geqslant 2}(S) \longrightarrow 0 \tag{4.5}$$

of functors in

- $\mathbf{Fct}(\langle\boldsymbol{\beta},\boldsymbol{\beta}^S\rangle,\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]\text{-Mod})$ *if $\ell\leqslant 2$;*
- $\mathbf{Fct}(\langle\boldsymbol{\beta},\boldsymbol{\beta}^S\rangle,\mathbb{Z}[Q^u_{(\mathbf{k},\ell)}(S)]\text{-Mod})$ *if $\ell\geqslant 3$ and $\star=u$;*
- $\mathbf{Fct}(\langle\boldsymbol{\beta},\boldsymbol{\beta}^S\rangle,\mathbb{Z}\text{-Mod})$ *if $\ell\geqslant 3$ and $\star=\ \ $.*

This holds also after any (non-zero) change of rings operation.

Proof. The roadmap of this proof is similar to that of Theorem 4.2, whose technical computations are also reused below for the analogous steps. As in the proof of Theorem 4.2, we stress that, when we deal with a *twisted* homological representation functor, the action on the ground ring $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]$ does not affect any point of the following work thanks to Convention 3.7.

We first consider the case where $n\geqslant 2$. Recall that we use [PS23, Prop. 2.2] for the generating set of $\mathbf{B}_n(S)$ and that we have introduced model graphs $\mathbb{I}_n$, $\mathbb{W}^{\Sigma}_g$ and $\mathbb{W}^{\mathcal{N}}_h$ in Notation 2.8, which are illustrated in Figures 2.1a and 2.1b. Let us abbreviate by writing $\mathbb{W}^S=\mathbb{W}^{\Sigma}_g$ if $S=\Sigma_{g,1}$ and $\mathbb{W}^S=\mathbb{W}^{\mathcal{N}}_h$ if $S=\mathcal{N}_{h,1}$. For each generator γ of $\mathbf{B}_n(S)$, the morphisms $(\tau_1\mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)})_{|\geqslant 2}(\gamma)$ for all $1\leqslant j\leqslant r$ and $\delta_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(\gamma)$ are induced by the action of $\mathrm{id}_1\natural\gamma$ on the Borel-Moore homology classes supported on the embedded graph $\mathbb{I}_{1+n}\vee\mathbb{W}^S\subset S$.

If $\gamma\neq\sigma_1$, the action of $\mathrm{id}_1\natural\gamma$ is supported on a subsurface containing the graph $\mathbb{I}_n\vee\mathbb{W}^S$ and disjoint from the left-most edge $(1,2)$, which therefore does not affect the first entries of the tuples $(w'_0,\ldots)$ and $(jw'_0,\ldots)$. Thus $(\tau_1\mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)})_{|\geqslant 2}(\gamma)$ does not interact with the action of $(\mathfrak{p}_{(\mathbf{k},\ell)})_n$ and a fortiori we have $(\mathfrak{p}_{(\mathbf{k},\ell)})_n\circ\delta_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(\gamma)=\big(\widetilde{\bigoplus}_{1\leqslant j\leqslant r}(\tau_1\mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)})_{|\geqslant 2}(\gamma)\big)\circ(\mathfrak{p}_{(\mathbf{k},\ell)})_n$. Hence the only remaining fact to check is that this last equality also holds for σ_1, which is done by repeating verbatim the corresponding point in the proof of Theorem 4.2, using Lemma 3.9. Furthermore, the analogous relations trivially hold also for $n=0$ (because $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(0)=0$) and $n=1$ (because the isomorphism $(\mathfrak{p}_{(\mathbf{k},\ell)})_1=(4.4)$ is $\mathbf{B}_1(S)$-equivariant by construction). Therefore, the collection of isomorphisms $\{(\mathfrak{p}_{(\mathbf{k},\ell)})_n\}_{n\in\mathbb{N}}$ assembles into a natural isomorphism $\mathfrak{p}_{(\mathbf{k},\ell)}$ of functors on $\boldsymbol{\beta}^S$ from (the restriction to $\boldsymbol{\beta}^S$ of) $\delta_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}$ to (the restriction to $\boldsymbol{\beta}^S$ of) $\widetilde{\bigoplus}_{1\leqslant j\leqslant r}(\tau_1\mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)})_{|\geqslant 2}$.

Since the canonical inclusion $\boldsymbol{\beta}^S\hookrightarrow\langle\boldsymbol{\beta},\boldsymbol{\beta}^S\rangle$ is the identity on objects, to check that $\mathfrak{p}_{(\mathbf{k},\ell)}$ extends to a natural isomorphism of functors on $\langle\boldsymbol{\beta},\boldsymbol{\beta}^S\rangle$, it suffices to check certain additional relations. Specifically, by Lemma 1.2, it suffices to check Relation (1.3), which we do now. First, we compute from (1.1) that $\tau_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}([1,\mathrm{id}_{1+n}])=\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}(\sigma_1^{-1})\circ\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}([1,\mathrm{id}_{2+n}])$ (using the canonical identification $b^{\boldsymbol{\beta}}_{1,1}=\sigma_1$). We recall from the description of (3.3) that $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}([1,\mathrm{id}_{2+n}])$

is induced by the embedding of $\mathbb{I}_{1+n}$ into $\mathbb{I}_{2+n}$ defined by sending each edge $(i,i+1)$ to the edge $(i+1,i+2)$ and by the identity on the wedge $\mathbb{W}^S$. Using Lemma 3.10 with the illustration of Figure 3.3, the analogous point in the proof of Theorem 4.2 repeats mutatis mutandis here. This proves that $(\mathfrak{p}_{(\mathbf{k},\ell)})_{m+n} \circ \delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2}([m,\mathrm{id}_{m+n}]) = \left(\widetilde{\bigoplus}_{1\leqslant j\leqslant r}(\tau_1\mathfrak{L}^\star_{(\mathbf{k}_j,\ell)})_{|\geqslant 2}([m,\mathrm{id}_{m+n}])\right) \circ (\mathfrak{p}_{(\mathbf{k},\ell)})_n$ for each $m \geqslant 1$ and $n \geqslant 2$. The analogous relation follows in the exact same way for $n=1$, the point being that $\widetilde{\bigoplus}_{1\leqslant j\leqslant r}(\tau_1\mathfrak{L}^\star_{(\mathbf{k}_j,\ell)})_{|\geqslant 2}([1,\mathrm{id}_2]) = \delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2}([1,\mathrm{id}_2]) \circ \Delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2}(1)$; the key relation also holds trivially for $n=0$ because $\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2}(0)=0$. Hence Relation (1.3) is satisfied for all $n\in\mathbb{N}$, and we deduce the short exact sequence (4.5) from Lemma 1.2.

Finally, the result for the change of rings operations follows from the right-exactness of any change of rings operation and Lemma 3.3. □

Remark 4.11 Since the effect of the $\mathbf{B}_n(S)$-action on $\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2}(n)$ on the $\mathbb{I}_n$-supported components of the basis elements is analogous to the $\mathbf{B}_n$-action on $\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2}(n)$ (see Corollary 2.7 and Figures 2.1a and 2.1b), it follows from the same argument as in Remark 4.3 that there is no obvious splitting for the short exact sequence (4.5).

Proof of Theorem C. We proceed by induction on $k\geqslant 0$. We first note that the functor $\mathfrak{L}^\star_{(0,\ell)}$ is very strong polynomial of degree 0 since it is constant.

Now, for $k\geqslant 1$, we have by Theorem 4.10 that the difference functor $\delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2}$ is an extension of the atomic functor $\tau_1\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2}(1)$ by a direct sum of functors of type $(\tau_1\mathfrak{L}^\star_{(\mathbf{k}_i,\ell)})_{|\geqslant 2}$ for $\mathbf{k}_i \in \{\mathbf{k}-1\}$. We also recall that $\mathfrak{L}^\star_{(\mathbf{k},\ell)}$ is an extension of the atomic functor $\mathfrak{L}^\star_{(\mathbf{k},\ell)}(1)$ by $\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2}$. Using the commutation property of the difference functor δ_1 and the translation functor τ_1, we see by the inductive assumption that each functor $\tau_1\mathfrak{L}^\star_{(\mathbf{k}_i,\ell)}$ is strong and weak polynomial of degree $k-1$. Also, the atomic functor $\mathfrak{L}^\star_{(\mathbf{k},\ell)}(1)$ is strong polynomial of degree 1 while weak polynomial of degree 0 by definition. We therefore deduce from Proposition 3.1 that the functors $\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2}$ and $\mathfrak{L}^\star_{(\mathbf{k},\ell)}$ are both strong and weak polynomial of degree at most k. Now, in the stable category $\mathbf{St}(\langle \boldsymbol{\beta},\boldsymbol{\beta}^S\rangle, \mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]\text{-Mod}^\star)$, by Theorem 4.10 and the commutation properties of δ_1 with colimits and with $\pi_{\langle \boldsymbol{\beta},\boldsymbol{\beta}^S\rangle}$, the functor $\delta_1^k\pi_{\langle \boldsymbol{\beta},\boldsymbol{\beta}^S\rangle}(\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2})$ surjects onto each $\delta_1^{k-1}\pi_{\langle \boldsymbol{\beta},\boldsymbol{\beta}^S\rangle}((\tau_1\mathfrak{L}^\star_{(\mathbf{k}_i,\ell)})_{|\geqslant 2})$ which is non-null by the inductive assumption. Also, the image $\pi_{\langle \boldsymbol{\beta},\boldsymbol{\beta}^S\rangle}(\mathfrak{L}^\star_{(\mathbf{k},\ell)}(1))$ is null, so $\pi_{\langle \boldsymbol{\beta},\boldsymbol{\beta}^S\rangle}(\mathfrak{L}^\star_{(\mathbf{k},\ell)}) \cong \pi_{\langle \boldsymbol{\beta},\boldsymbol{\beta}^S\rangle}(\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2})$. So, the weak degrees of $\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2}$ and $\mathfrak{L}^\star_{(\mathbf{k},\ell)}$ are both exactly k. Furthermore, the strong degree is always greater than or equal to the weak one: this is a direct consequence of the commutation property $\delta_1\circ\pi_{\langle \boldsymbol{\beta},\boldsymbol{\beta}^S\rangle} = \pi_{\langle \boldsymbol{\beta},\boldsymbol{\beta}^S\rangle}\circ\delta_1$. Hence, the strong degrees of $\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2}$ and $\mathfrak{L}^\star_{(\mathbf{k},\ell)}$ are also both k.

The last property to be checked is that $\kappa_1(\delta_1^m(\mathfrak{L}^\star_{(\mathbf{k},\ell)}))=0$ for all $0\leqslant m\leqslant k$. We have already checked the case of $m=0$, which follows from the evident injectivity of $i_1\mathfrak{L}^\star_{(\mathbf{k},\ell)}(n)\colon \mathfrak{L}^\star_{(\mathbf{k},\ell)}(n) \to \tau_1\mathfrak{L}^\star_{(\mathbf{k},\ell)}(n)$ for each n. We consider the long exact sequence (3.2) associated to the extension $\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2} \hookrightarrow \mathfrak{L}^\star_{(\mathbf{k},\ell)} \twoheadrightarrow \mathfrak{L}^\star_{(\mathbf{k},\ell)}(1)$. We note that $\mathfrak{L}^\star_{(\mathbf{k},\ell)}(1) = \kappa_1(\mathfrak{L}^\star_{(\mathbf{k},\ell)}(1))$ while $\kappa_1\mathfrak{L}^\star_{(\mathbf{k},\ell)}=0$. Using Theorem 4.10, we deduce from the definition of the connecting map of the snake lemma that the cokernel of the injection $\mathfrak{L}^\star_{(\mathbf{k},\ell)}(1) \hookrightarrow \delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 2}$ induced by (3.2) is the (unique) extension of the atomic functor $\delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)}(1)$ (using the projection $\Delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)}(1)$) by the functor $\bigoplus_{1\leqslant j\leqslant r}(\tau_1\mathfrak{L}^\star_{(\mathbf{k}_j,\ell)})_{|\geqslant 2}$ (similar to Definition 4.9 using Lemma 4.8). But this cokernel is also formally isomorphic to $(\delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)})_{|\geqslant 1}$. We thus obtain the short exact sequence $(\delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)})_{|\geqslant 1} \hookrightarrow \delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)} \twoheadrightarrow \delta_1(\mathfrak{L}^\star_{(\mathbf{k},\ell)}(1))$ with the explicit description of $(\delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)})_{|\geqslant 1}$ as an extension, and we consider its long exact sequence (3.2). Since $\delta_1^2(\mathfrak{L}^\star_{(\mathbf{k},\ell)}(1)) = \kappa_1((\delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)})_{|\geqslant 1})=0$ as a direct consequence of the definitions and of the inductive assumption, we have an exact sequence $\kappa_1\delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)} \hookrightarrow \delta_1(\mathfrak{L}^\star_{(\mathbf{k},\ell)}(1)) \to \delta_1((\delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)})_{|\geqslant 1}) \twoheadrightarrow \delta_1^2\mathfrak{L}^\star_{(\mathbf{k},\ell)}$. As above, using Theorem 4.10 and the above description of $(\delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)})_{|\geqslant 1}$, it is routine to check that the middle morphism μ is an injection induced by the map $i_1(\delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)})(0)$. Therefore, we have $\kappa_1\delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)}=0$, i.e. the desired result for $m=1$. Finally, we consider the long exact sequence (3.2) applied to the short exact sequence $\delta_1(\mathfrak{L}^\star_{(\mathbf{k},\ell)}(1)) \hookrightarrow \delta_1((\delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)})_{|\geqslant 1}) \twoheadrightarrow \delta_1^2\mathfrak{L}^\star_{(\mathbf{k},\ell)}$. Since $\delta_1^2(\mathfrak{L}^\star_{(\mathbf{k},\ell)}(1))=0$ and $\kappa_1(\delta_1^2\mathfrak{L}^\star_{(\mathbf{k},\ell)|\geqslant 1}) \cong \delta_1(\mathfrak{L}^\star_{(\mathbf{k},\ell)}(1))$, we deduce that $\kappa_1(\delta_1^2\mathfrak{L}^\star_{(\mathbf{k},\ell)})=0$, i.e. the desired result for $m=2$. Also, by the commutation property of δ_1 with τ_1 and with colimits, it then follows from the above description of $(\delta_1\mathfrak{L}^\star_{(\mathbf{k},\ell)})_{|\geqslant 1}$ as an extension and from Theorem 4.10 that $\delta_1^3\mathfrak{L}^\star_{(\mathbf{k},\ell)} \cong \bigoplus_{1\leqslant j\leqslant j'\leqslant r}\tau_1^2\delta_1\mathfrak{L}^\star_{(\mathbf{k}_{j,j'},\ell)|\geqslant 2}$. Then, by the inductive assumption, each $\tau_1^2\delta_1\mathfrak{L}^\star_{(\mathbf{k}_{j,j'},\ell)|\geqslant 2}$

is a very strong polynomial functor of degree $k-3$ and we deduce that $\kappa_1(\delta_1^m(\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}))=0$ for all $3\leqslant m\leqslant k$, which ends the proof. $\square$

Remark 4.12 We conjecture that Theorem 4.10 repeats verbatim for the entire functors (i.e. not considering the "cut version" subfunctor $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)|\geqslant 2}$) and thus that they are strong and weak polynomial with the same degrees. We have in fact verified this for the functor $\mathfrak{L}_{(k,2)}(\Sigma_{g,1})$ for each $k\geqslant 1$ with the trivial partition $\mathbf{k}=k$ using explicit formulas for the action of $\mathbf{B}_n(\Sigma_{g,1})$ from [PS]. However, this seems much more difficult to prove in full generality for $\mathbf{k}$. The main difficulty in proving the short exact sequences analogous to (4.5) is for the object $n=1$: checking that these are short exact sequences of $\mathbf{B}_1(S)$-modules involves a subtle study of the actions of the generators a_i and b_i of $\mathbf{B}_1(S)$, for which diagrammatic arguments of the type of Lemma 3.9 seem insufficient. This task thus would require technical and heavy further computations. In any case, such finer results would not improve the polynomiality results of Theorem C, which are already optimal.

4.2.2. Vertical-type alternatives

We now focus on the vertical-type alternatives of the homological representation functors for the surface braid groups, namely $\mathfrak{L}^{v}_{(\mathbf{k},2)}(\Sigma_{g,1})$, $\mathfrak{L}^{v}_{(\mathbf{k},\ell)}(\Sigma_{g,1})$ and $\mathfrak{L}^{u,v}_{(\mathbf{k},\ell)}(\Sigma_{g,1})$ for $g\geqslant 1$, and $\mathfrak{L}^{v}_{(\mathbf{k},2)}(\mathcal{N}_{h,1})$, $\mathfrak{L}^{v}_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ and $\mathfrak{L}^{u,v}_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ for $h\geqslant 1$; see §1.3.2. We recall from Lemma 2.13 that, for each of these functors $\mathfrak{L}^{\star,v}_{\mathbf{k}}$ and each $n\in\mathbb{N}$, the $\mathbb{Z}[Q^{\star}_{(\mathbf{k},\ell)}(S)]$-module $\mathfrak{L}^{\star,v}_{\mathbf{k}}(n)$ has a free generating set over $\mathbb{Z}[Q^{\star}_{(\mathbf{k},\ell)}(S)]$ indexed by the set of ("vertical") tuples of type $(w_1,\ldots,[w_{n+2g-2},w_{n+2g-1}])^v$ if $S=\Sigma_{g,1}$ and $(w_1,\ldots,[w_{n+h-1}])^v$ if $S=\mathcal{N}_{h,1}$, which has the same dimension as the (original, non-vertical) module $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(n)$. Similarly to the case of the classical braid groups in Theorem 4.5, the polynomiality properties of these alternatives $\mathfrak{L}^{\star,v}_{\mathbf{k}}$ thoroughly differ from those of their original counterparts studied in §4.2.1:

Theorem 4.13 *For each partition $\mathbf{k}\vdash k\geqslant 1$ and $\ell\geqslant 3$, the functors $\mathfrak{L}^{v}_{(\mathbf{k},1)}(\Sigma_{g,1})$, $\mathfrak{L}^{v}_{(\mathbf{k},2)}(\Sigma_{g,1})$, $\mathfrak{L}^{u,v}_{(\mathbf{k},\ell)}(\Sigma_{g,1})$, $\mathfrak{L}^{v}_{(\mathbf{k},\ell)}(\Sigma_{g,1})$, $\mathfrak{L}^{v}_{(\mathbf{k},1)}(\mathcal{N}_{h,1})$ $\mathfrak{L}^{v}_{(\mathbf{k},2)}(\mathcal{N}_{h,1})$, $\mathfrak{L}^{u,v}_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ and $\mathfrak{L}^{v}_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ are not strong polynomial, but they are weak polynomial of degree 0. These results still hold after any (non-zero) change of rings operation on the functors.*

Proof. The proof is analogous to that of Theorem 4.5: Lemma 3.11 shows that $\delta_1\mathfrak{L}^{\star,v}_{(\mathbf{k},\ell)}$ is a direct sum of infinitely many atomic functors with $\delta_1\mathfrak{L}^{\star,v}_{(\mathbf{k},\ell)}(n)$ non-trivial for all $n\geqslant 1$. $\square$

Remark 4.14 Similarly to Remark 4.6, the first step of the proof of Theorem 4.10 repeats mutatis mutandis for the vertical-type alternatives, inducing a short exact sequence of functors defined on $\boldsymbol{\beta}^S$ analogous to (4.5) for each $\mathfrak{L}^{\star,v}_{(\mathbf{k},\ell)}$.

Fact 4.15 (Dual representation functors.) *We recall from Corollary 2.10 and Remark 2.11 that, for $S:=\mathbb{D}$ or $\Sigma_{g,1}$ or $\mathcal{N}_{h,1}$ with $g,h\geqslant 1$, the $\mathbf{B}_n(S)$-representation $H^{\partial}_k(C_{\mathbf{k}}(\mathbb{D}_n\natural S);\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)])$ introduced in Notation 2.9 is the dual of the $\mathbf{B}_n(S)$-representation $H^{BM}_k(C_{\mathbf{k}}(\mathbb{D}_n\natural S);\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)])$. Gathering these representations and assigning for each morphism $[1,\mathrm{id}_{n+1}]\in\langle\boldsymbol{\beta},\boldsymbol{\beta}^S\rangle$ the same map as that for the vertical-type alternatives $\mathfrak{LB}^{\star,v}_{(\mathbf{k},\ell)}$, $\mathfrak{L}^{\star,v}_{(\mathbf{k},\ell)}(\Sigma_{g,1})$ and $\mathfrak{L}^{\star,v}_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$, it is not difficult to check (via the strategy of Lemma 1.2) that we define functors $\mathfrak{LB}^{\star,\partial}_{(\mathbf{k},\ell)}$, $\mathfrak{L}^{\star,\partial}_{(\mathbf{k},\ell)}(\Sigma_{g,1})$ and $\mathfrak{L}^{\star,\partial}_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ respectively of the form $\langle\boldsymbol{\beta},\boldsymbol{\beta}^S\rangle\to\mathbb{Z}[Q^{\star}_{(\mathbf{k},\ell)}(S)]$-Mod. Then the reasoning and non-polynomiality results of Theorems 4.5 and 4.13 may be repeated verbatim for these functors.*

Alternatively, for orientable surfaces, the functors $f_!\mathfrak{LB}^{\star,v}_{(\mathbf{k},\ell)}\otimes\mathrm{F}(R)$ and $f_!\mathfrak{L}^{\star,v}_{(\mathbf{k},\ell)}(\Sigma_{g,1})\otimes\mathrm{F}(R)$ encode these dual representations, by (2.13), and the non-polynomiality results are then given by Theorems 4.5 and 4.13. Here, $f_!\colon\mathbb{Z}[Q^{\star}_{(\mathbf{k},\ell)}(S)]$-Mod $\to R$-Mod denotes any change of rings operation such that R is an integral domain and the resulting local system over R is u-homogeneous (see Definition 2.14) for $u\in R^{\times}$ such that $[i]_u\neq 0$ for all $i\geqslant 1$. For instance, the case of the Lawrence-Bigelow functors is detailed in Example 2.17 for $f_!=\mathrm{id}$.

4.3. For mapping class group functors

4.3.1. For the classical homological representation functors

In this section, we prove polynomiality properties for the functors $\mathfrak{L}_{(\mathbf{k},1)}(\mathbf{\Gamma})$, $\mathfrak{L}_{(\mathbf{k},2)}(\mathbf{\Gamma})$,[1] $\mathfrak{L}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})$ and $\mathfrak{L}^{u}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})$ defined in §1.3.3 for any $\mathbf{k} \vdash k \geqslant 1$ and $\ell \geqslant 1$. The arguments being analogous for orientable and non-orientable surfaces, we pool the key steps and common arguments for these two cases. We use the standard notation $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$ for all of the above functors, $\mathcal{M}$ for either $\mathcal{M}_2^+$ or $\mathcal{M}_2^-$, S for either $\mathbb{T}$ or $\mathbb{M}$ and $\mathrm{MCG}(S^{\natural n})$ for either $\mathbf{\Gamma}_{n,1}$ or $\boldsymbol{\mathcal{N}}_{n,1}$.

We recall from Corollary 2.7 that, for each $n \in \mathbb{N}$, the $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]$-module $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(n)$ has a free generating set over $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]$ indexed by a certain set of tuples $w = ([w_1, w_2], \ldots, [w_{2n-1}, w_{2n}])$ if $S^{\natural n} = \Sigma_{n,1}$ and $w = ([w_1], \ldots, [w_n])$ if $S^{\natural n} = \mathbb{N}_{n,1}$. It follows from these module structures that $\kappa_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)} = 0$ and $\delta_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(n)$ is the free $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]$-module with generating set indexed the tuples such that $|w_1| + |w_2| \geqslant 1$ if $S^{\natural n} = \Sigma_{n,1}$ respectively $|w_1| \geqslant 1$ if $S^{\natural n} = \mathbb{N}_{n,1}$. For $n \geqslant 1$, we consider the following $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]$-module morphisms.

- If $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)} = \mathfrak{L}_{(\mathbf{k},1)}(\mathbf{\Gamma})$ or $\mathfrak{L}_{(\mathbf{k},2)}(\mathbf{\Gamma})$: for each $1 \leqslant j \leqslant r$, we define two morphisms $\tau_1 \mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)}(n) \hookrightarrow \delta_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(n)$, given by mapping $([w_1, w_2], \ldots, [w_{2n+1}, w_{2n+2}])$ to $([jw_1, 0], \ldots, [w_{2n+1}, w_{2n+2}])$ and $([0, jw_2], \ldots, [w_{2n+1}, w_{2n+2}])$ respectively. Furthermore, for each pair of non-negative integers (j_1, j_2) such that $1 \leqslant j_1 \leqslant j_2 \leqslant r$, we define a morphism $\tau_1 \mathfrak{L}^{\star}_{(\mathbf{k}_{j_1,j_2},\ell)}(n) \hookrightarrow \delta_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(n)$ by mapping $([w_1, w_2], \ldots, [w_{2n+1}, w_{2n+2}])$ to $([jw_1, jw_2], \ldots, [w_{2n+1}, w_{2n+2}])$.
- If $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)} = \mathfrak{L}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})$ or $\mathfrak{L}^{u}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})$: for each $1 \leqslant j \leqslant r$, we define $\tau_1 \mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)}(n) \hookrightarrow \delta_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(n)$ by assigning $([w_1], \ldots, [w_n]) \mapsto ([jw_1], \ldots, [w_n])$.

Finally, for $n = 0$, we set the above morphisms to be the trivial morphism. We denote by $(\mathfrak{i}_{(\mathbf{k},\ell)})_n$ the direct sum over $1 \leqslant j \leqslant r$ of all the above morphisms associated to $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$. To abbreviate, let us denote by $\bigoplus_{\mathbf{k}'} \tau_1 \mathfrak{L}^{\star}_{(\mathbf{k}',\ell)}$ the functor $\bigoplus_{1 \leqslant j_1 \leqslant j_2 \leqslant r} \tau_1 \mathfrak{L}_{(\mathbf{k}_{j_1,j_2},\ell)}(\mathbf{\Gamma}) \oplus \bigoplus_{1 \leqslant j \leqslant r} (\tau_1 \mathfrak{L}_{(\mathbf{k}_j,\ell)}(\mathbf{\Gamma})^{\oplus 2})$ when $S^{\natural n} = \Sigma_{n,1}$ and the functor $\bigoplus_{1 \leqslant j \leqslant r} \tau_1 \mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)}(\boldsymbol{\mathcal{N}})$ when $S^{\natural n} = \mathbb{N}_{n,1}$. It follows from the free generating sets described in Corollary 2.7 that, for each $n \in \mathbb{N}$, the map $(\mathfrak{i}_{(\mathbf{k},\ell)})_n$ is a $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]$-module isomorphism $\bigoplus_{\mathbf{k}'} \tau_1 \mathfrak{L}^{\star}_{(\mathbf{k}',\ell)}(n) \cong \delta_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(n)$. The key technical result (Theorem 4.16) in this section states that this assembles into an isomorphism of functors on $\mathfrak{U}\mathcal{M}$.

As one last preliminary piece of notation, we consider the canonical $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]$-module embedding $(\Delta'_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)})_n \colon \delta_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(n) \hookrightarrow \tau_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(n)$ defined by sending each generating tuple w to itself. The second part of the statement of Theorem 4.16 is that these assemble into a functor that provides a splitting of the (short) exact sequence (3.1).

Theorem 4.16 *For each partition $\mathbf{k} \vdash k \geqslant 1$ and integers $\ell \geqslant 1$ and $\ell' \leqslant 2$, the exact sequence (3.1) induces the following isomorphisms in the functor categories $\mathbf{Fct}(\mathfrak{U}\mathcal{M}_2^+, \mathbb{Z}[Q_{(\mathbf{k},2)}(\mathbb{T})]\text{-Mod})$ and $\mathbf{Fct}(\mathfrak{U}\mathcal{M}_2^-, \mathbb{Z}[Q_{(\mathbf{k},\ell)}(\mathbb{M})]\text{-Mod}^{\star})$ respectively:*

$$\tau_1 \mathfrak{L}_{(\mathbf{k},\ell')}(\mathbf{\Gamma}) \cong \mathfrak{L}_{(\mathbf{k},\ell')}(\mathbf{\Gamma}) \oplus \left(\bigoplus_{\mathbf{k}'' \in \{\mathbf{k}-2\}} \tau_1 \mathfrak{L}_{(\mathbf{k}'',\ell')}(\mathbf{\Gamma}) \right) \oplus \left(\bigoplus_{1 \leqslant j \leqslant r} \tau_1 \mathfrak{L}_{(\mathbf{k}_j,\ell')}(\mathbf{\Gamma}) \right) \oplus \left(\bigoplus_{1 \leqslant j \leqslant r} \tau_1 \mathfrak{L}_{(\mathbf{k}_j,\ell')}(\mathbf{\Gamma}) \right), \tag{4.6}$$

$$\tau_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}}) \cong \mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}}) \oplus \left(\bigoplus_{1 \leqslant j \leqslant r} \tau_1 \mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)}(\boldsymbol{\mathcal{N}}) \right), \tag{4.7}$$

with $\star \in \{\ \ , u\}$. These isomorphisms still hold after any (non-zero) change of rings operation.

Proof. The strategy consists in showing that, in the category $\mathbf{Fct}(\mathfrak{U}\mathcal{M}, \mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]\text{-Mod})$, the morphisms $\{(\mathfrak{i}_{(\mathbf{k},\ell)})_n\}_{n \in \mathbb{N}}$ assemble into an isomorphism $\mathfrak{i}_{(\mathbf{k},\ell)} \colon \bigoplus_{\mathbf{k}'} \tau_1 \mathfrak{L}^{\star}_{(\mathbf{k}',\ell)} \xrightarrow{\sim} \delta_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$, while the morphisms $\{(\Delta'_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)})_n\}_{n \in \mathbb{N}}$ define a section $\Delta'_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)} \colon \delta_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)} \to \tau_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$ of $\Delta_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$.

[1] We recall that $\mathfrak{L}_{(\mathbf{k},2)}(\mathbf{\Gamma}) = \mathfrak{L}_{(\mathbf{k},\ell)}(\mathbf{\Gamma})$ for $\ell \geqslant 3$ and $\mathfrak{L}_{(\mathbf{k},\ell')}(\mathbf{\Gamma}) = \mathfrak{L}^{u}_{(\mathbf{k},\ell')}(\mathbf{\Gamma})$ for $\ell' \leqslant 2$; see §1.3.3 and Lemma 1.8.

First, we prove the commutation of $(\mathfrak{i}_{(\mathbf{k},\ell)})_n$ and $(\Delta'_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)})_n$ with respect to the action of $\mathrm{MCG}(S^{\natural n})$. We recall that we introduce model graphs $\mathbb{W}^{\Sigma}_g$ and $\mathbb{W}^{\mathbb{N}}_h$ in Notation 2.8, which are illustrated in Figures 2.1c and 2.1d. Let us write $\mathbb{W}^S_n = \mathbb{W}^{\Sigma}_{2n}$ in the orientable setting and $\mathbb{W}^S_n = \mathbb{W}^{\mathbb{N}}_n$ in the non-orientable setting. For each generator γ of $\mathrm{MCG}(S^{\natural n})$, the morphisms $\tau_1\mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)}(\gamma)$, $\tau_1\mathfrak{L}^{\star}_{(\mathbf{k}'',\ell)}(\gamma)$ and $\delta_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(\gamma)$ are induced by the action of $\mathrm{id}_1\natural\gamma$ on the Borel-Moore homology classes supported on the embedded graph $\mathbb{W}^S_{1+n} \subset S^{\natural 1+n}$. It follows from a disjoint support argument (Lemma 3.13) that the action of the mapping class group $\mathrm{MCG}(S^{\natural n})$ does not affect the first two (if $S = \mathbb{T}$) or one (if $S = \mathbb{M}$) entries of a tuple corresponding to a generator. We thus deduce that $(\mathfrak{i}_{(\mathbf{k},\ell)})_n$ and $(\Delta'_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)})_n$ commute with the action of γ, since they only affect the first two or one entries of a tuple. Therefore, the isomorphisms $\{(\mathfrak{i}_{(\mathbf{k},\ell)})_n\}_{n\in\mathbb{N}}$ and $\{(\Delta'_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)})_n\}_{n\in\mathbb{N}}$ define natural isomorphisms $\mathfrak{i}_{(\mathbf{k},\ell)}$ and $\Delta'_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$ in $\mathbf{Fct}(\mathcal{M}, \mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]\text{-Mod})$.

We now prove that $\mathfrak{i}_{(\mathbf{k},\ell)}$ and $\Delta'_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$ are actually natural isomorphisms of functors $\mathfrak{U}\mathcal{M} \to \mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]\text{-Mod}$ by using the approach of Lemma 1.2. We fix an integer $n \geqslant 1$, the proof being trivial for $n = 0$. We compute from (1.1) that $\tau_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}([S, \mathrm{id}_{S^{\natural 1+n}}]) = \mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(\sigma_1^{-1})\circ\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}([S, \mathrm{id}_{S^{\natural 2+n}}])$ where $\sigma_1 \in \mathrm{Aut}_{\mathcal{M}}(S\natural S)$ is the braiding of $\mathcal{M}$ (see Figure 3.7). We recall from the description of (3.3) that the morphism $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}([S, \mathrm{id}_{S^{\natural 2+n}}])$ is the map induced by the embedding of $\mathbb{W}^S_{1+n}$ into $\mathbb{W}^S_{2+n}$ given by sending the i-th edge $(\mathbb{S}^1 - \mathrm{pt})_i$ to the $(i+2)$-nd edge $(\mathbb{S}^1 - \mathrm{pt})_{i+2}$ (if $S = \mathbb{T}$) or the $(i+1)$-st edge $(\mathbb{S}^1 - \mathrm{pt})_{i+1}$ (if $S = \mathbb{M}$). In particular, in the image of $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}([S, \mathrm{id}_{S^{\natural 2+n}}])$, there are no configuration points on the two first edges $(\mathbb{S}^1 - \mathrm{pt})_1$ and $(\mathbb{S}^1 - \mathrm{pt})_2$ if $S = \mathbb{T}$ or on the first interval $(\mathbb{S}^1 - \mathrm{pt})_1$ if $S = \mathbb{M}$. Then the morphism $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(\sigma_1^{-1})$ corresponds to the action of σ_1^{-1} on $\mathbb{W}^S_{2+n}$. It follows from equations (3.10) and (3.12) of Lemma 3.15 that for any generator w of $\bigoplus_{\mathbf{k}'}\tau_1\mathfrak{L}^{\star}_{(\mathbf{k}',\ell)}(n)$, both $\delta_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}([S, \mathrm{id}_{S^{\natural 1+n}}])((\mathfrak{i}_{(\mathbf{k},\ell)})_n(w))$ and $(\mathfrak{i}_{(\mathbf{k},\ell)})_{1+n}(\bigoplus_{\mathbf{k}'}\tau_1\mathfrak{L}^{\star}_{(\mathbf{k}',\ell)}([S, \mathrm{id}_{S^{\natural 1+n}}])(w))$ are equal to:

$$\begin{cases} \sum_{1\leqslant j_1+j_2\leqslant r}([j_1w_1, j_2w_2], [0,0], [w_3, w_4], \ldots, [w_{2n+3}, w_{2n+4}]) & \text{if } S = \mathbb{T}; \\ \sum_{1\leqslant j\leqslant r}([jw_1], [0], [w_2], \ldots, [w_{n+2}]) & \text{if } S = \mathbb{M}. \end{cases}$$

The same arguments using Lemma 3.15 also prove that $\tau_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}([S, \mathrm{id}_{S^{\natural 1+n}}])((\Delta'_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)})_n(w'))$ and $(\Delta'_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)})_{1+n}(\delta_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}([S, \mathrm{id}_{S^{\natural 1+n}}])(w'))$ are both equal to:

$$\begin{cases} ([w'_1, w'_2], [0,0], [w'_3, w'_4], \ldots, [w'_{2n+3}, w'_{2n+4}]) & \text{if } S = \mathbb{T}; \\ ([w'_1], [0], [w'_2], \ldots, [w'_{n+2}]) & \text{if } S = \mathbb{M}; \end{cases}$$

for any generator w' of $\delta_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(n)$. It then straightforwardly follows from the above equalities and a clear induction on $m \geqslant 1$ that the collections of isomorphisms $\{(\mathfrak{i}_{(\mathbf{k},\ell)})_n\}_{n\in\mathbb{N}}$ and $\{(\Delta'_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)})_n\}_{n\in\mathbb{N}}$ commute with the action of $[m, \mathrm{id}_{S^{\natural m+n}}]$ for each $m \geqslant 1$. Hence Relation (1.3) is satisfied for all $n \in \mathbb{N}$ and we apply Lemma 1.2 for $\mathfrak{i}_{(\mathbf{k},\ell)}$ and $\Delta'_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$.

Thus $\mathfrak{i}_{(\mathbf{k},\ell)}$ is an isomorphism in $\mathbf{Fct}(\mathfrak{U}\mathcal{M}, \mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]\text{-Mod})$ between $\delta_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$ and $\bigoplus_{\mathbf{k}'}\tau_1\mathfrak{L}^{\star}_{(\mathbf{k}',\ell)}$. Also, it follows from the definitions that $(\Delta_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)})_n \circ (\Delta'_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)})_n = \mathrm{id}_{\delta_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(n)}$, so $\Delta'_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$ is a section of $\Delta_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$ in $\mathbf{Fct}(\mathfrak{U}\mathcal{M}, \mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]\text{-Mod})$. Since $\kappa_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)} = 0$ (because $i_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$ is clearly injective), we deduce that the exact sequence (3.1) is a split short exact sequence, which provides the isomorphisms (4.6) and (4.7).

Whenever we deal with a *twisted* homological representation functor, we note that the action on the ground ring $\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]$ does not affect any of the above reasoning, while $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$ and $\bigoplus_{1\leqslant j\leqslant r}\tau_1\mathfrak{L}^{\star}_{(\mathbf{k}_j,\ell)}$ are equipped with the same action as $\tau_1\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$ (via the change of rings operation of Convention 3.7 for each summand of the latter; see Observation 1.11).

Finally, the result after a change of rings operation follows from the right-exactness of any change of rings operation and Lemma 3.3. □

Proof of Theorem D. Let $0 \leqslant m \leqslant k$. Using the commutation property of δ_1 and τ_1, we deduce from Theorem 4.16 that $\tau_1^m\mathfrak{L}^{\star}_{(\mathbf{k},\ell)} \cong \mathfrak{L}^{\star}_{(\mathbf{k},\ell)} \oplus \delta_1^m\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$, where $\delta_1^m\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$ is a direct sum of functors of the form $\tau_1^m\mathfrak{L}^{\star}_{(\mathbf{k}',\ell)}$ for $\mathbf{k}' \in \{\mathbf{k} - m'\}$ with $m \leqslant m' \leqslant 2m$ in the orientable setting and $m' = m$ in the non-orientable setting. In particular, $\delta_1\mathfrak{L}^{\star}_{(1,\ell)}$ is the constant functor at $\mathbb{Z}[Q_{(1,\ell)}]$. Hence $\mathfrak{L}^{\star}_{(1,\ell)}$ is both split and weak polynomial of degree 1. Using the above properties of $\tau_1^m\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$, $\delta_1^m\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$

and $\kappa_1 \mathfrak{L}^{\star}_{(\mathbf{k},\ell)} = 0$, along with the commutation property of κ_1 and τ_1, it then follows from a clear induction on k that $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}$ is both split and weak polynomial of degree k. □

4.3.2. Vertical-type alternatives

We now deal with the vertical-type alternatives of the homological representation functors for the mapping class groups of surfaces introduced in §1.3.3. We consider the functors $\mathfrak{L}^{v}_{(\mathbf{k},\ell')}(\boldsymbol{\Gamma})$ for orientable surfaces and the functors $\mathfrak{L}^{v}_{(\mathbf{k},\ell')}(\boldsymbol{\mathcal{N}})$ for non-orientable surfaces for $\ell' \leqslant 2$. In particular, we do not consider the functors $\mathfrak{L}^{v}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})$ with $\ell \geqslant 3$ for non-orientable surfaces because the proof of Theorem 4.17 relies on some technical arguments of §3.3.3.2: these decisively use the specific structure of the transformation groups $Q_{(\mathbf{k},2)}(\mathbb{T})$ and $Q_{(\mathbf{k},2)}(\mathbb{M})$, described in [PS23, Cor. 3.6], whereas the groups $Q_{(\mathbf{k},\ell)}(\mathbb{M})$ are not known for $\ell \geqslant 3$. We however conjecture that all of the following arguments, and a fortiori the results of Theorem 4.17, also hold for the homological representation functors $\mathfrak{L}^{v}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})$ with $\ell \geqslant 3$.

We recall from Lemma 2.13 that, for each of these functors $\mathfrak{L}^{v}_{(\mathbf{k},\ell')}$ and each $n \in \mathbb{N}$, the $\mathbb{Z}[Q_{(\mathbf{k},\ell')}]$-module $\mathfrak{L}^{v}_{(\mathbf{k},\ell')}(n)$ has a free generating set over $\mathbb{Z}[Q_{(\mathbf{k},\ell')}]$ indexed by the set of ("vertical") tuples of type $([w_1,w_2],\ldots,[w_{2n-1},w_{2n}])^v$ in the orientable setting and $([w_1],\ldots,[w_n])^v$ in the non-orientable setting. In particular, it has the same dimension as the corresponding ("non-vertical") module of §4.3.1. In contrast to the vertical-type alternatives of the homological representation functors for classical braid groups in §4.1.2 and for surface braid groups in §4.2.2, these vertical-type alternatives $\mathfrak{L}^{v}_{(\mathbf{k},\ell')}(\boldsymbol{\Gamma})$ and $\mathfrak{L}^{v}_{(\mathbf{k},\ell')}(\boldsymbol{\mathcal{N}})$ have the same polynomiality properties as their original counterparts studied in §4.3.1:

Theorem 4.17 *For each partition $\mathbf{k} \vdash k \geqslant 1$, the exact sequence (3.1) induces the analogous isomorphisms to (4.6) and (4.7) for the functors $\mathfrak{L}^{v}_{(\mathbf{k},1)}(\boldsymbol{\Gamma})$, $\mathfrak{L}^{v}_{(\mathbf{k},2)}(\boldsymbol{\Gamma})$, $\mathfrak{L}^{v}_{(\mathbf{k},1)}(\boldsymbol{\mathcal{N}})$ and $\mathfrak{L}^{v}_{(\mathbf{k},2)}(\boldsymbol{\mathcal{N}})$. Therefore, these functors are split and weak polynomial of degree k. These results still hold after any (non-zero) change of rings operation.*

Proof. We fix $\ell \in \{1,2\}$. The analogous isomorphisms to (4.6) and (4.7) for the vertical-type alternatives follow mutatis mutandis from the proof of Theorem 4.16 by defining analogues $\{(\mathfrak{i}^{v}_{(\mathbf{k},\ell)})_n\}_{n\in\mathbb{N}}$ and $\{(\Delta'_1\mathfrak{L}^{v}_{(\mathbf{k},\ell)})_n\}_{n\in\mathbb{N}}$ of the morphisms $\{(\mathfrak{i}_{(\mathbf{k},\ell)})_n\}_{n\in\mathbb{N}}$ and $\{(\Delta'_1\mathfrak{L}_{(\mathbf{k},\ell)})_n\}_{n\in\mathbb{N}}$. The proof that these define natural transformations $\mathfrak{i}^{v}_{(\mathbf{k},\ell)}$ and $\Delta'_1\mathfrak{L}^{v}_{(\mathbf{k},\ell)}$ in $\mathbf{Fct}(\mathcal{M},\mathbb{Z}[Q_{(\mathbf{k},\ell)}]\text{-Mod})$ is a verbatim repetition of the first part of the proof of Theorem 4.16, again using Lemma 3.13. Then, the proof that $\mathfrak{i}^{v}_{(\mathbf{k},\ell)}$ and $\Delta'_1\mathfrak{L}^{v}_{(\mathbf{k},\ell)}$ are natural transformations in $\mathbf{Fct}(\mathfrak{U}(\mathcal{M})^{\dagger},\mathbb{Z}[Q_{(\mathbf{k},\ell)}]\text{-Mod})$ is the same as the second part of the proof of Theorem 4.16, except that we now use equations (3.11) and (3.13) of Lemma 3.15 to understand the action of the braiding. Finally, the proof of the polynomiality result repeats mutatis mutandis that of Theorem D. □

Remark 4.18 We could define the functors $\mathfrak{L}^{v}_{(\mathbf{k},1)}(\boldsymbol{\Gamma})$, $\mathfrak{L}^{v}_{(\mathbf{k},2)}(\boldsymbol{\Gamma})$, $\mathfrak{L}^{v}_{(\mathbf{k},1)}(\boldsymbol{\mathcal{N}})$ and $\mathfrak{L}^{v}_{(\mathbf{k},2)}(\boldsymbol{\mathcal{N}})$ as objects of $\mathfrak{U}\mathcal{M}_2^{+}$ and $\mathfrak{U}\mathcal{M}_2^{-}$ respectively (i.e. *without* the opposite convention for the braiding of $\mathcal{M}_2$ induced by the $\dagger$ endofunctor; see §1.3.3); however, in this setting it is not clear that there are isomorphisms analogous to (4.6) and (4.7). On the other hand, the polynomiality results still hold with this opposite convention, via certain dimension reduction arguments on the objects of the difference functors.

Fact 4.19 (Dual representation functors.) *As in Fact 4.15, we note that, for either $S = \mathbb{T}$ or $S = \mathbb{M}$, the $\mathrm{MCG}(S^{\natural n})$-representation $H^{\partial}_k(C_{\mathbf{k}}(S^{\natural n});\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)])$ introduced in Notation 2.9 is the dual of the $\mathrm{MCG}(S^{\natural n})$-representation $H^{BM}_k(C_{\mathbf{k}}(S^{\natural n});\mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)])$; see Corollary 2.10 and Remark 2.11. Assigning for each morphism $[S,\mathrm{id}_{S^{\natural 1+n}}] \in \mathfrak{U}\mathcal{M}$ the same map as for the vertical-type alternatives, they extend to functors $\mathfrak{L}^{\star,\partial}_{(\mathbf{k},\ell)}(\boldsymbol{\Gamma})$ and $\mathfrak{L}^{\star,\partial}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})$ of the form $\mathfrak{U}(\mathcal{M})^{\dagger} \to \mathbb{Z}[Q_{(\mathbf{k},\ell)}(S)]\text{-Mod}$ encoding the dual representations to those of $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(\boldsymbol{\Gamma})$ and $\mathfrak{L}^{\star}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})$. The polynomiality results of Theorem 4.17 may then be repeated verbatim for these functors.*

5. Applications

We now explain the main applications of the polynomiality results proved in §4.

5.1. Faithfulness results for classical braid group representations

The short exact sequences of (4.1) and (4.2) provide new connections between the different Lawrence-Bigelow representations. Note that the natural transformation $i_1 \colon Id \to \tau_1$ corresponds to considering the $\mathbf{B}_{n+1}$-representations $\mathfrak{LB}_{(\mathbf{k},\ell)}(1+n)$ and $\mathfrak{LB}^u_{(\mathbf{k},\ell)}(1+n)$ as $\mathbf{B}_n$-representations via the injection $\mathbf{B}_n \hookrightarrow \mathbf{B}_{n+1}$ defined by $\sigma_i \mapsto \sigma_{i+1}$ for $1 \leqslant i \leqslant n-1$. Recall from [Big02, §4] that the $\mathbf{B}_n$-representation $\mathfrak{LB}_{(2,2)}(n)$ is faithful for each n (see also [Big01; Kra02]), from which we may now deduce Theorem G:

Proof of Theorem G. In [PS21, §5.2.1.2], we deduce from the faithfulness of the $\mathbf{B}_n$-representation $\mathfrak{LB}_{(2,2)}(n)$ that the representation $\mathfrak{LB}_{(2,\ell)}(n)$ is faithful for each $\ell \geqslant 3$, and so is $\mathfrak{LB}^u_{(2,\ell)}(n)$ by repeating verbatim the same reasoning. Thus $\tau_1\mathfrak{LB}_{(2,\ell)}(n)$ and $\tau_1\mathfrak{LB}^u_{(2,\ell)}(n)$ are faithful as $\mathbf{B}_{n+1}$-representations, so in particular they are faithful as $\mathbf{B}_n$-representations. Then, we deduce the faithfulness of the $\mathbf{B}_n$-representations $\tau_1\mathfrak{LB}_{(\mathbf{k},\ell)}(n)$ and $\tau_1\mathfrak{LB}^u_{(\mathbf{k},\ell)}(n)$ by identifying $\tau_1\mathfrak{LB}_{(2,\ell)}(n)$ and $\tau_1\mathfrak{LB}^u_{(2,\ell)}(n)$ as their quotients thanks to (4.1) and (4.2). □

5.2. Analyticity of a quantum representation

Jackson and Kerler [JK11] introduce a representation $\mathbb{V}$ over the group ring $\mathbb{L} := \mathbb{Z}[\mathtt{s}^{\pm 1}, \mathtt{q}^{\pm 1}]$, called the *generic Verma module*, of $\mathbb{U}_q(\mathfrak{sl}_2)$, the quantum enveloping algebra of the Lie algebra $\mathfrak{sl}_2$. Since $\mathbb{U}_q(\mathfrak{sl}_2)$ is a quasitriangular Hopf algebra, the representation $\mathbb{V}$ comes equipped with an automorphism $S \in \mathrm{Aut}_{\mathbb{L}}(\mathbb{V} \otimes \mathbb{V})$. This induces a $\mathbf{B}_n$-representation on $\mathbb{V}^{\otimes n}$ given by sending $\sigma_i \in \mathbf{B}_n$ to $\mathrm{id}_{i-1} \otimes S \otimes \mathrm{id}_{n-i-1}$, which we call *the Verma quantum representation*; see [JK11, §1]. For $k \geqslant 1$, the *weight space* $V_{n,k} \subseteq \mathbb{V}^{\otimes n}$ is the eigenspace of the action of a certain generator $K \in \mathbb{U}_q(\mathfrak{sl}_2)$ corresponding to the eigenvalue $\mathtt{s}^n\mathtt{q}^{-2k}$ and the *highest weight space* $W_{n,k}$ is its intersection with the kernel of the action of another generator $E \in \mathbb{U}_q(\mathfrak{sl}_2)$. The $\mathbf{B}_n$-action on $\mathbb{V}^{\otimes n}$ restricts to sub-$\mathbf{B}_n$-representations on $V_{n,k}$ and $W_{n,k}$ for each $k \geqslant 1$. The first one is the *quantum representation* of $\mathbf{B}_n$ of *weight* k, while the second one is the *quantum representation* of $\mathbf{B}_n$ of *highest weight* k. The relation between the variables $\mathtt{s}$ and $\mathtt{q}$ and the generators q and t of $Q_{(2,2)}(\mathbb{D}) = \mathbb{Z}^2$ (defining the representation $\mathfrak{LB}_{(2,2)}(n)$) is given by the ring morphism $\theta \colon \mathbb{K} := \mathbb{Z}[q^{\pm 1}, t^{\pm 1}] \to \mathbb{L}$ defined by $(q,t) \mapsto (\mathtt{s}^2, -\mathtt{q}^{-2})$.[2] In particular, $\mathbb{L}$ is a left $\mathbb{K}$-module via θ; the change of rings operation $- \otimes_{\mathbb{K}} \mathbb{L}$ corresponds to adjoining square roots of q and t. For each n and $k \geqslant 1$, we have:

Lemma 5.1 *There is an isomorphism of $\mathbf{B}_n$-representations $V_{n,k} \cong \tau_1\mathfrak{LB}_k(n) \otimes_{\mathbb{K}} \mathbb{L}$.*

Proof. Let $\mathbb{D}'_n$ denote the closed disc minus n interior points and minus a point on its boundary. To obtain an alternative description of $\tau_1\mathfrak{LB}_k(n)$, the construction of [PS21, §2] may be applied using the space of configurations of k unordered points in $\mathbb{D}'_n$: we thus obtain a $\mathbb{Z}[\mathbb{Z}^2]$-module $H_k^{\mathrm{BM}}(C_k(\mathbb{D}'_n); \mathbb{Z}[\mathbb{Z}^2])$, which is a $\mathbf{B}_n$-representation; see [PS21, §2.4]. Gluing $\mathbb{D}'_0$ to $\mathbb{D}'_n$ so that the two boundary punctures coincide induces an embedding $\mathbb{D}'_n \hookrightarrow \mathbb{D}_{1+n}$, which in turn induces an embedding $C_k(\mathbb{D}'_n) \hookrightarrow C_k(\mathbb{D}_{1+n})$. This latter embedding defines a (covariant) map on Borel-Moore homology since its image is closed, thus it is a proper map, and Borel-Moore homology is covariantly functorial with respect to proper maps [Bre97, Proposition V.4.5]. We also note that the local coefficient system that we define on $C_k(\mathbb{D}'_n)$ is the restriction of the one that we consider on $C_k(\mathbb{D}_{1+n})$. Therefore there is a well-defined map

$$H_k^{\mathrm{BM}}(C_k(\mathbb{D}'_n); \mathbb{Z}[\mathbb{Z}^2]) \longrightarrow \tau_1\mathfrak{LB}_k(n) \tag{5.1}$$

of $\mathbf{B}_n$-representations over $\mathbb{Z}[\mathbb{Z}^2] = \mathbb{K}$. The fact that this map is an isomorphism follows from the evident bijection that it induces on the free bases as $\mathbb{K}$-modules obtained from Lemma 2.1.

We now consider the subspace $C_k^-(\mathbb{D}_n) \subset C_k(\mathbb{D}_n)$ of all configurations that intersect a particular fixed point on the boundary. Martel [Mar22, §2] introduces the $\mathbf{B}_n$-representation given by

[2] We note that [JK11] and [Mar22] use slightly different notations from those used here, where we have been consistent with the usual notations for the variables q and t of the homological representations [Big01; Big02]. In [JK11], the variables $q, t, \mathtt{s}, \mathtt{q}$ are denoted $\mathtt{q}, \mathtt{t}, s, q$ respectively. In [Mar22], instead of the variables $q, t, \mathtt{s}, \mathtt{q}$ related by $q = \mathtt{s}^2$ and $t = -\mathtt{q}^{-2}$, the author has variables $t, \mathtt{t}, s, q$ related by $t = -\mathtt{t} = -q^{-2}$ and the variable s is also sometimes denoted by q^α. In particular one should not confuse the $\mathtt{t}$ of [JK11] (which corresponds to t in [Mar22]) with the $\mathtt{t}$ of [Mar22] (which corresponds to $-\mathtt{t}$ in [JK11]).

the $\mathbb{K}$-module $H_k^{\mathrm{BM}}(C_k(\mathbb{D}_n), C_k^-(\mathbb{D}_n); \mathbb{K})$. Since $C_k(\mathbb{D}'_n)$ is an open subspace of $C_k(\mathbb{D}_n)$ with closed complement $C_k^-(\mathbb{D}_n)$, the inclusion $(C_k(\mathbb{D}'_n), \varnothing) \hookrightarrow (C_k(\mathbb{D}_n), C_k^-(\mathbb{D}_n))$ is an open embedding. Relative Borel-Moore homology is contravariantly functorial with respect to open embeddings (since it is the composition of reduced homology with the contravariant functor from locally-compact, Hausdorff spaces and open embeddings to based spaces and maps given by one-point compactification), so we have a map

$$H_k^{BM}(C_k(\mathbb{D}_n), C_k^-(\mathbb{D}_n); \mathbb{K}) \longrightarrow H_k^{BM}(C_k(\mathbb{D}'_n); \mathbb{K}). \tag{5.2}$$

This is a map of $\mathbf{B}_n$-representations over $\mathbb{K}$ since the $\mathbf{B}_n$-action (up to homotopy) on $C_k(\mathbb{D}_n)$ preserves its partition into $C_k^-(\mathbb{D}_n)$ and $C_k(\mathbb{D}'_n)$. The fact that this map is an isomorphism follows from the evident bijection that it induces on the free bases as $\mathbb{K}$-modules obtained from Lemma 2.1 for the right-hand side and [Mar22, Prop. 3.6] for the left-hand side (see also [Mar22, Cor. 3.9]).[3]

Now, [Mar22, Th. 1.5] provides an isomorphism

$$V_{k,n} \cong H_k^{\mathrm{BM}}(C_k(\mathbb{D}_n), C_k^-(\mathbb{D}_n); \mathbb{Z}[\mathbb{Z}^2]) \otimes_{\mathbb{K}} \mathbb{L} \tag{5.3}$$

of $\mathbf{B}_n$-representations over $\mathbb{L}$. The desired isomorphism of the lemma is then the composition of (5.1), (5.2) and (5.3) (tensoring the first two isomorphisms over $\mathbb{K}$ with $\mathbb{L}$). □

Corollary 5.2 *For each $n \geqslant 2$, there is an isomorphism of $\mathbf{B}_n$-representations over $\mathbb{L}$*

$$\mathbb{V}^{\otimes n} \cong \bigoplus_{k \geqslant 0} \tau_1 \mathfrak{LB}_k(n) \otimes_{\mathbb{K}} \mathbb{L}. \tag{5.4}$$

We may therefore define the ***Verma module representation functor*** *$\mathfrak{Ver} \colon \mathfrak{U}\boldsymbol{\beta} \to \mathbb{L}\text{-Mod}$ to be the colimit $\bigoplus_{k\geqslant 0} \tau_1 \mathfrak{LB}_k \otimes_{\mathbb{K}} \mathbb{L}$. This functor $\mathfrak{Ver}$ is* ***analytic****, i.e. it is a colimit of polynomial functors, and* ***exponential****, i.e. it is a strong monoidal functor $(\mathfrak{U}\boldsymbol{\beta}, \natural, 0) \to (\mathbb{L}\text{-Mod}, \otimes, \mathbb{L})$. However, the functor $\mathfrak{Ver}$ is* ***not polynomial****.*

Proof. The isomorphisms (5.4) follow directly from Lemma 5.1 and the decomposition of the Verma module representations into weight spaces. The analyticity of the functor $\mathfrak{Ver}$ follows from its definition and Theorem 4.2, from which we also deduce that $\delta_1^m \mathfrak{Ver} \cong \bigoplus_{k\geqslant 0} \tau_1^{m+1} \mathfrak{LB}_k \otimes_{\mathbb{K}} \mathbb{L}$ for all $m \geqslant 1$. Hence there is a natural embedding $\mathfrak{Ver} \hookrightarrow \delta_1^m \mathfrak{Ver}$ for all $m \geqslant 1$, which proves that the functor $\mathfrak{Ver}$ is not polynomial. That it is an exponential functor straightforwardly follows from the isomorphism $\mathfrak{Ver}(n) \cong \mathbb{V}^{\otimes n}$. □

Remark 5.3 Analogous arguments to those of Corollary 5.2 may be repeated verbatim for functors for the mapping class groups of surfaces extending the Magnus representations (see for instance [Sak12, §4] for the definition of this representation) or the representations induced by actions on discrete Heisenberg groups introduced by [BPS21].

5.3. Homological stability

Another key application of the polynomiality results of §4 is homological stability. Namely, fixing a strict monoidal groupoid $(\mathcal{G}, \natural, 0)$, an object X of $\mathcal{G}$, a left-module $(\mathcal{M}, \natural)$ and an object A of $\mathcal{M}$, we denote by $\langle \mathcal{G}, \mathcal{M} \rangle_{X,A}$ the full subcategory of $\langle \mathcal{G}, \mathcal{M} \rangle$ with objects $\{X^{\natural n} \natural A\}_{n\in\mathbb{N}}$ and by G_n the automorphism group $\mathrm{Aut}_{\langle \mathcal{G}, \mathcal{M}\rangle}(X^{\natural n} \natural A)$ for all $n \in \mathbb{N}$. The family of groups $\{G_n\}_{n\in\mathbb{N}}$ is said to satisfy *homological stability (with twisted coefficients)* if for any strong polynomial functor $F \colon \langle \mathcal{G}, \mathcal{M} \rangle_{X,A} \to \mathbb{Z}\text{-Mod}$ of degree d for which there exists $N_F \in \mathbb{N}$ such that $\kappa_{X^{\natural n}}(\delta_X^i F) = 0$ for all $i \in \mathbb{N}$ and $n \geqslant N_F - i$,[4] the natural maps

$$H_*(G_n, F(X^{\natural n} \natural A)) \longrightarrow H_*(G_{1+n}, F(X^{\natural 1+n} \natural A))$$

[3] In fact there is a natural isomorphism $H_*^{BM}(X, X \smallsetminus U; \mathcal{L}) \cong H_*^{BM}(U; \mathcal{L})$ for any open inclusion $U \subset X$ if X is Hausdorff, locally compact and paracompact, and if the local system $\mathcal{L}$ is finitely generated over a principal ideal domain. This follows from [Bre97, Corollary V.5.10, page 312] (see the note mdp.ac/notes/2301-relative-Borel-Moore-homology.pdf for details on how to deduce this special case of that much more general result). Although this is a more conceptual reason for the isomorphism (5.2), it does not apply directly in our setting, since $\mathbb{K}$ is not a principal ideal domain. The explicit free generating sets for each side of (5.2) allow us to avoid this subtlety by simply checking directly that it is a bijection.

[4] The functor F is then said to be a *coefficient system of degree d at N_F*.

are isomorphisms for all $n \geqslant B(*, d, N_F)$, where $B(*, d, N_F) \in \mathbb{N}$ depends on $*$ and d. The lower d and N_F are, the lower (and so the better) the stability bound $B(*, d, N_F)$ is. If F is in addition *split* polynomial, the stability bound $B(*, d, N_F)$ is even lower.

Homological stability with constant coefficients for the classical braid groups is due to Arnold [Arn70] (see also [Seg73]), and to McDuff [McD75] and Segal [Seg79] for braid groups on connected, open surfaces. For a certain more restrictive family of twisted coefficients, homological stability is proven by the first author [Pal18]. For general (strong polynomial) twisted coefficients, homological stability is proven by Randal-Williams and Wahl [RW17, Th. 5.22]. For the mapping class groups of orientable surfaces $\{\mathbf{\Gamma}_{g,1}\}_{g\in\mathbb{N}}$, the first homological stability properties are due to Harer [Har85] (for constant coefficients and some particular twisted coefficients), while the first general framework and results for (strong polynomial) twisted coefficients are due to Ivanov [Iva93]. The stability bound $B(*, d, N_F)$ is improved by Boldsen [Bol12] (both for constant and twisted coefficients) and by Randal-Williams [Ran16] (for constant coefficients). Randal-Williams and Wahl [RW17, Th. 5.26] extract the general framework for proving homological stability properties (both for constant and twisted coefficients) for this family of groups, subsuming these previous results. For the mapping class groups of non-orientable surfaces $\{\boldsymbol{\mathcal{N}}_{h,1}\}_{h\in\mathbb{N}}$, homological stability with constant coefficients is proved in [Wah08, Th. A] (with improvements to the range in [Ran16, §1.4]) and the general case of (strong polynomial) twisted coefficients is covered by [RW17, Th. 5.29].

Proof of Theorem E. This is an immediate corollary of Theorems B, C and D, which state that a large family of homological representation functors are very strong or split polynomial, applied to [RW17, Th. A] (see also [RW17, Th. 4.20] for the improvement to the range when the functors are split polynomial). □

5.4. Classification

A fundamental reason for the notion of weak polynomial functors to be introduced in [DV19] is that, contrary to the category $\mathcal{P}ol_d^{str}(\langle \mathcal{G}, \mathcal{M}\rangle, \mathcal{A})$, the category $\mathcal{P}ol_d(\langle \mathcal{G}, \mathcal{M}\rangle, \mathcal{A})$ is *localising*. This allows us to define the quotient categories for all $d \in \mathbb{N}$:

$$\mathcal{P}ol_{d+1}(\langle \mathcal{G}, \mathcal{M}\rangle, \mathcal{A})/\mathcal{P}ol_d(\langle \mathcal{G}, \mathcal{M}\rangle, \mathcal{A}). \tag{5.5}$$

A refined description of the category $\mathcal{P}ol_d^{str}(\langle \mathcal{G}, \mathcal{M}\rangle, \mathcal{A})$ is out of reach in general even for small values of d. In contrast, understanding the quotient categories (5.5) is more attainable: for example, when $\mathcal{G} = \mathcal{M} = FB$ (the category of finite sets and bijections) [DV19, Prop. 5.9] gives a general equivalence of these quotients in terms of module categories. Also, these quotient categories shed new light on the behaviour of the successive homological representation functors, thanks to Theorem F, which we may now prove:

Proof of Theorem F. First, each functor $\mathcal{P}_{d+1} : (5.5)_{d+1} \to (5.5)_d$ is canonically induced from the difference functors $\delta_1 : \mathcal{P}ol_{d+1}(\langle \mathcal{G}, \mathcal{M}\rangle, \mathcal{A}) \to \mathcal{P}ol_d(\langle \mathcal{G}, \mathcal{M}\rangle, \mathcal{A})$. We know from Theorems 4.2, 4.10 and 4.16 that each of the functors $\mathfrak{L}_{(\mathbf{k},\ell)}$ under consideration is weak polynomial of degree k. These results, combined with the short exact sequences of Theorems 4.2 and 4.10 (for the surface braid groups) and from the isomorphisms of Theorems 4.16 and 4.17 (for mapping class groups of surfaces), we deduce that $\mathfrak{L}_{(\mathbf{k},\ell)} \cong \tau_1 \mathfrak{L}_{(\mathbf{k},\ell)}$ in the quotient categories. □

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