Annex file to the article: “Polynomiality of surface braid and mapping class group representations”

Martin Palmer and Arthur Soulié

11th July 2023

1. Introduction

This file is an annex to the article [PS23]. It first deals with recollections of presentations of surface braid groups in §2. Then we prove in §3 some properties of the transformation groups of the homological representation functors studied in [PS23]. We finally show in §4 the relationship between the Moriyama representations [Mor07] with some homological representations for mapping class groups from [PS23, §1.3.3] and their duals.

Disclaimer 1 The results of this annex file are a potpourri of “folklore” facts and classical properties, and are not strictly necessary for the proofs of the results of [PS23]. Namely, the only required points are the following:

- The presentations of surface braid groups of Proposition 2.2: this is classical knowledge following from the previous literature on this topic [Bel04; DPS22], which we simply adapt to our conventions (see Remark 2.3).
- Corollary 3.8 computing the transformation groups of the mapping class groups homological representation functors: such calculations are easy, straightforwardly following from the definitions and some classical computations of the abelianisation of a semi-direct product (see Lemma 3.7).
- The identification of the Moriyama representations in Proposition 4.1: this is essentially a consequence of Poincaré duality that we carefully detail here.

However, we explain these three points here in more detail for the convenience of the reader and for the sake of completeness. Furthermore, the other results of this annex file add some concreteness to the homological representation functors, and may help the reader in understanding the objects we consider in [PS23].

General notation and conventions. This file is not intended to be read independently of [PS23]. In particular, we follow the framework of [PS23], although we often recall its notation for the convenience of the reader.

We denote by $\mathfrak{S}_n$ the symmetric group on a set of n elements. For an integer $n \geqslant 1$, an *ordered partition of* n means an ordered r-tuple $\mathbf{n} = \{n_1, \ldots, n_r\}$ of integers $n_i \geqslant 1$ (for some $r \geqslant 1$ called the *length* of $\mathbf{n}$) such that $n = \sum_{1 \leqslant i \leqslant r} n_i$ (and without the condition $n_i \geqslant n_{i+1}$). For simplicity, we denote the trivial partition $\mathbf{n} = \{n\}$ by n. The lower central series of a group G is the descending chain of subgroups $\{\varGamma_\ell(G)\}_{\ell \geqslant 1}$ defined by $\varGamma_1(G) := G$ and $\varGamma_{\ell+1}(G) := [G, \varGamma_\ell(G)]$, the subgroup of G generated by the commutators $[g, h]$ for $g \in G$ and $h \in \varGamma_\ell(G)$. The induced canonical projection $G \twoheadrightarrow G/\varGamma_\ell(G)$ is denoted by $/\varGamma_\ell(G)$. When there is no ambiguity, we omit G from the notation.

2. Presentations of surface braid groups

Presentations of braid groups on surfaces with one boundary component may be found in [HL02, §4] and in [Bel04, Th. 1.1 and A.2]; see also [DPS22, §6.3]. These are classical results that are useful to justify some details in [PS23]. We re-write them with our own conventions, detailed in Remark 2.3. We fix three non-negative integers $k \geqslant 0$, $g \geqslant 0$ and $h \geqslant 1$. We denote by $\Sigma_{g,1}^k$ an orientable surface of genus g with one boundary component with k punctures in the interior, and by $\mathcal{N}_{h,1}^k$ a non-orientable surface of genus h with one boundary component and k punctures in the interior.

Notation 2.1 We write $x \rightleftarrows y$ to denote the relation saying that x and y commute.

Proposition 2.2 *The braid group on n strands on the orientable surface $\Sigma_{g,1}^k$, denoted by $\mathbf{B}_n(\Sigma_{g,1}^k)$, admits the presentation with generators $\mathcal{S} = \{\sigma_i\}_{1\leqslant i\leqslant n-1}$, $A = \{a_i\}_{1\leqslant i\leqslant g}$, $B = \{b_i\}_{1\leqslant i\leqslant g}$ and $X = \{\xi_i\}_{1\leqslant i\leqslant k}$ and relations given by the braid relations for the elements of $\mathcal{S}$, to which are added the following families of relations (where x and y denote either a or b, and $1 \leqslant r, s \leqslant g$):*

$$\begin{cases} (BS1) & \sigma_i \rightleftarrows x_r & \textit{for all } r \textit{ and all } 1 \leqslant i \leqslant n-2\ ; \\ (BS2) & x_r \rightleftarrows \sigma_{n-1} y_s \sigma_{n-1}^{-1} & \textit{for } s < r\ ; \\ (BS3) & (\sigma_{n-1} x_r)^2 = (x_r \sigma_{n-1})^2 & \textit{for all } r\ ; \\ (BS4) & [\sigma_{n-1} b_r \sigma_{n-1}^{-1}, a_r^{-1}] = \sigma_{n-1}^2 & \textit{for all } r; \\ (BS5) & \xi_j \rightleftarrows \sigma_i & \textit{for all } 1 \leqslant j \leqslant k \textit{ and all } 1 \leqslant i \leqslant n-2\ ; \\ (BS6) & x_r \rightleftarrows \sigma_{n-1} \xi_j \sigma_{n-1}^{-1} & \textit{for all } 1 \leqslant j \leqslant k \textit{ and all } 1 \leqslant r \leqslant g\ ; \\ (BS7) & \xi_i \rightleftarrows \sigma_{n-1} \xi_j \sigma_{n-1}^{-1} & \textit{for } i < j. \end{cases} \tag{2.1}$$

The braid group on n strands on the non-orientable surface $\mathcal{N}_{h,1}^k$, denoted by $\mathbf{B}_n(\mathcal{N}_{h,1}^k)$, admits the presentation with generators $\mathcal{S} = \{\sigma_i\}_{1\leqslant i\leqslant n-1}$, $C = \{c_i\}_{1\leqslant i\leqslant h}$ and $X = \{\xi_i\}_{1\leqslant i\leqslant k}$ and relations given by the braid relations for the elements of $\mathcal{S}$, to which are added the following families of relations (where $1 \leqslant r, s \leqslant h$):

$$\begin{cases} (BN1) & \sigma_i \rightleftarrows c_r & \textit{for all } r \textit{ and all } 1 \leqslant i \leqslant n-2\ ; \\ (BN2) & c_r \rightleftarrows \sigma_{n-1} c_s \sigma_{n-1}^{-1} & \textit{for } s < r\ ; \\ (BN3) & [\sigma_{n-1} c_r \sigma_{n-1}^{-1}, c_r^{-1}] = \sigma_{n-1}^2 & \textit{for all } r; \\ (BN4) & \xi_j \rightleftarrows \sigma_i & \textit{for all } 1 \leqslant j \leqslant k \textit{ and all } 1 \leqslant i \leqslant n-2\ ; \\ (BN5) & c_r \rightleftarrows \sigma_{n-1} \xi_j \sigma_{n-1}^{-1} & \textit{for all } 1 \leqslant j \leqslant k \textit{ and all } 1 \leqslant r \leqslant h\ ; \\ (BN6) & \xi_i \rightleftarrows \sigma_{n-1} \xi_j \sigma_{n-1}^{-1} & \textit{for } i < j\ ; \\ (BN7) & (\sigma_{n-1} \xi_j)^2 = (\xi_j \sigma_{n-1})^2 & \textit{for all } 1 \leqslant j \leqslant k. \end{cases} \tag{2.2}$$

Remark 2.3 There is an obvious correspondence between the elements of the sets A, B, C and X in the presentations of Proposition 2.2 and the generators of the fundamental group of the underlying surface; see [Bel04]. Namely, we fix $\{A', B'\} := \{\alpha_i, \beta_i \mid 1 \leqslant i \leqslant g\}$ a system of meridians and longitudes for the surface $\Sigma_{g,1}$, $C' := \{\varsigma_i \mid 1 \leqslant i \leqslant h\}\}$ a system of curves passing through each of the crosscaps of $\mathcal{N}_{h,1}$ and $X' := \{\gamma_i \mid 1 \leqslant i \leqslant k\}\}$ a system of curves encircling the k punctures for each surface. Then the sets A and B correspond to moving a configuration point along the curves of $\{A', B'\}$, the set C corresponds to moving a configuration point along the curves of C' and the set X corresponds to moving a configuration point along the curves of X'.

On another note, we use the opposite convention to that of [Bel04; DPS22]. More precisely, our numbering for the generators of $\mathcal{S} = \{\sigma_i\}_{1\leqslant i\leqslant n-1}$ is the converse of the one used by Bellingeri, so that the respective roles of σ_1 and σ_{n-1} in the mixed relations in the above presentations are switched compared to those of [Bel04] and [DPS22, §6.3.1]. We made this arbitrary choice to be consistent with the module structures for the homological representations detailed in [PS23, §2]: our model and convention is so that σ_{n-1} is the braid generator that naturally interacts with the fundamental group generators; see [PS23, Fig. 3.1 (a) and (b)] for instance.

Now we consider a partition $\mathbf{k} = \{k_1; \ldots; k_r\} \vdash k$. Hence there is an isomorphism $\mathbf{B}_{\mathbf{k}}(S) \cong \mathbf{B}_{\{k_1;\ldots;k_{r-1}\}}(\mathbb{D}_{k_r} \natural S) \rtimes \mathbf{B}_{k_r}(S)$ deduced from the (Fadell-Neuwirth) split short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \mathbf{B}_{\{k_1;\ldots;k_{r-1}\}}(\mathbb{D}_{k_r} \natural S) \arrow[r] & \mathbf{B}_{\mathbf{k}}(S) \arrow[r] & \mathbf{B}_{k_r}(S) \arrow[r] \arrow[l, bend right, dashed] & 1. \end{tikzcd} \tag{2.3}$$

We recall that there is a classical method of constructing a presentation of a group extension from a presentation of the quotient and a presentation of the kernel; see [HEO05, §2.4.3] and [DPS22,

Appendix B]. For instance, the presentation of the group $\mathbf{B}_{k,n}(\Sigma_{g,1})$ is detailed in [BGG17, Prop. 3.2] following this method, while that of $\mathbf{B}_{k,n}(\mathcal{N}_{h,1})$ is sketched in [DPS22, Prop. 6.58]. It is routine to generalise this work to give a full presentation for the partition $\mathbf{k}$, while an analogous study may directly be made for the non-orientable surface $\mathcal{N}_{h,1}$ following the same method. We thus obtain from Proposition 2.2 the following result for the partitioned surface braid groups.

Proposition 2.4 *Let $\mathbf{k} = \{k_1; \ldots; k_r\}$ be a partition of $k \geqslant 1$. The surface braid group $\mathbf{B}_{\mathbf{k}}(S)$ admits a presentation whose generating sets are:*

- $X^{(\rho)} = \{\xi_i^{(\rho)} \mid 1 \leqslant i \leqslant \Sigma_\rho\}$ *with* $\Sigma_\rho := \sum_{\rho+1 \leqslant l \leqslant r} k_l$*, for each block* $1 \leqslant \rho \leqslant r-1$*;*
- $\mathcal{S}^{(\rho')} = \{\sigma_i^{(\rho')}\}_{1 \leqslant i \leqslant k_{\rho'}-1}$ *for each block* $1 \leqslant \rho' \leqslant r$ *such that* $r_{\rho'} \geqslant 2$*;*
- *if* $S = \Sigma_{g,1}$*:* $A^{(\rho)} = \{a_i^{(\rho)}\}_{1 \leqslant i \leqslant g}$ *and* $B^{(\rho)} = \{b_i^{(\rho)}\}_{1 \leqslant i \leqslant g}$ *for each block* $1 \leqslant \rho \leqslant r$*;*
- *if* $S = \mathcal{N}_{h,1}$*:* $C^{(\rho)} = \{c_i^{(\rho)}\}_{1 \leqslant i \leqslant h}$ *for each block* $1 \leqslant \rho \leqslant r$*.*

The relations between generators of the same blocks are those of (2.1) and (2.2), while the relations between generators of different blocks are analogous to those of (c.1)–(c.8) in [BGG17, Prop. 3.2].

3. Properties of the transformation groups

This section deals with some properties of the homological representation functors of [PS23, §1]. It aims to compute the transformation groups (see Lemmas 3.3 and 3.5 and Corollary 3.8), prove Q-stability properties (see Definition 3.2 below, Proposition 3.4 and Corollary 3.8), explain some restrictions on the parameter ℓ for the lower central series (see Corollary 3.9) and characterise the transformation group rings (see Corollary 3.9). Throughout §3 we consider an integer $\ell \geqslant 1$ corresponding to a lower central series index, and an integer $k \geqslant 1$ and a partition $\mathbf{k} = \{k_1; \ldots; k_r\} \vdash k$. We also denote by r' the number of indices $i \leqslant r$ in $\mathbf{k}$ such that $k_i \geqslant 2$.

Transformation groups. We briefly recollect here the construction of the transformation groups that define the homological representation functors of [PS23, §1.2]. We recall that we consider a family of groups $\{G_n\}_{n\in\mathbb{N}}$, which will be either surface braid groups or mapping class groups, the configuration space $\left\{(x_1, \ldots, x_k) \in \mathcal{S}_n^{\times k} \mid x_i \neq x_j \text{ if } i \neq j\right\}/\mathfrak{S}_{\mathbf{k}}$, denoted by $C_{\mathbf{k}}(\mathcal{S}_n)$, associated to the partition $\mathbf{k}$ of k points in a surface $\mathcal{S}_n$ which is defined depending on the setting (see [PS23, §1.2]). For each n, there is a split short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \mathbf{B}_{\mathbf{k}}(\mathcal{S}_n) \arrow[r] & G_{\mathbf{k},n} \arrow[r] & G_n \arrow[l, bend right, dashed] \arrow[r] & 1, \end{tikzcd} \tag{3.1}$$

defining the following key diagram (see [PS23, §1.2, (1.7)]):

$$\begin{tikzcd} 1 \arrow[r] & \mathbf{B}_{\mathbf{k}}(\mathcal{S}_n) \arrow[r] \arrow[d, two heads] & G_{\mathbf{k},n} \arrow[r] \arrow[d, two heads] & G_n \arrow[l, bend right, dashed] \arrow[r] \arrow[d, two heads] & 1 \\ 1 \arrow[r] & Q_{(\mathbf{k},\ell,n)}(\mathcal{S}) \arrow[r] & G_{\mathbf{k},n}/\varGamma_\ell \arrow[r] & G_n/\varGamma_\ell \arrow[l, bend right, dashed] \arrow[r] & 1. \end{tikzcd} \tag{3.2}$$

The universal property of $G_{\mathbf{k},n}/\varGamma_\ell$ and $G_n/\varGamma_\ell$ as cokernels and the universal property of a kernel ensure that there exists a canonical map $q_{(\mathbf{k},\ell,n)} \colon Q_{(\mathbf{k},\ell,n)}(\mathcal{S}) \to Q_{(\mathbf{k},\ell,n+1)}(\mathcal{S})$ making the evident diagram induced from (3.2) (comparing n with $n+1$) commutative. The colimit of the groups $\{(Q_{(\mathbf{k},\ell,n)}(\mathcal{S}))\}_{n\in\mathbb{N}}$ with respect to the maps $q_{(\mathbf{k},\ell,n)}$ is denoted by $Q_{(\mathbf{k},\ell)}(\mathcal{S})$.

Remark 3.1 The above framework for transformation groups defined in [PS23, §1.2] differs slightly from that of [PS21]. In particular, in [PS23], we introduce and use the colimit $Q_{(\mathbf{k},\ell)}$ to define the homological representation functors and define the untwisted transformation group $Q^u_{(\mathbf{k},\ell)}$. However, these modifications do not affect the construction.

In many examples of interest, we have the following phenomenon:

Definition 3.2 ([PS21, Def. 5.14]) If the colimit $Q_{(\mathbf{k},\ell)}(\mathcal{S})$ is isomorphic to $Q_{(\mathbf{k},\ell,n)}(\mathcal{S})$ (via the canonical map) for all n sufficiently large, we say that we have *Q-stability* in this setting.

Although **we do not need this Q-stability property for the work of [PS23]**, this phenomenon remains of interest to point out, for instance allowing us to compute the colimit $Q_{(\mathbf{k},\ell)}(\mathcal{S})$.

3.1. Surface braid groups

We first deal with the computation of the transformation groups of the homological representation functors defined in [PS23, §1.3]. We use the notations of Proposition 2.4 and 3.1. We consider a compact, connected, smooth surface S with one boundary component.

Abelian quotients. We start with the functors defined with the Γ_2-term of the lower central series. We recall from [DPS22, Prop. 3.5] that the abelianisation of the group $\mathbf{B}_{\mathbf{k},n}(S)$ may be described via the corresponding generating set of Proposition 2.4. In $\mathbf{B}_{\mathbf{k},n}(S)^{\mathrm{ab}}$:

- for any S: we denote by $t_{i'}$ the common image of the generators of $\mathcal{S}^{(i')}$ for each $1 \leqslant i' \leqslant r$ such that $r_{i'} \geqslant 2$;
- if $S = \mathbb{D}$: we denote by q_i the common image of all $\xi_j^{(i)} \in X^{(i)}$ with $j \geqslant 1 + \sum_{i+1 \leqslant l \leqslant r} k_l$ for each $1 \leqslant i \leqslant r$, and by s_{i_1,i_2} the common image of all $\xi_j^{(i_1)}$ with $j \in \{j' + \sum_{i_1+1 \leqslant l \leqslant i_2-1} k_l \mid 1 \leqslant j' \leqslant i_2\}$ for each pair $1 \leqslant i_1 < i_2 \leqslant r$.
- if $S = \Sigma_{g,1}$: the images $\{A_j^{(i)}, B_j^{(i)}\}_{1 \leqslant j \leqslant g}$ of the sets $A^{(\rho)}$ and $B^{(\rho)}$ for each $1 \leqslant i \leqslant r$;
- if $S = \mathcal{N}_{h,1}$: the images $\{C_j^{(i)}\}_{1 \leqslant j \leqslant h}$ of the set $C^{(\rho)}$ for each $1 \leqslant i \leqslant r$; $C_j^{(i)}$.

By definition, the following result for the transformation groups of the functors of [PS23, §1.3.1–§1.3.2] is then a straightforward consequence of the computations of the abelianisations of $\mathbf{B}_{\mathbf{k},n}(S)^{\mathrm{ab}}$ and $\mathbf{B}_n(S)^{\mathrm{ab}}$ for $n \geqslant 3$; see for instance [DPS22, Prop. 6.47].

Lemma 3.3 *For all $n \geqslant 3$, we have:*

- *if $S = \mathbb{D}^2$: $Q_{(\mathbf{k},2)}(\mathbb{D}) = Q_{(\mathbf{k},2,n)}(\mathbb{D}) = \mathbb{Z}^{r'} \times \mathbb{Z}^{r(r-1)/2} \times \mathbb{Z}^r = \langle t_1, \ldots, t_{r'} \rangle \times \langle s_{r_1,r_2} \rangle_{1 \leqslant r_1 < r_2 \leqslant r} \times \langle q_1, \ldots, q_r \rangle$;*
- *otherwise: $Q_{(\mathbf{k},2)}(S) = Q_{(\mathbf{k},2,n)}(S) = (\mathbb{Z}/2)^{r'} \times H_1(S;\mathbb{Z})^r$.*

In particular, the Q-stability property for the functors of [PS23, §1.3] with parameters $(\mathbf{k}, 2)$ directly follows from the observation that $Q_{(\mathbf{k},\ell,n)}(\mathbb{D}) = Q_{(\mathbf{k},\ell,n+1)}(\mathbb{D})$ for all $n \geqslant 3$. However, we will more generally prove that this property holds for the transformation groups associated to any functor $\mathfrak{LB}_{(\mathbf{k},\ell)}$ of [PS23, §1.3.1] in Proposition 3.4 below.

Further Γ_ℓ-quotients. We now consider more generally the surface braid group homological representations defined with any parameter $\ell \geqslant 2$. First, the following result proves the Q-stability property for the transformation groups associated to the functors $\mathfrak{LB}_{(\mathbf{k},\ell)}$ of [PS23, §1.3.1], $\mathfrak{L}_{(\mathbf{k},\ell)}(\Sigma_{g,1})$ and $\mathfrak{L}_{(\mathbf{k},\ell)}(\mathcal{N}_{h,1})$ of [PS23, §1.3.2]. This is a generalisation of [PS21, Prop. 5.35].

Proposition 3.4 *We fix integers $\ell \geqslant 2$ and $n \geqslant 4$. Then we have $Q_{(\mathbf{k},\ell,n)}(S) = Q_{(\mathbf{k},\ell,n+1)}(S)$.*

Proof. We take up the presentations, notations and conventions of §2. The data which depends on n in the presentation of $\mathbf{B}_{\mathbf{k},n}(S)$ are the set of braid generators $\mathcal{S}^{(n)}$ of the n-th block, and, for each block $1 \leqslant \rho \leqslant r$, the subset of $X^{(\rho)}$ of the pure braid generators $\{\chi_i^{(\rho)} := \xi_{\Sigma_\rho + i}^{(\rho)} \mid 1 \leqslant i \leqslant n\}$ where Σ_ρ denotes the sum $\sum_{\rho+1 \leqslant l \leqslant r} k_l$. For a group G, we generically denote by γ_ℓ the projection onto the ℓ-nilpotent quotient G/Γ_ℓ.

Since the assignment $S \mapsto \mathbf{B}_n(S)$ is functorial with respect to embeddings of surfaces, we have a canonical injection $\mathbf{B}_n \hookrightarrow \mathbf{B}_n(S) \hookrightarrow \mathbf{B}_{\mathbf{k},n}(S)$. In particular, this morphism sends $\Gamma_\infty(\mathbf{B}_n)$ to $\Gamma_\infty(\mathbf{B}_{\mathbf{k},n}(S))$. Since $\Gamma_\infty(\mathbf{B}_n) = \Gamma_2(\mathbf{B}_n)$ (see for instance [DPS22, Ex. 2.3]), we know that $\sigma_i \sigma_j^{-1} \in \Gamma_\infty(\mathbf{B}_n)$ for all $1 \leqslant i, j \leqslant n-1$. A fortiori, we deduce that $\sigma_i^{(n)} \equiv \sigma_j^{(n)} \pmod{\Gamma_\infty(\mathbf{B}_n(S))}$ and we denote by $\sigma^{(n)} \in \mathbf{B}_n(S)/\Gamma_\ell$ the common image of all the $\sigma_i^{(n)}$ under γ_ℓ.

Furthermore, for each $1 \leqslant \rho \leqslant r$, we have the relations $\sigma_i^{(n)} \chi_i^{(\rho)} (\sigma_i^{(n)})^{-1} = (\chi_i^{(\rho)})^{-1} \chi_{i+1}^{(\rho)} \chi_i^{(\rho)}$, $\sigma_i^{(n)} \chi_{i+1}^{(\rho)} (\sigma_i^{(n)})^{-1} = \chi_i^{(\rho)}$ and $\sigma_i^{(n)} \chi_j^{(\rho)} (\sigma_i^{(n)})^{-1} = \chi_j^{(\rho)}$ if $j \notin \{i, i+1\}$ by Proposition 2.4. These are the typical relations between pure braids and Artin generators of a given block induced by the injection $\mathbf{B}_{\mathbf{k},n} \hookrightarrow \mathbf{B}_{\mathbf{k},n}(S)$; see [BGG17, Prop. 3.2] for the case of $S = \Sigma_{g,1}$ and $\mathbf{k} = k$. Then, we deduce from these relations that:

- $\gamma_\ell(\chi_{i+1}^{(\rho)}) = \gamma_\ell((\sigma_i^{(n)})^{-1} \chi_i^{(\rho)} \sigma_i^{(n)}) = \gamma_\ell((\sigma_{i-1}^{(n)})^{-1} \chi_i^{(\rho)} \sigma_{i-1}^{(n)}) = \gamma_\ell(\chi_i^{(\rho)})$ for all $2 \leqslant i \leqslant n-1$;
- $\gamma_\ell(\chi_2^{(\rho)}) = \gamma_\ell((\sigma_1^{(n)})^{-1} \chi_1^{(\rho)} \sigma_1^{(n)}) = \gamma_\ell((\sigma_3^{(n)})^{-1} \chi_1^{(\rho)} \sigma_3^{(n)}) = \gamma_\ell(\chi_1^{(\rho)})$.

We denote by $\chi^{(\rho)} \in \mathbf{B}_{\mathbf{k},n}(S)/\Gamma_\ell$ the common image of all the $\chi_i^{(\rho)}$ under γ_ℓ.

Therefore, the presentation of $\mathbf{B}_{\mathbf{k},n}(S)/\Gamma_\ell$ is independent of n. In particular, it is routine to check that there is a well-defined map $\gamma'_\ell\colon \mathbf{B}_{\mathbf{k},n+1}(S) \to \mathbf{B}_{\mathbf{k},n}(S)/\Gamma_\ell$ defined by $\sigma_i^{(n+1)} \mapsto \sigma^{(n)}$, $\chi_j^{(\rho)} \mapsto \chi^{(\rho)}$ (with $1 \leqslant j \leqslant n+1$) and the assignment of γ_ℓ for the other generators. Then γ'_ℓ induces an inverse to the canonical map $\mathbf{B}_{\mathbf{k},n}(S)/\Gamma_\ell \to \mathbf{B}_{\mathbf{k},n+1}(S)/\Gamma_\ell$, which is thus an isomorphism. We also know the analogous result for $\mathbf{B}_n(S)/\Gamma_\ell$ by [BGG17, Prop. 3.13] (see also [DPS22, Prop. 6.43]), whence the result. □

In addition, there are many situations where we can actually compute the transformation groups of the surface braid group homological representation functors. We recall that when $\ell = 2$ this is done in Lemma 3.3 and the following result gathers the results for other situations.

Lemma 3.5 *We assume that the partition is such that $k_l \geqslant 3$ for all $1 \leqslant l \leqslant r$.*

For the classical braid groups, we have for all $\ell \geqslant 2$

$$Q_{(\{2;\mathbf{k}\},\ell)}(\mathbb{D}) = \mathbb{Z}^{\binom{r+2}{2}-1} \times ((\mathbb{Z}^2/2^{\ell-2}\bar{\Delta})^{r+1} \rtimes \mathbb{Z}),$$

where $\bar{\Delta} = (1,-1) \in \mathbb{Z}^2$ and $1 \in \mathbb{Z}$ acts on each copy of $\mathbb{Z}^2/2^{\ell-2}\bar{\Delta}$ by swapping coordinates.

For the surfaces different from the disc, we have the following computations for $\ell = 3$. For orientable surfaces, for all $g \geqslant 1$:

$$Q_{(\mathbf{k},3)}(\Sigma_{g,1}) = ((\mathbb{Z}^{r(r-1)/2} \times \mathbb{Z}^r \times \mathbb{Z}^{rg}) \rtimes \mathbb{Z}^{rg}) \times \mathbb{Z}^r. \tag{3.3}$$

In more detail, the right-hand side of (3.3) may be written as

$$\left(\langle s_{r_1,r_2} \rangle_{1 \leqslant r_1 < r_2 \leqslant r} \times \langle t_1, \ldots, t_r \rangle \times \prod_{1 \leqslant \rho \leqslant r} \langle A_1^{(\rho)}, \ldots, A_g^{(\rho)} \rangle \right) \rtimes \prod_{1 \leqslant \rho \leqslant r} \langle B_1^{(\rho)}, \ldots, B_g^{(\rho)} \rangle \times \langle q_1, \ldots, q_r \rangle$$

where the action defining the semi-direct product structure is determined by

- $[A_i^{(\rho)}, B_i^{(\rho)}] = t_\rho^2$ *for all* $1 \leqslant \rho \leqslant r$;
- $[A_i^{(r_1)}, B_i^{(r_2)}] = [A_i^{(r_2)}, B_i^{(r_1)}] = s_{r_1,r_2}$ *for all* $1 \leqslant r_1 < r_2 \leqslant r$;
- *all other pairs of generators commute.*

We deduce that $Q^u_{(\mathbf{k},3)}(\Sigma_{g,1}) = Q_{(\mathbf{k},3)}(\Sigma_{g,1})/\langle q_1, \ldots, q_r \rangle$. For non-orientable surfaces, for all $h \geqslant 1$:

$$Q_{(\mathbf{k},3)}(\mathcal{N}_{h,1}) = \big((\mathbb{Z}^{r-1} \times (\mathbb{Z}^{r-2} \times \cdots \times (\mathbb{Z}^2 \times (\mathbb{Z} \times \mathbb{Z}^h) \rtimes \mathbb{Z}^h) \rtimes \cdots \rtimes \mathbb{Z}^h) \rtimes \mathbb{Z}^h) \rtimes \mathbb{Z}^h\big) \times (\mathbb{Z}/2)^r \times \mathbb{Z}^r. \tag{3.4}$$

In more detail, the right-hand side of (3.4) may be written as

$$\Big(\Big(\langle s_{1,r_2} \rangle_{2 \leqslant r_2 \leqslant r} \times \cdots \times \Big(\langle s_{r-1,r} \rangle \times \langle C_1^{(r)}, \ldots, C_h^{(r)} \rangle\Big) \rtimes \cdots \rtimes \langle C_1^{(2)}, \ldots, C_h^{(2)} \rangle\Big) \rtimes \langle C_1^{(1)}, \ldots, C_h^{(1)} \rangle\Big) \times \langle t_1, \ldots, t_r \rangle \times \langle q_1, \ldots, q_r \rangle$$

where the action defining the semi-direct product structure is determined by

- $[C_i^{(r_1)}, C_i^{(r_2)}] = [C_i^{(r_2)}, C_i^{(r_1)}] = s_{r_1,r_2}$ *for all* $1 \leqslant r_1 < r_2 \leqslant r$;
- *all other pairs of generators commute.*

We deduce that $Q^u_{(\mathbf{k},3)}(\mathcal{N}_{h,1}) = Q_{(\mathbf{k},3)}(\mathcal{N}_{h,1})/\langle q_1, \ldots, q_r \rangle$.

Proof. That $Q_{(\{2;\mathbf{k}\},\ell)}(\mathbb{D}) = Q^u_{(\{2;\mathbf{k}\},\ell)}(\mathbb{D})$ and its explicit computation for each $\ell \geqslant 3$ is done in [PS22, §4, Cor. 5.4].

For the surfaces different from the disc, we first recall that the quotient $\mathbf{B}_n(\Sigma_{g,1})/\Gamma_3$ is computed by [BGG17, Prop. 3.13], while $\mathbf{B}_n(\mathcal{N}_{h,1})/\Gamma_3 = \mathbf{B}_n(\mathcal{N}_{h,1})^{\mathrm{ab}}$ by [DPS22, Th. 6.42]. Now the computation of $\mathbf{B}_{\mathbf{k},n}(S)/\Gamma_3$ follows the same steps as the proof of [DPS22, Prop. 6.58] which computes $\mathbf{B}_{k,n}(\mathcal{N}_{h,1})/\Gamma_3$ and generalises mutatis mutandis as follows.

- Using the presentation from Proposition 2.4, let N be the normal closure of the $\sigma_i^{(\rho)}(\sigma_{i+1}^{(\rho)})^{-1}$ for $i < k_\rho$ together with the $\xi_j^{(\rho)}(\xi_{j+1}^{(\rho)})^{-1}$ for $j < \Sigma_\rho$, for each $1 \leqslant \rho \leqslant n$. Therefore, $N \subseteq \Gamma_3(\mathbf{B}_{\mathbf{k},n}(S))$ because its generators are in $\Gamma_3(\mathbf{B}_{\mathbf{k},n}(S))$; see for instance [DPS22, Lem. 6.49] and its proof.

- We consider the partition $\{\mathbf{k}, n\}$ as a partition $\mathbf{k}'$ of k with length $r+1$ and we make the identification $s_{i,n} := q_i$ for each $1 \leqslant i \leqslant r$. It is routine, although lengthy (and an inductive generalisation of the analogous point in the proof of [DPS22, Prop. 6.58]), to check from the presentation of Proposition 2.4 that the quotient $\mathbf{B}_{\mathbf{k},n}(S)/N$ is

$$\left(\langle s_{r_1,r_2} \rangle_{1 \leqslant r_1 < r_2 \leqslant r+1} \times \langle t_1, \ldots, t_{r+1} \rangle \times \prod_{1 \leqslant \rho \leqslant r+1} \langle A_1^{(\rho)}, \ldots, A_g^{(\rho)} \rangle \right) \rtimes \prod_{1 \leqslant \rho \leqslant r+1} \langle B_1^{(\rho)}, \ldots, B_g^{(\rho)} \rangle \tag{3.5}$$

if $S = \Sigma_{g,1}$, and

$$\Big(\langle s_{1,r_2} \rangle_{2 \leqslant r_2 \leqslant r+1} \times \cdots \times (\langle s_{r,r+1} \rangle \times \langle C_1^{(r+1)}, \ldots, C_h^{(r+1)} \rangle) \rtimes \cdots \rtimes \langle C_1^{(1)}, \ldots, C_h^{(1)} \rangle \Big) \times \langle t_1, \ldots, t_{r+1} \rangle \tag{3.6}$$

if $S = \mathcal{N}_{h,1}$.

- The proof thus follows from the observation that, using the second point, $\mathbf{B}_{\mathbf{k},n}(S)/N$ is a 2-nilpotent group. Indeed, in both cases, the commutator subgroup is generated by the elements s_{r_1,r_2} and t_i^2 (if $S = \Sigma_{g,1}$ for the latter): all these generators are also clearly central in $\mathbf{B}_{\mathbf{k},n}(S)/N$, which proves our claim.

The computations of $Q_{(\mathbf{k},3)}(\Sigma_{g,1})$ and $Q_{(\mathbf{k},3)}(\mathcal{N}_{h,1})$ then directly follow from the above description of $\mathbf{B}_{k,n}(\mathcal{N}_{h,1})/\Gamma_3$ and $\mathbf{B}_n(\Sigma_{g,1})/\Gamma_3$.

Finally, we compute the untwisted quotients $Q^u_{(\mathbf{k},3)}(\Sigma_{g,1})$ and $Q^u_{(\mathbf{k},3)}(\mathcal{N}_{h,1})$ as follows. First we know from the presentation of $\mathbf{B}_{\mathbf{k},n}(S)/\Gamma_3$ (see (3.5) and (3.6)) that the only generators of $Q_{(\mathbf{k},3)}(S)$ on which the action of $\mathbf{B}_n(S)$ (given by conjugation) is not trivial are $A_i^{(\rho)}$ and $B_i^{(\rho)}$ for all i and each $1 \leqslant \rho \leqslant r$ if $S = \Sigma_{g,1}$, or the $C_j^{(\rho)}$ for all j and each $1 \leqslant \rho \leqslant r$ if $S = \mathcal{N}_{h,1}$. Then, we deduce from the presentation of Proposition 2.4 that for all $1 \leqslant \rho \leqslant r$:

$$q_\rho B_i^{(\rho)} = A_i^{(r+1)} B_i^{(\rho)} (A_i^{(r+1)})^{-1}$$

and its analogue swapping A and B for all $1 \leqslant i \leqslant g$, and

$$q_\rho C_j^{(\rho)} = C_j^{(r+1)} C_j^{(\rho)} (C_j^{(r+1)})^{-1}$$

for all $1 \leqslant j \leqslant h$. This proves that the quotienting submodule defining the coinvariants is $\langle q_1, \ldots, q_r \rangle$ in both case, which ends the proof. □

Remark 3.6 If $k_i \leqslant 2$ for some $1 \leqslant i \leqslant r$, it is not clear to our knowledge that the quotient $Q^u_{(\mathbf{k},\ell)}(S)$ of $Q_{(\mathbf{k},\ell)}(S)$ is *proper* for each $\ell \geqslant 3$. However, there are conjectures on such results; see for instance [PS21, Conjecture 5.38].

3.2. Mapping class groups

We now study the mapping class group transformation groups for the homological representation functors of [PS23, §1.3.3]. For the sake of completeness, we recall that the Lickorish generators together with the Dehn twist along a simple closed curve encircling the boundary component generate the mapping class group $\mathbf{\Gamma}_{g,1}$ with $g \geqslant 1$ (see for instance [FM12, §4.4]), and that standard generating sets for the mapping class group $\boldsymbol{\mathcal{N}}_{h,1}$ are worked out by Stukow in [Stu06, Th. A.7] for $h = 2$ and in [Stu10, Th. 5.2] for $h \geqslant 3$ while $\boldsymbol{\mathcal{N}}_{0,1} = \boldsymbol{\mathcal{N}}_{1,1}$ are trivial by [Eps66].

We first make the following general decomposition for the abelianisations of the mapping class groups:

Lemma 3.7 *For S a compact, connected, smooth, non-planar surface with one boundary component, we have:*

$$\mathrm{MCG}(\mathbb{D}_k \natural S, \mathbf{k})^{\mathrm{ab}} \cong (\mathbb{Z}/2)^{r'} \times (H_1(S;\mathbb{Z})^r)_{\mathrm{MCG}(S)} \times \mathrm{MCG}(S)^{\mathrm{ab}}, \tag{3.7}$$

where each of the first r' $\mathbb{Z}/2$-summands is generated by the image in the abelianisation $\sigma^{(\rho')}$ (with $1 \leqslant \rho' \leqslant r$ such that $k_{\rho'} \geqslant 2$) of a standard braid generator (considered as a mapping class) interchanging two points in the corresponding ρ'-th block of the partition.

Proof. Using the computation of $\mathbf{B_k}(S)^{\mathrm{ab}}$ (see for instance [DPS22, Prop. 6.47]), it follows from the general formula for the calculation of the abelianisation of a semi-direct product that:

$$\mathrm{MCG}(\mathbb{D}_k \natural S, \mathbf{k})^{\mathrm{ab}} \cong ((\mathbb{Z}/2)^{r'} \times H_1(S;\mathbb{Z})^r)_{\mathrm{MCG}(S)} \times \mathrm{MCG}(S)^{\mathrm{ab}}.$$

We note that the splitting of the Birman short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \mathbf{B_k}(\mathcal{S}_n) \arrow[r] & \mathrm{MCG}(\mathbb{D}_k \natural S, \mathbf{k}) \arrow[r] & \mathrm{MCG}(S) \arrow[r] \arrow[l, bend right, dashed] & 1 \end{tikzcd} \tag{3.8}$$

(see [PS21, Cor. 4.19, §5.1.3] for instance) is induced by the embedding of surfaces $S \hookrightarrow \mathbb{D}_k \natural S$. Therefore, each $\sigma^{(\rho')}$ may be represented as a mapping class supported in the subsurface $\mathbb{D}_k$ on which $\mathrm{MCG}(S)$ acts trivially, whence the result. □

A fortiori, we have the following calculations for the transformation groups of the homological representation functors $\mathfrak{L}_{(\mathbf{k},2)}(\mathbf{\Gamma})$ and $\mathfrak{L}_{(\mathbf{k},2)}(\boldsymbol{\mathcal{N}})$ of [PS23, §1.3.3]. Following the notations of [PS23, §1.1.2.1 and §1.3.3] where $\mathbb{T} \cong \Sigma_{1,1}$ and $\mathbb{M} \cong \mathcal{N}_{1,1}$, we denote by $\mathbf{\Gamma}^{\mathbf{k}}_{g,1}$ the mapping class group $\mathrm{MCG}(\Sigma_{g,1}, \mathbf{k})$ and by $\boldsymbol{\mathcal{N}}^{\mathbf{k}}_{h,1}$ the mapping class group $\mathrm{MCG}(\mathcal{N}_{h,1}, \mathbf{k})$.

Corollary 3.8 *The transformation groups of the homological representation functors for mapping class groups are such that:*

- *For orientable surfaces:* $Q_{(\mathbf{k},2,g)}(\mathbb{T}) = (\mathbb{Z}/2)^{r'}$ *for all* $g \geqslant 3$.
- *For non-orientable surfaces:* $Q_{(\mathbf{k},2,h)}(\mathbb{M}) = (\mathbb{Z}/2)^{r'} \times (\mathbb{Z}/2)^r$ *for all* $h \geqslant 7$. *Here, for each* $1 \leqslant \rho \leqslant r$, *the* ρ*-th* $\mathbb{Z}/2$*-summand is generated by the image of a puncture slide* $c^{(\rho)}$ *that sends a puncture of the* ρ*-th block through the core of a cross-cap.*

In particular, the functors $\mathfrak{L}_{(\mathbf{k},2)}(\mathbf{\Gamma})$ *and* $\mathfrak{L}_{(\mathbf{k},2)}(\boldsymbol{\mathcal{N}})$ *satisfy the* Q*-stability property.*

Proof. By Lemma 3.7, the results follow from the computations of $(H_1(S;\mathbb{Z})^r)_{\mathrm{MCG}(S)}$ for $S = \mathbb{T}^{\natural g}$ and $\mathbb{M}^{\natural h}$. Let us denote by δ the discrete partition $\{1, ..., 1\}$. We know from [Kor02, Th. 5.1] that the abelianisations of $\mathbf{\Gamma}_{g,1}$ and $\mathbf{\Gamma}^{\delta}_{g,1}$ are trivial for $g \geqslant 3$, while it follows from [Stu10, Th. 6.21] that $(\boldsymbol{\mathcal{N}}^{\delta}_{h,1})^{\mathrm{ab}} \cong (\mathbb{Z}/2)^r \times (\boldsymbol{\mathcal{N}}_{h,1})^{\mathrm{ab}}$ for $h \geqslant 7$. We then deduce from (3.7) that $(H_1(\mathbb{T}^{\natural g};\mathbb{Z})^r)_{\mathbf{\Gamma}_{g,1}} = 0$ and that $(H_1(\mathcal{N}_{h,1};\mathbb{Z})^r)_{\boldsymbol{\mathcal{N}}_{h,1}} \cong (\mathbb{Z}/2)^r$, whence the result. □

Finally, following the methods developed in [DPS22], it is not difficult to obtain results about the lower central series of some *partitioned* mapping class groups of punctured surfaces. Following the terminology of [DPS22], the lower central series $\mathit{\Gamma}_*(G)$ of a group G is said to *stop* if there exists an integer $i \geqslant 1$ such that $\mathit{\Gamma}_i(G) = \mathit{\Gamma}_{i+1}(G)$. We say that it *stops at* $\mathit{\Gamma}_i$ if i is the smallest integer for which this holds.

Corollary 3.9 *Let* $g \geqslant 1$. *The lower central series* $\mathit{\Gamma}_*(\mathbf{\Gamma}^{\mathbf{k}}_{g,1})$ *stops before* $\mathit{\Gamma}_2$. *It stops at* $\mathit{\Gamma}_1$ *if* $g \geqslant 3$ *and* $\mathbf{k}$ *is a discrete partition.*

Proof. First, we note that when $g \geqslant 3$ and $\mathbf{k}$ is the discrete partition δ, we know from [Kor02, Th. 5.1] that the abelianisation of $\mathbf{\Gamma}^{\delta}_{g,1}$ is trivial, so its lower central series stops at $\mathit{\Gamma}_1$. We therefore just have to show that the lower central series of $\mathbf{\Gamma}^{\mathbf{k}}_{g,1}$ stops before $\mathit{\Gamma}_2$ in all cases, which we will do by the usual geometric disjoint support trick introduced in [DPS22]: for a group G and a generating set S of G^{ab}, if for each pair $(s,t) \in S^2$, we can find representatives $\tilde{s}, \tilde{t} \in G$ of s and t such that $\tilde{s}$ and $\tilde{t}$ commute, then $\mathit{\Gamma}_2(G) = \mathit{\Gamma}_3(G)$; see [DPS22, Cor. 2.6]. We recall from Lemma 3.8 and Corollary 3.7 that $(\mathbf{\Gamma}^{\mathbf{k}}_{g,1})^{\mathrm{ab}} \cong \bigoplus_{k_\rho \geqslant 2} \langle \sigma^{(\rho)} \rangle$ with $\sigma^{(\rho)}$ the image of a standard braid generator interchanging two points of the ρ-th block. It is geometrically clear that the generators $\sigma^{(\rho')}$ for distinct ρ' have pairwise disjoint support, because there always exits a subdisc of the surface containing only the points of the ρ'-th block. Hence each pair of generators of the abelianisation of $\mathbf{\Gamma}^{\mathbf{k}}_{g,1}$ may be represented by homeomorphisms of the surface with disjoint support, whence the result. □

Remark 3.10 In general, whether or not the lower central series of the mapping class group $\boldsymbol{\mathcal{N}}^{\mathbf{k}}_{h,1}$ stops is an open question: from its abelianisation $(\mathbb{Z}/2)^{r'} \times (\mathbb{Z}/2)^r \times (\mathbb{Z}/2)$, it is not immediately clear how to decide this via the methods of [DPS22]. Thus we cannot at present compute $Q_{(\mathbf{k},\ell)}(\mathbb{M})$

for $\ell \geqslant 3$. The only exception is the case of $\mathbf{k}=1$: the lower central series of $\boldsymbol{\mathcal{N}}_{h,1}^{(1)}$ stops at $\varGamma_2$ by [DPS22, Cor. 2.2] because $(\boldsymbol{\mathcal{N}}_{h,1}^{(1)})^{\mathrm{ab}} \cong \mathbb{Z}/2$. It is therefore a priori relevant to consider higher $\ell \geqslant 3$ as a parameter for the homological representation functors $\mathfrak{L}_{(\mathbf{k},\ell)}(\boldsymbol{\mathcal{N}})\colon \mathfrak{U}\mathcal{M}_2^- \to \mathbb{Z}[Q_{(\mathbf{k},2)}(\mathbb{M})]\text{-Mod}$ of [PS23, §1.3.3].

4. Comparison of representations: the Moriyama representations

Moriyama [Mor07] considered the $\mathbf{\Gamma}_{g,1}$-representation given by its action on the relative homology group $H_n(\Sigma_{g,1}^{\times n}, \Delta \cup A_g)$, where Δ denotes the "fat diagonal" of $\Sigma_{g,1}^{\times n}$ where at least two points coincide and A_g denotes the subspace of $\Sigma_{g,1}^{\times n}$ where at least one point is equal to p_0, a chosen basepoint on $\partial\Sigma_{g,1}$. In this section, we make the connection of these mapping class group representations with those encoded by certain homological representation functors introduced in [PS23, §1.3.3] and their duals; see (4.1) and Proposition 4.1.

Let us write $\Sigma'_{g,1} = \Sigma_{g,1} \smallsetminus \{p_0\}$. Since $\Sigma_{g,1}^{\times n}$ is a compactification of $F_n(\Sigma'_{g,1}) = \Sigma_{g,1}^{\times n} \smallsetminus (\Delta \cup A_g)$, the Borel-Moore homology of $F_n(\Sigma'_{g,1})$ is isomorphic to the relative homology group $H_*(\Sigma_{g,1}^{\times n}, \Delta \cup A_g)$. Thus Moriyama's representation may be viewed as an action on Borel-Moore homology. Denoting by $F_n(X,Y) \subseteq F_n(X)$ the subspace of configurations that intersect $Y \subseteq X$ non-trivially, by Poincaré duality we have $H_*^{\mathrm{BM}}(F_n(\Sigma'_{g,1})) \cong H^n(F_n(\Sigma'_{g,1}), F_n(\Sigma'_{g,1}, \partial\Sigma'_{g,1}))$ since $F_n(\Sigma'_{g,1})$ is a connected, orientable manifold. Using the fact that $\Sigma'_{g,1} \subset \Sigma_{g,1}$ and $\{p_1\} \subset \partial\Sigma'_{g,1}$ are isotopy equivalences, where $p_1 \in \partial\Sigma'_{g,1}$ is another (different) point on the boundary of $\Sigma_{g,1}$, this is naturally isomorphic to $H^n(F_n(\Sigma_{g,1}), F_n(\Sigma_{g,1}, \{p_1\}))$.

A special case of [PS23, Lem. 2.1] implies that $H_*^{\mathrm{BM}}(F_n(\Sigma'_{g,1}))$ is concentrated in degree $* = n$, so by the universal coefficient theorem, Moriyama's representation $H_n(\Sigma_{g,1}^{\times n}, \Delta \cup A_g)$ is dual to the relative cohomology group $H^n(\Sigma_{g,1}^{\times n}, \Delta \cup A_g)$. Analogous identifications to those above, replacing H_*^{BM} with compactly-supported cohomology H_c^*, etc., apply to these dual representations.

In summary, we have identifications:

$$\begin{array}{ccccc} H_n(\Sigma_{g,1}^{\times n}, \Delta \cup A_g) & \cong & H_n^{\mathrm{BM}}(F_n(\Sigma'_{g,1})) & \cong & H^n(F_n(\Sigma_{g,1}), F_n(\Sigma_{g,1}, \{p_1\})) \\ {\scriptstyle dual}\updownarrow & & & & \\ H^n(\Sigma_{g,1}^{\times n}, \Delta \cup A_g) & \cong & H_c^n(F_n(\Sigma'_{g,1})) & \cong & H_n(F_n(\Sigma_{g,1}), F_n(\Sigma_{g,1}, \{p_1\})), \end{array} \tag{4.1}$$

where the top row are models for Moriyama's representation and the bottom row are models for its dual. Finally, the representations on the top row (i.e. Moriyama's representation) are isomorphic to the representations encoded by the homological representation functor $\mathfrak{L}_{(\{1,\ldots,1\},1)}(\mathbf{\Gamma})$ introduced in [PS23, §1.3.3]:

Proposition 4.1 *The restriction of the functor $\mathfrak{L}_{(\{1,\ldots,1\},1)}(\mathbf{\Gamma})$ to the g-th automorphism group $\mathbf{\Gamma}_{g,1}$ is the n-th Moriyama representation $\mathbf{\Gamma}_{g,1} \to \mathrm{Aut}_{\mathbb{Z}}(H_n(\Sigma_{g,1}^{\times n}, \Delta \cup A_g))$.*

Proof. The restriction of the functor $\mathfrak{L}_{(\{1,\ldots,1\},1)}(\mathbf{\Gamma})$ to the g-th automorphism group $\mathbf{\Gamma}_{g,1}$ is given by the natural action of the mapping class group on $H_n^{\mathrm{BM}}(F_n(\Sigma'_{g,1}))$. In fact, in [PS23, §1.3.3], we remove a closed interval from the boundary of $\Sigma_{g,1}$, instead of just a point, but the resulting configuration spaces are homotopy equivalent. The result then follows from the above discussion; see in particular (4.1). □

References

[Bel04] P. Bellingeri. *On presentations of surface braid groups*. *J. Algebra* 274.2 (2004), pp. 543–563 (↑ 1, 2).

[BGG17] P. Bellingeri, E. Godelle and J. Guaschi. *Abelian and metabelian quotient groups of surface braid groups*. *Glasg. Math. J.* 59.1 (2017), pp. 119–142 (↑ 3–5).

[DPS22] J. Darné, M. Palmer and A. Soulié. *When the lower central series stops: a comprehensive study for braid groups and their relatives.* To appear in the Memoirs of the American Mathematical Society; ArXiv: 2201.03542. 2022 (↑ 1–8).

[Eps66] D. B. A. Epstein. *Curves on 2-manifolds and isotopies*. *Acta Math.* 115 (1966), pp. 83–107 (↑ 6).

[FM12] B. Farb and D. Margalit. *A primer on mapping class groups*. Vol. 49. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 2012, pp. xiv+472 (↑ 6).

[HEO05] D. F. Holt, B. Eick and E. A. O'Brien. *Handbook of computational group theory*. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2005, pp. xvi+514 (↑ 2).

[HL02] R. Häring-Oldenburg and S. Lambropoulou. *Knot theory in handlebodies*. Vol. 11. 6. Knots 2000 Korea, Vol. 3 (Yongpyong). 2002, pp. 921–943 (↑ 1).

[Kor02] M. Korkmaz. *Low-dimensional homology groups of mapping class groups: a survey*. *Turkish J. Math.* 26.1 (2002), pp. 101–114 (↑ 7).

[Mor07] T. Moriyama. *The mapping class group action on the homology of the configuration spaces of surfaces*. *J. Lond. Math. Soc. (2)* 76.2 (2007), pp. 451–466 (↑ 1, 8).

[PS21] M. Palmer and A. Soulié. *Topological representations of motion groups and mapping class groups – a unified functorial construction*. ArXiv: 1910.13423v4. 2019, v4: 2021 (↑ 3, 4, 6, 7).

[PS22] M. Palmer and A. Soulié. *The pro-nilpotent Lawrence-Krammer-Bigelow representation*. ArXiv: 2211.01855. 2022 (↑ 5).

[PS23] M. Palmer and A. Soulié. *Polynomiality of surface braid and mapping class group representations*. ArXiv: 2302.08827. 2023 (↑ 1–4, 6–8).

[Stu06] M. Stukow. *Dehn twists on nonorientable surfaces*. *Fund. Math.* 189.2 (2006), pp. 117–147 (↑ 6).

[Stu10] M. Stukow. *Generating mapping class groups of nonorientable surfaces with boundary*. *Adv. Geom.* 10.2 (2010), pp. 249–273 (↑ 6, 7).

Martin Palmer, *Institutul de Matematică Simion Stoilow al Academiei Române, 21 Calea Griviței, 010702 București, Romania.* Email address: `mpanghel@imar.ro`

Arthur Soulié, *Center for Geometry and Physics, Institute for Basic Science (IBS), Pohang 37673, Korea.* Email address: `artsou@hotmail.fr, arthur.soulie@ibs.re.kr`