Annex file to the article: "Polynomiality of surface braid and mapping class group representations"

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This file is an annex to the article [PS23]. It deals in §1 with some properties of the lower central series of mapping class groups of surfaces, and in §2 with the identification of some of the homological representation functors studied in [PS23, §1.3.3] for mapping class groups of orientable surfaces. The article [PS23] is independent of these discussions, but we explain these points here in order to offer a more concrete understanding of the homological representation functors considered in [PS23]. This file is not intended to be read independently of [PS23]. In particular, we follow the framework and notation of [PS23], some of which we now recall.

Notation 0.1 For an integer $n \geqslant 1$, an *ordered partition of* n means an ordered r-tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$ of integers $\lambda_i \geqslant 1$ (for some $r \geqslant 1$ called the *length* of λ) such that $n = \sum_{1 \leqslant i \leqslant r} \lambda_i$ (and without the condition $\lambda_i \geqslant \lambda_{i+1}$). We also write r' for the number of $i \in \{1, \ldots, r\}$ such that $\lambda_i \geqslant 2$. For simplicity, we denote the trivial partition $\lambda = (n)$ by n.

The lower central series of a group G is the descending chain of subgroups $\{\varGamma_\ell(G)\}_{\ell \geqslant 1}$ defined by $\varGamma_1(G) := G$ and $\varGamma_{\ell+1}(G) := [G, \varGamma_\ell(G)]$, the subgroup of G generated by the commutators $[g, h]$ for $g \in G$ and $h \in \varGamma_\ell(G)$. The induced canonical projection $G \twoheadrightarrow G/\varGamma_\ell(G)$ is denoted by $/\varGamma_\ell(G)$. When there is no ambiguity, we omit G from the notation.

Let S be a smooth, connected, compact surface with one boundary component ∂S, and with a finite set of $k \geqslant 0$ points removed from its interior (in other words with *punctures*), and let $\bar{S}$ be the surface obtained from S by filling in each puncture with a *marked point*. For a partition $\lambda \vdash k$, the *mapping class group* $\mathrm{MCG}(S, \lambda)$ is the group of isotopy classes of diffeomorphisms of the surface $\bar{S}$ that restrict to the identity on the boundary and that fix the k marked points setwise while respecting the partition λ. When the surface S is orientable, mapping classes automatically preserve orientations due to the fact that they are the identity on ∂S.

We fix integers $k \geqslant 0$, $g \geqslant 0$ and $h \geqslant 1$. We denote by $\Sigma^k_{g,1}$ (resp. $\mathcal{N}^k_{h,1}$) a connected, compact, orientable (resp. non-orientable) surface of genus g (resp. h) with one boundary component and k punctures in its interior. We denote by $\mathbf{\Gamma}^\lambda_{g,1}$ the mapping class group $\mathrm{MCG}(\Sigma^k_{g,1}, \lambda)$ and by $\boldsymbol{\mathcal{N}}^\lambda_{h,1}$ the mapping class group $\mathrm{MCG}(\mathcal{N}^k_{h,1}, \lambda)$. When $k = 0$, we omit it, as well as λ, from the notation.

1. Lower central series of mapping class groups

We study here the lower central series of mapping class groups of surfaces, which are involved in the definition of the homological representation functors of [PS23, §1.3.3]. We first recall the following general decomposition of abelianisations of mapping class groups; see [PS24, Prop. 4.8] for a proof.

Proposition 1.1 *For a compact, connected, smooth, non-planar surface S with one boundary component, we have:*

$$\mathrm{MCG}(\Sigma^k_{0,1} \natural S, \lambda)^{\mathrm{ab}} \cong (\mathbb{Z}/2)^{r'} \oplus (H_1(S;\mathbb{Z})^r)_{\mathrm{MCG}(S)} \oplus \mathrm{MCG}(S)^{\mathrm{ab}}. \tag{1.1}$$

Each of the first r' $\mathbb{Z}/2$-summands is generated by the image in the abelianisation $\sigma^{(\rho)}$ (with $1 \leqslant \rho \leqslant r$ such that $\lambda_\rho \geqslant 2$) of a standard braid generator (considered as a mapping class) interchanging two points in the corresponding ρ-th block of the partition. The generator is known as the writhe *(modulo 2) of the ρ-th block of strands.*

Following the methods developed in [DPS22], it is not difficult to obtain results about the lower central series of the partitioned mapping class groups $\mathbf{\Gamma}^{\lambda}_{g,1}$. Using the terminology of [DPS22], the lower central series $\varGamma_*(G)$ of a group G is said to *stop* if there exists an integer $i \geqslant 1$ such that $\varGamma_i(G) = \varGamma_{i+1}(G)$. We say that it *stops at* $\varGamma_i$ if i is the smallest integer for which this holds.

Proposition 1.2 *Let $g \geqslant 1$. The lower central series $\varGamma_*(\mathbf{\Gamma}^{\lambda}_{g,1})$ stops before $\varGamma_2$. It stops at $\varGamma_1$ if $g \geqslant 3$ and λ is a discrete partition.*

Proof. First, we note that when $g \geqslant 3$ and λ is the discrete partition δ, we know from [Kor02, Th. 5.1] that the abelianisation of $\mathbf{\Gamma}^{\delta}_{g,1}$ is trivial, so its lower central series stops at $\varGamma_1$. We therefore just have to show that the lower central series of $\mathbf{\Gamma}^{\lambda}_{g,1}$ stops before $\varGamma_2$ in all cases, which we will do by the geometric disjoint support trick introduced in [DPS22]: for a group G and a generating set S of G^{ab}, if for each pair $(s,t) \in S^2$ we can find representatives $\tilde{s}, \tilde{t} \in G$ of s and t such that $\tilde{s}$ and $\tilde{t}$ commute, then $\varGamma_2(G) = \varGamma_3(G)$; see [DPS22, Cor. 2.6].

By Proposition 1.1 with $S = \Sigma_{g,1}$, the abelianisation $(\mathbf{\Gamma}^{\lambda}_{g,1})^{\mathrm{ab}}$ decomposes into three summands, namely $(\mathbb{Z}/2)^{r'}$, the coinvariants $(H_1(\Sigma_{g,1};\mathbb{Z})^r)_{\mathbf{\Gamma}_{g,1}}$ and $(\mathbf{\Gamma}_{g,1})^{\mathrm{ab}}$. By [Kor02, Th. 5.1], in the case of a discrete partition $\lambda = \delta$, we have $(\mathbf{\Gamma}^{\delta}_{g,1})^{\mathrm{ab}} \cong (\mathbf{\Gamma}_{g,1})^{\mathrm{ab}}$ for all $g \geqslant 1$, so it follows that the middle term $(H_1(\Sigma_{g,1};\mathbb{Z})^r)_{\mathbf{\Gamma}_{g,1}}$ is zero in all cases. We may thus simplify the decomposition for any partition λ to $(\mathbf{\Gamma}^{\lambda}_{g,1})^{\mathrm{ab}} \cong (\mathbb{Z}/2)^{r'} \oplus (\mathbf{\Gamma}_{g,1})^{\mathrm{ab}}$. A $\mathbb{Z}/2$-basis for the first summand is given by $\sigma^{(\rho)}$ for each $\rho \in \{1,\ldots,r\}$ such that $\lambda_\rho \geqslant 2$: here, $\sigma^{(\rho)}$ is the class of a standard braid generator interchanging two points of the ρ-th block. By [Kor02, Th. 5.1], the second summand is trivial if $g \geqslant 3$ and is cyclic generated by a Dehn twist if $g \in \{1,2\}$. It is geometrically clear that all of these generators may be realised by diffeomorphisms of the surface with disjoint support: one simply has to choose pairwise disjoint subdiscs separating the different blocks of the partition of the punctures, and (in the case $g \in \{1,2\}$) ensure that these are also disjoint from a tubular neighbourhood of the curve whose associated Dehn twist is the remaining generator. □

Remark 1.3 When $S = \Sigma_{g,1}$, it follows from Proposition 1.1 and the proof of Proposition 1.2 that $Q_{(\lambda,2)}(\mathbb{T}) \cong (\mathbf{\Gamma}_{g,1})^{\mathrm{ab}} \cong (\mathbb{Z}/2)^{r'}$ for $g \geqslant 3$, where the $\mathbb{Z}/2$-summands are generated by the writhe (modulo 2) of the ρ-th blocks of strands for $\rho \in \{1,\ldots,r\}$ such that $\lambda_\rho \geqslant 2$.

For $S = \mathcal{N}_{h,1}$, we first consider the discrete partition δ, where we know from [Stu10, Th. 6.21] that $(\boldsymbol{\mathcal{N}}^{\delta}_{h,1})^{\mathrm{ab}} \cong (\mathbb{Z}/2)^r \times \mathbb{Z}/2$ for $h \geqslant 7$. Using Proposition 1.1 for the case $\lambda = \delta$, we deduce that $(H_1(\mathcal{N}_{h,1};\mathbb{Z})^r)_{\boldsymbol{\mathcal{N}}_{h,1}} \cong (\mathbb{Z}/2)^r$. Applying Proposition 1.1 again, now for arbitrary λ, we then obtain that $(\boldsymbol{\mathcal{N}}^{\lambda}_{h,1})^{\mathrm{ab}} \cong (\mathbb{Z}/2)^{r'} \times (\mathbb{Z}/2)^r \times \mathbb{Z}/2$, generated by (i) standard braid generators in the blocks of punctures of size at least 2 (i.e. the writhe modulo 2), (ii) sliding one puncture from each block through a crosscap, (iii) one additional crosscap-slide ("Y-homeomorphism") which generates $(\boldsymbol{\mathcal{N}}_{h,1})^{\mathrm{ab}} \cong \mathbb{Z}/2$. In particular, this abelianisation is independent of h, and we therefore deduce that $\mathcal{Q}_{(\lambda,2)}(\boldsymbol{\mathcal{N}}) \cong (\mathbb{Z}/2)^{r'} \times (\mathbb{Z}/2)^r$, with the above description (i) and (ii) of the generators; see also [PS24, Cor. 4.9].

Remark 1.4 In general, whether or not the lower central series of the mapping class group $\boldsymbol{\mathcal{N}}^{\lambda}_{h,1}$ stops is an open question. Namely, with the above description of Remark 1.3 of the abelianisation $(\boldsymbol{\mathcal{N}}^{\lambda}_{h,1})^{\mathrm{ab}}$ for $h \geqslant 7$, it is not immediately clear how to apply methods of [DPS22] in this setting. Thus we cannot at the present time compute for $\ell \geqslant 3$ the group $Q_{(\lambda,\ell)}(\mathbb{M})$ associated to the homological representation functors of [PS23, §1.3.3] associated to the mapping class groups of non-orientable surfaces.

One exception is the case of $h \geqslant 7$ and $\lambda = 1$: in this case, it follows from [Stu10, Thm. 6.21] that the abelianisation is $(\mathbb{Z}/2)^2$, generated by a Y-homeomorphism and sliding the unique puncture through a crosscap; these may be realised disjointly. Similarly, if $h \geqslant 7$ and $\lambda = k$ (with the trivial partition) for $k \geqslant 3$: in this case the abelianisation is $(\mathbb{Z}/2)^3$, generated by the previous two generators and a standard braid generator; these may also be realised disjointly; see [Stu10, Thm. 6.21].

It is therefore a priori relevant to consider higher $\ell \geqslant 3$ as a parameter for the homological representation functors $\mathfrak{L}_{(\lambda,\ell)}(\boldsymbol{\mathcal{N}})\colon \langle \mathcal{M}_2^-, \mathcal{M}_2^- \rangle \to \mathbb{Z}[Q_{(\lambda,2)}(\mathbb{M})]\text{-Mod}$ of [PS23, §1.3.3].

2. Comparison of representations: Moriyama representations

Moriyama [Mor07] considered the $\mathbf{\Gamma}_{g,1}$-representation given by its action on the relative homology group $H_n(\Sigma_{g,1}^{\times n}, \Delta \cup A_g; \mathbb{Z})$, where Δ denotes the "fat diagonal" of $\Sigma_{g,1}^{\times n}$ where at least two points coincide and A_g denotes the subspace of $\Sigma_{g,1}^{\times n}$ where at least one point is equal to p_0, a chosen basepoint on $\partial\Sigma_{g,1}$. In this section, we relate these mapping class group representations to those encoded by certain homological representation functors introduced in [PS23, §1.3.3] and their duals; see (2.1) and Proposition 2.1.

Let us write $\Sigma'_{g,1} = \Sigma_{g,1} \smallsetminus \{p_0\}$ where $p_0 \in \partial\Sigma_{g,1}$. Since $\Sigma_{g,1}^{\times n}$ is a compactification of $F_n(\Sigma'_{g,1}) = \Sigma_{g,1}^{\times n} \smallsetminus (\Delta \cup A_g)$, the Borel-Moore homology of $F_n(\Sigma'_{g,1})$, denoted by $H_*^{\mathrm{BM}}(F_n(\Sigma'_{g,1}); \mathbb{Z})$, is isomorphic to the relative homology group $H_*(\Sigma_{g,1}^{\times n}, \Delta \cup A_g; \mathbb{Z})$. Thus Moriyama's representation may be viewed as an action on Borel-Moore homology. Denoting by $F_n(X, Y) \subseteq F_n(X)$ the subspace of configurations that intersect $Y \subseteq X$ non-trivially, by Poincaré duality (see [Bre97, §5, Th. 16.30] for instance) we have $H_*^{\mathrm{BM}}(F_n(\Sigma'_{g,1}); \mathbb{Z}) \cong H^{2n-*}(F_n(\Sigma'_{g,1}), F_n(\Sigma'_{g,1}, \partial\Sigma'_{g,1}); \mathbb{Z})$ since $F_n(\Sigma'_{g,1})$ is a connected, orientable manifold. Using the fact that the inclusions $\Sigma'_{g,1} \subset \Sigma_{g,1}$ and $\{p_1\} \subset \partial\Sigma'_{g,1}$ are isotopy equivalences, where $p_1 \in \partial\Sigma'_{g,1}$ is another point on the boundary of $\Sigma_{g,1}$, distinct from p_0, this is naturally isomorphic to $H^{2n-*}(F_n(\Sigma_{g,1}), F_n(\Sigma_{g,1}, \{p_1\}); \mathbb{Z})$.

A special case of [PS23, Thm. 2.1] implies that $H_*^{\mathrm{BM}}(F_n(\Sigma'_{g,1}); \mathbb{Z})$ is concentrated in degree $* = n$, so by the universal coefficient theorem, Moriyama's representation $H_n(\Sigma_{g,1}^{\times n}, \Delta \cup A_g; \mathbb{Z})$ is dual to the relative cohomology group $H^n(\Sigma_{g,1}^{\times n}, \Delta \cup A_g; \mathbb{Z})$. Analogous identifications to those above, replacing H_*^{BM} with compactly-supported cohomology H_c^*, etc., apply to these dual representations. In summary, we have identifications:

$$\begin{array}{ccccc} H_n(\Sigma_{g,1}^{\times n}, \Delta \cup A_g; \mathbb{Z}) & \cong & H_n^{\mathrm{BM}}(F_n(\Sigma'_{g,1}); \mathbb{Z}) & \cong & H^n(F_n(\Sigma_{g,1}), F_n(\Sigma_{g,1}, \{p_1\}); \mathbb{Z}) \\ {\scriptstyle dual}\updownarrow & & & & \\ H^n(\Sigma_{g,1}^{\times n}, \Delta \cup A_g; \mathbb{Z}) & \cong & H_c^n(F_n(\Sigma'_{g,1}); \mathbb{Z}) & \cong & H_n(F_n(\Sigma_{g,1}), F_n(\Sigma_{g,1}, \{p_1\}); \mathbb{Z}), \end{array} \tag{2.1}$$

where the top row are models for Moriyama's representation and the bottom row are models for its dual. Finally, the representations on the top row (i.e. Moriyama's representation) are isomorphic to the representations encoded by the homological representation functor $\mathfrak{L}_{((1,\ldots,1),1)}(\mathbf{\Gamma})$ introduced in [PS23, §1.3.3]:

Proposition 2.1 *The restriction of the functor $\mathfrak{L}_{((1,\ldots,1),1)}(\mathbf{\Gamma})$ to the g-th automorphism group $\mathbf{\Gamma}_{g,1}$ is the n-th Moriyama representation $\mathbf{\Gamma}_{g,1} \to \mathrm{Aut}_{\mathbb{Z}}(H_n(\Sigma_{g,1}^{\times n}, \Delta \cup A_g; \mathbb{Z}))$.*

Proof. The restriction of the functor $\mathfrak{L}_{((1,\ldots,1),1)}(\mathbf{\Gamma})$ to the g-th automorphism group $\mathbf{\Gamma}_{g,1}$ is given by the natural action of the mapping class group on $H_n^{\mathrm{BM}}(F_n(\Sigma'_{g,1}); \mathbb{Z})$. In fact, in [PS23, §1.3.3], we remove a closed interval from the boundary of $\Sigma_{g,1}$, instead of just a point, but the resulting configuration spaces are homotopy equivalent. The result then follows from the above discussion; see in particular (2.1). □

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