

TRIPLE-CROSSING NUMBER AND MOVES ON TRIPLE-CROSSING LINK DIAGRAMS

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ABSTRACT. Every link in the 3-sphere has a projection to the plane where the only singularities are pairwise transverse triple points. The associated diagram, with height information at each triple point, is a *triple-crossing diagram* of the link. We give a set of diagrammatic moves on triple-crossing diagrams analogous to the Reidemeister moves on ordinary diagrams. The existence of n -crossing diagrams for every $n > 1$ allows the definition of the n -crossing number. We prove that for any nontrivial, nonsplit link, other than the Hopf link, its triple-crossing number is strictly greater than its quintuple-crossing number.

1. INTRODUCTION

The classical theory of knots and links is often approached via link *diagrams* and the well-known *Reidemeister moves*. A link diagram is a projection of the link to a 2-sphere, which we think of as a plane union a point at infinity, having only transverse double points as singularities and with an indication at each double point of which strand is “on top.” This is done by erasing a small section of the under-crossing strand. The Reidemeister moves consist of local transformations which can be used to alter a diagram while not changing the underlying link. In general, we will refer to any such diagrammatic change as a *diagram move*. The important result, proven by both Reidemeister [8] and Alexander and Briggs [6], is that two diagrams represent the same link if and only if they are related by a sequence of Reidemeister moves.

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Of course, we also allow any ambient isotopy of the diagram in the projection 2-sphere. Thus we may treat the objects of knot theory as equivalence classes of diagrams.

When working with oriented links, there is a corresponding set of oriented Reidemeister moves. In [7] it is shown that a set of four oriented Reidemeister moves, called $\Omega 1a$, $\Omega 1b$, $\Omega 2a$, and $\Omega 3a$, and their inverses, are sufficient to pass between any two oriented diagrams of the same link. Later in this paper, we will extend these moves to four moves which we call $C1a$, $C1b$, $C2a$, and $C3a$ pictured in Figure 6. If the dashed curves in the figure (which will be explained later) are ignored, then $C1a$, $C1b$, $C2a$, and $C3a$ reduce to $\Omega 1a$, $\Omega 1b$, $\Omega 2a$, and $\Omega 3a$, respectively. If we ignore the orientations in $\Omega 1a$, $\Omega 1b$, $\Omega 2a$, and $\Omega 3a$ we obtain a set of three unoriented moves sufficient to pass between all unoriented diagrams of the same unoriented link.

Recently, several papers have explored the topic of link diagrams with *multicrossings*. See, for example, [1]–[5]. In these diagrams, n strands are allowed to cross at a single point in the plane (still pairwise transversely), creating what is known as an n -crossing. Now each n -crossing must be accompanied with a labeling of the strands, $1, 2, \dots, n$, from top to bottom in order to depict the link in space. Many of the obvious results analogous to classical diagrams have been proven. For example, given any $n > 1$, every link has an n -diagram, that is, one with only n -crossings. However, until now, no analog of the Reidemeister moves have been found for multicrossing diagrams. In Sections 2–5 we describe a set of moves on 3-diagrams and prove that they are sufficient to pass between all 3-diagrams of the same link, as long as the “natural” orientation of the 3-diagrams define the same oriented link, up to a certain equivalence.

Because every link has an n -diagram for every n , the n -crossing number, $c_n(L)$, of a link L may be defined as the smallest number of n -crossings in any n -diagram of L . See [1]. It is known that $c_2(L) > c_3(L)$ and $c_2(L) > c_4(L)$. Moreover, it is known that $c_n(L) \geq c_{n+2}(L)$ and $c_n(L) \geq c_{2n}(L)$ for all $n \geq 2$ and all L . See [3]. In Section 6 of this paper we prove that for any nontrivial, nonsplit link L , other than the Hopf link, $c_3(L) > c_5(L)$.

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2. MOVES ON 3-DIAGRAMS

Before describing our set of 3-diagram moves, we begin by noticing that an unoriented 3-diagram can be given a natural orientation by

using the checkerboard coloring of its complement. In fact, this is true for all $(2n + 1)$ -diagrams for any $n \geq 1$, and is quite different from the case of $(2n)$ -diagrams.

Lemma 1. *The complementary regions of any n -diagram may be colored black and white, checkerboard fashion.*

Proof. By slightly perturbing the strands near each multicrossing of an n -diagram D , each n -crossing can be separated into $n(n - 1)/2$ classical crossings, creating a classical diagram d of the same link. Now d can be checkerboard colored and that coloring induces one on D when the perturbed strands are returned to their original positions. \square

If D is a $(2n + 1)$ -diagram, we can use the checkerboard coloring to orient D by orienting the boundary edges of each black region counterclockwise. Because each multicrossing involves an odd number of strands, the orientations match up to give an orientation of D . We call this a *natural* orientation of D . Swapping the colors reverses the orientation. Thus there are two possible natural orientations. If D is disconnected, meaning that the associated projection¹ P is a disconnected subset of the projection plane, then we will refer to the diagram associated to a maximal, connected, subset of P as a *subdiagram* of D . If D is disconnected, then a *piecewise natural* orientation of D is one that is natural on each subdiagram of D . If D is the union of j disjoint subdiagrams, then there are 2^j piecewise natural orientations of D .

In any 3-diagram, call a 3-crossing *Type A* if, as we encircle the crossing, the strands are oriented in–out–in–out–in–out. Considering Type A crossings provides an alternate approach to orienting a 3-diagram.

Lemma 2. *An orientation of a 3-diagram D is piecewise natural if and only if all crossings are Type A.*

Proof. If the orientation of a diagram is piecewise natural, then it is easy to see that every crossing is Type A. Conversely, suppose D is oriented so that every crossing is Type A and that D' is a subdiagram of D . Let R be a complementary region of D' . Because all the crossings are of Type A, it follows that the boundary edges of R are all oriented in the same direction as we go around the boundary of R . Therefore, if the complementary regions of D' are checkerboard colored black and white, then the orientations on the boundaries of the black regions are either all clockwise or all counterclockwise. Thus the orientation of D is piecewise natural. \square

¹By a *projection*, we mean an actual projection with transverse double points (or multi-points), as opposed to a *diagram* where heights have been indicated at every multi-point.

Figures 1, 2 and 3 illustrate the *1-move*, *2-move*, *basepoint move*, and *band move* on 3-diagrams. The 1-move can be thought of as first performing a Type I Reidemeister move on the strand on the right and then sliding the 1-gon over the strand on the left. We may think of the 2-move as performing a Type II Reidemeister move on the two outer strands with the center strand lying above.

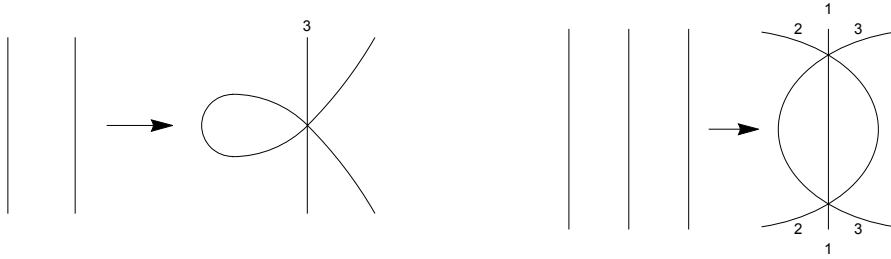


FIGURE 1. The 1-move and 2-move.

A basepoint move is illustrated in Figure 2, and is defined as follows.² Suppose the 3-diagram D contains a trivial 3-string tangle T_1 consisting of three parallel arcs, two of which are boundary parallel (the *outer arcs*) and one of which is not (the *central arc*). The endpoints of the three arcs come in *opposite* pairs in the obvious way. Suppose that α is an arc of D whose endpoints are a pair of opposite endpoints of T_1 belonging to the outer arcs. Suppose further that α is an *over-arc* of D , that is, for every crossing that α passes through, it lies on top. Moreover, assume that α passes through at least one 3-crossing and let T_2 be a small disk centered at that crossing. To make the move, we replace T_1 with a 3-crossing T'_1 where the central arc of T_1 becomes the lowest strand of T'_1 , the endpoints of α are joined to form the highest strand, and the other pair of endpoints are joined to form the middle strand. Additionally, the crossing at T_2 is replaced with a trivial tangle T'_2 having as central strand what was the lowest strand of the crossing. Figure 2 shows how a basepoint move might appear. Notice that α may pass through many 3-crossings and T_2 does not need to be the 3-crossing nearest to T_1 . Moreover, the heights of the strands in T_2 and the choice of endpoints of α in T_1 may not appear the same as in Figure 2.

To see that the basepoint move does not change the underlying link, simply pick up the arc α and then lay it down inside the tangle T_1 . This

²Why we call this a basepoint move will become clear in Section 4.

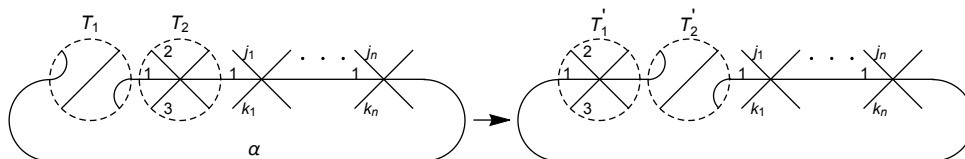


FIGURE 2. A possible basepoint move.

creates a 2-crossing in T_1 and reduces all the 3-crossings to 2-crossings. We obtain the same diagram if, after the move, the over-arc is again picked up and laid down inside T'_2 .

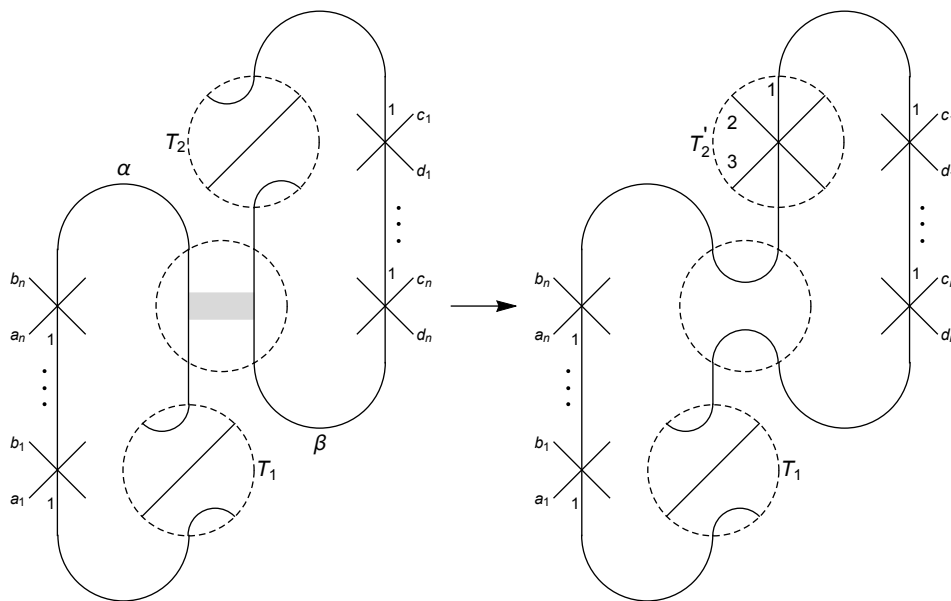


FIGURE 3. A possible band move.

Figure 3 depicts a possible band move. Suppose that D contains two trivial 3-string tangles T_1 and T_2 , each consisting of three parallel arcs, two disjoint over-arcs α and β , and a potential band move between α and β . (Recall that if γ and δ are a pair of disjoint embedded arcs in the plane then a band move on γ and δ is defined as follows. Consider an embedding of $I \times I$ (the band) into the plane such that $(\gamma \cup \delta) \cap (I \times I) = \partial I \times I$. We can now alter $\gamma \cup \delta$ by replacing $\partial I \times I$ with $I \times \partial I$.) Futhermore, suppose the endpoints of α are a pair of opposite

endpoints in T_1 , not the ends of the central arc, and the same is true of β with respect to T_2 . To perform the move, we perform the band move, joining α to β . Additionally, we replace T_2 with a 3-crossing T'_2 where the central arc of T_2 becomes the lowest strand of T'_2 , the endpoints of β are joined to form the highest strand of T'_2 , and the other opposite pair of endpoints are joined to form the middle strand of T'_2 . As with the basepoint move, the two 3-diagrams can be seen to represent the same link if we pass to 2-diagrams by lifting up the over-arcs and laying them down as the over-crossing strand of a single 2-crossing. Again, the situation need not look exactly as in Figure 3.

In addition to the moves illustrated in Figures 1, 2 and 3, we include one more move. Let α be an arc of a 3-diagram D that passes through no crossings and suppose α' is another arc whose interior does not intersect D , which has the same boundary points as α , and which has no self-crossings. Replacing α with α' is called a *trivial pass move*. If D is connected, and we think of the diagram as lying on a 2-sphere rather than a plane, then the trivial pass move can be achieved by isotopy of the diagram. But if D is disconnected, then trivial pass moves may be nontrivial. For example, trivial pass moves allow one subdiagram of D to be picked up and put down in a different complementary region of the rest of D . With classical diagrams, this can be accomplished with Reidemeister moves, but for 3-diagrams, a trivial pass move may not be a consequence of our other moves.

We may now state our main theorem:

Theorem 3. *Two unoriented 3-diagrams D_1 and D_2 are related by a sequence of 1-moves, 2-moves, basepoint moves, band moves, and trivial pass moves, if and only if natural orientations on D_1 and D_2 define the same oriented link, up to reversal of maximal nonsplit sublinks.*

If D_1 and D_2 represent nonsplit links, the trivial pass move is not needed and the statement of the theorem can be made somewhat simpler.

Corollary 4. *Two unoriented 3-diagrams D_1 and D_2 of nonsplit links are related by a sequence of 1-moves, 2-moves, basepoint moves, and band moves, if and only if natural orientations on D_1 and D_2 define the same oriented link, up to complete reversal.*

Corollary 5. *Two unoriented 3-diagrams of knots are related by a sequence of 1-moves, 2-moves, basepoint moves, and band moves, if and only if they define the same unoriented knot.*

Remark: Notice that a number of variations on all of the 3-diagram moves exist that still preserve the underlying link with natural orientation. For example, in the 1-move, the 1-gon could lie under the other strand. In the 2-move, the central strand could lie between or below the other two strands. Notice that in both the basepoint move and the band move, the over-arcs could be replaced with *under-arcs* instead (that is, arcs that pass beneath all other crossings). In Figures 2 and 3, all the heights of 1 would then become 3, etc.. These variations on the moves still preserve the link type, but are not needed in the statement of Theorem 3. Instead, it is possible to derive these variations from the other moves.

We will prove Theorem 3 in Section 5.

3. STATE MARKERS AND THE EVEN STATE

Given a classical link projection, each crossing divides the plane locally into four regions. A *state marker* at a given crossing is a choice of two opposite regions indicated by placing dots in the corners of the two regions near the crossing. A *state* for the projection is a choice of state marker at each crossing. An *even state* is one where each complementary region of the projection contains an even number of dots.

A state marker at a crossing determines a way to smooth that crossing, namely, smooth it in the way that connects the two regions marked with the dots. If a crossing is a *self-crossing*, that is, the two strands there belong to the same component, then exactly one of the two state markers will determine a smoothing which splits the component into two components. The other state marker determines a smoothing which will maintain a single component. Call the former the *fission* state marker.

If the projection D is oriented, then we may use the orientation to determine a state marker at each crossing as illustrated in Figure 4. If the crossing is a self-crossing, then the orientation induced state marker and the fission state marker coincide. If we choose the state marker at every crossing by using the orientation, we call this the *orientation induced state* of D .

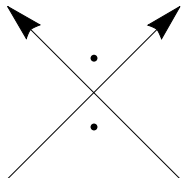


FIGURE 4. The state marker induced by the orientation.

Theorem 6. *Suppose D is a classical link projection. Then all the following are true.*

- (1) *An orientation induced state of D is even.*
- (2) *If D is connected, then orientations \mathcal{O}_1 and \mathcal{O}_2 of D induce the same state if and only if they are equal or one is the reverse of the other.*
- (3) *Every even state of D is the orientation induced state for some orientation of D .*
- (4) *If D is a connected diagram of a link with k components, then D has 2^{k-1} distinct even states.*

Proof. To prove (1), suppose D is oriented, that S is the orientation induced state, and that R is a complementary region of D . As we traverse the boundary of R , we encounter an even number of vertices where the orientation of the edges on ∂R reverses from clockwise to counter-clockwise or vice versa. These are exactly the vertices that contribute a dot to R . Thus R contains an even number of dots and S is an even state.

For (2), because reversing all orientations in Figure 4 does not change the placement of the dots, it follows that if one orientation is the reverse of the other then both induce the same state. Conversely, if orientations \mathcal{O}_1 and \mathcal{O}_2 induce the same state, then at each crossing they either agree or are the reverse of each other. Because D is connected, it follows that they are equal at every crossing, or opposite at every crossing. (This is false if D is not connected.)

Let S be an even state of D . We will prove (3) by induction on the number of double points in D . If D has one double point, the result is obvious. Now suppose that D has more than one double point. Pick one and smooth it as in Figure 5, to obtain a new projection D' . By forgetting the state marker at the crossing that was removed by smoothing, we obtain a state S' of D' . The state S' is even because the regions b and d each contain an even number of dots and, hence,

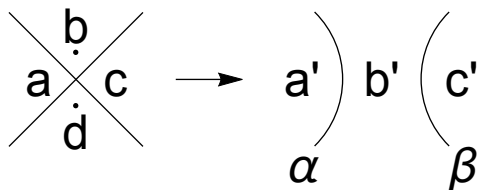


FIGURE 5. Smoothing a crossing.

so does the region b' in D' . Because D' has one fewer crossing than D , there exists an orientation of D' for which S' is the induced state. If b and d are really the same region, then D' is disconnected with α and β lying in disjoint subdiagrams of D' . If necessary, we may reverse all the orientations of the subdiagram containing α so that α and β are oriented parallel to each other. As already proven, this will not change the even state S' . If instead, b and d are distinct regions, then we may traverse the boundary of b' , starting at α and reach β . One way takes us essentially around the boundary of b , the other way around the boundary of d . But in either case, we pass an odd number of dots because both b and d each contain an even number of dots. As we pass each dot, the orientation of the edges on the boundary of b' is reversed. Thus when we get to β we see that it is oriented parallel to α . Hence, in both cases, there is an orientation of D' for which S' is the induced state and moreover, α and β are parallel. We may now use this orientation to obtain an orientation of D that induces S .

Finally, for (4), if D is a connected projection of a link with k components, we now see that the map from the set of orientations of D to the set of even states of D is a 2-to-1 surjection. The total number of orientations of D is 2^k and hence the total number of even states is 2^{k-1} . \square

In the case of a knot we have

Corollary 7. *If D is a knot projection, then D has a unique even state.*

4. CROSSING CIRCLES

Given a classical diagram D , a *crossing circle* for D is a circle C embedded in the projection 2-sphere that intersects D only at crossings. Additionally, at each such crossing, the two strands of D are each transverse to C . The crossing circle is *trivial* if it contains no crossings of D . A finite set of disjoint crossing circles that together contain all crossings of D , and moreover, where a crossing has been chosen on each

nontrivial crossing circle, will be called a *crossing circle cover* of D , or more concisely, a *CCC* of D . We will refer to the chosen crossing on each crossing circle as the *basepoint*. If no basepoints have been chosen on the crossing circles, we will refer to the collection of circles as an *unmarked CCC*.

Each *CCC* of a diagram D determines an even state of the underlying projection by choosing the state marker at each crossing that labels the regions containing the crossing circle. Moreover, it is easy to see that every even state determines an unmarked *CCC*, although not uniquely. Because there are an even number of dots in each region, we may connect them in pairs with disjoint arcs inside each region and then piece the arcs together to obtain an unmarked *CCC*. There may be more than one way to do this. However, all unmarked *CCC*'s associated to the same even state are related by band moves and the introduction or deletion of a trivial crossing circle. Thus, Theorem 6 implies that every diagram has a *CCC*. In the case of a knot diagram, the even state is unique. Thus any two *CCC*'s of the same knot diagram differ by band moves, change of basepoints, and insertion or deletion of trivial crossing circles. For links, we must additionally require that the two *CCC*'s determine the same even state in order that they be so related. We summarize these statements in the following result.

Lemma 8. *Two CCC's of a classical diagram D determine the same even state of D if and only if they differ by band moves, change of basepoints, and insertion or deletion of trivial crossing circles.*

If we consider oriented diagrams, then we can demand that any *CCC* of the diagram be *compatible* with the orientation, that is, the state determined by the *CCC* matches the orientation induced state. Lemma 8 now implies that if two *CCC*'s are both compatible with the same oriented diagram, then they differ by band moves, change of basepoints, and insertion or deletion of trivial crossing circles.

Theorem 9. *Suppose that D_1 and D_2 are two oriented 2-diagrams, each equipped with a compatible CCC. Then D_1 and D_2 represent the same oriented link, up to orientation reversal on maximal nonsplit sublinks, if and only if one can be changed to the other by the following moves:*

- (1) *Reversing the orientation on any subdiagram of a diagram.*
- (2) *Insertion or deletion of a trivial crossing circle.*
- (3) *The four Reidemeister moves, and their inverses, on compatible CCC-equipped diagrams shown in Figure 6.*

- (4) *Changing the CCC by moving the basepoint on one crossing circle.*
- (5) *Changing the CCC by a band move. If the band move splits a nontrivial crossing circle into two nontrivial crossing circles, we must introduce a new basepoint. If the band move joins two nontrivial crossing circles together, then one basepoint is removed.*

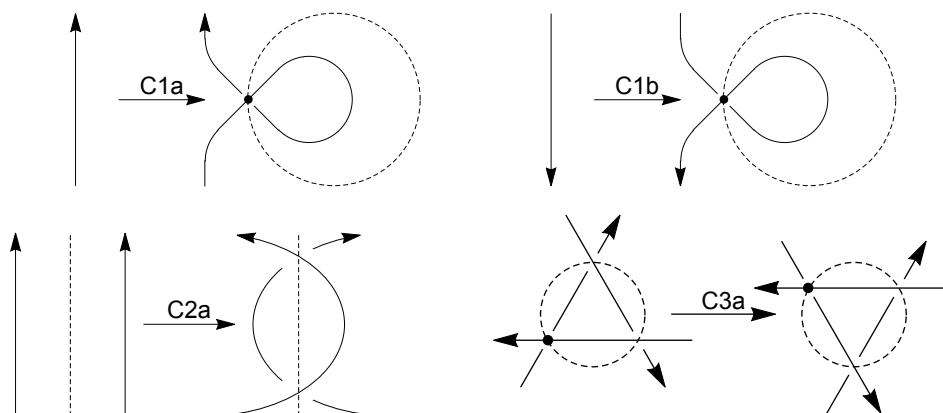


FIGURE 6. Reidemeister moves on oriented diagrams with compatible CCC's. Move C2a is shown in the case where the crossing circle on the left is nontrivial. The case with trivial crossing circle, if shown, would include a basepoint on one of the two crossings introduced by the move.

Proof. We first show that if two diagrams are related by any one of the five kinds of moves, then the oriented links that they define are the same up to orientation reversal of maximal nonsplit sublinks. In the case of the first move, notice that reversing the orientation on one subdiagram of a diagram reverses the orientation of a union of maximal nonsplit sublinks. Notice that the even state induced by the orientation is unchanged, so the CCC remains compatible. Each of the other moves preserves the oriented link exactly.

Now suppose that D_1 and D_2 are oriented diagrams that define oriented links L_1 and L_2 , respectively, that differ only by reversing maximal nonsplit sublinks. In general, if L is an oriented link, let $|L|$ be the unoriented link obtained from L by forgetting the orientation. Thus

$|L_1| = |L_2|$. Let D be some unoriented diagram of $|L_1|$ such that every subdiagram of D defines a maximal non-split sublink of $|L_1|$. We may now orient D somehow to obtain an oriented diagram \overline{D}_1 representing L_1 and perhaps some other way to obtain an oriented diagram \overline{D}_2 representing L_2 . Notice that \overline{D}_1 and \overline{D}_2 differ by at most complete reversal on subdiagrams. Because D_1 and \overline{D}_1 are two oriented diagrams of the same oriented link L_1 , there exists a sequence of $\Omega 1a, \Omega 1b, \Omega 2a$, and $\Omega 3a$ moves and their inverses that take D_1 to \overline{D}_1 . The same is true for D_2 and \overline{D}_2 . Our goal, for both $i = 1$ and $i = 2$, is to replace these oriented Reidemeister moves with the corresponding moves of Figure 6, thus carrying the *CCC* from D_i to one which is compatible with \overline{D}_i . However, we will see that to do this may also require moves (1), (2), (4), or (5) of the theorem. Once we show how to do this, we can then apply move (1) to \overline{D}_1 to obtain a *CCC* on \overline{D}_2 . We now have two *CCC*'s on \overline{D}_2 : the one coming from D_1 via \overline{D}_1 and the one coming from D_2 . But both are compatible with the same oriented diagram and hence, by the remark following Lemma 8, must be related by moves (2), (4), or (5). Thus, we will finally obtain a path from D_1 to D_2 using the moves given in the theorem.

We now return to the problem of “extending” the sequence of oriented Reidemeister moves from D_1 to \overline{D}_1 . It suffices to consider the case where d is an oriented diagram with compatible *CCC* and we wish to perform a single Reidemeister move (or its inverse) from the set $\{\Omega 1a, \Omega 1b, \Omega 2a, \Omega 3a\}$ to d , changing it into d' . If the *CCC* for d does not appear exactly as depicted in Figure 6, then we may need to first change the *CCC* by moves (2), (4), or (5). For example, suppose a crossing c is to be eliminated by the inverse of $\Omega 1a$ or $\Omega 1b$ but the crossing circles do not appear exactly as in $C1a$ or $C1b$, respectively. If trivial crossing circles lie within the 1-gon, then we may remove them with move (2). Or, if the crossing circle that contains c also contains other crossings, then a band move can be used to split off a crossing circle that contains only c . On the other hand, if we wish to perform $\Omega 1a$ or $\Omega 1b$, we can immediately apply $C1a$ or $C1b$; the *CCC* for d already appears as is needed.

The situation is a bit more subtle when extending an $\Omega 2a$ move to a $C2a$ move. Let α and β be the two strands involved in the Reidemeister move. There may be strands of crossing circles that lie “in between” α and β . Using band moves, we can clear out this region, one pair of crossing circle strands at a time, leaving either one or no crossing circles between them. If one crossing circle strand remains then we may perform the $C2a$ move (and introduce a basepoint at one of the

two new crossings if necessary). If no crossing circles lie in between α and β then it must be because α and β lie in different subdiagrams of D . For if they are both part of the same subdiagram, let R be the complementary region that lies between α and β . Note that R may contain subdiagrams in its interior. Let S be the set of all crossings c of the boundary of R such that the crossing circle that contains c locally meets the interior of R . As we traverse any boundary component of R , either the one containing both α and β , or perhaps one belonging to another subdiagram, the orientations on the edges of ∂R reverse each time we pass a crossing in S . Hence, each boundary component of R contains an even number of crossings in S . Moreover, as we traverse the boundary component of R traveling from α only as far as β , we must pass an odd number of crossings in S . Therefore, any arc from α to β lying inside R and intersecting crossing circles transversely must intersect an odd number of crossing circles. Hence, if no crossing circles lie between α and β , then they lie in different subdiagrams and we may introduce a trivial crossing circle that separate α and β . The *CCC* now appears as needed to perform the *C2a* move. In this case, a basepoint must be added to one of the newly introduced crossings. Finally, if we wish to extend the inverse of $\Omega 2a$ to the inverse of *C2a*, and the crossing circle that contains the two crossings also contains other crossings, we can first move the basepoint, if necessary, to not lie at either of the two crossings that are to be eliminated. If this is not the case, then the basepoint lies at one of the two crossings and after eliminating the 2-gon, the crossing circle is trivial.

Finally, notice that to extend $\Omega 3a$ (or its inverse) to *C3a* (or its inverse) we can use band moves and deletion of trivial crossing circles, if necessary, to first alter the *CCC* so that it appears as shown in Figure 6. \square

Using the fact that every classical diagram has a *CCC*, it is shown in [3] that every link has a 3-diagram. The construction is as follows. Given a link L , let D be a 2-diagram of L equipped with a *CCC*. For each non-trivial crossing circle C , let c be the chosen crossing on C . Now replace the over-crossing strand at c with the crossing circle. That is, as we come into the crossing c on the over-crossing strand, instead of continuing on the over-crossing strand, detour around the crossing circle on top of all the other crossings on C and in such a way as to eliminate the crossing c . All of the other crossings on the crossing circle are now turned into 3-crossings. If we do this for each crossing circle in the *CCC*, we obtain a 3-diagram of the same link. Note that trivial crossing circles are ignored or, if one prefers, removed at the beginning

of the process. An example is shown in Figure 7. Alternatively, we could replace the under-crossing strand at c by detouring around C *beneath* all the other crossings along C . We call the former the *over CCC* construction, and the latter the *under CCC* construction. If the *CCC* contains more than one circle, we could even mix the two constructions. Notice that in any case, one 2-crossing has been lost for each crossing circle. This implies that the 3-crossing number of a link is strictly less than the 2-crossing number. Let Ψ be the map from the set of 2-diagrams equipped with *CCC*'s into the set of 3-diagrams which is defined by the over *CCC* construction.

If D is an oriented 2-diagram with compatible *CCC*, then the over *CCC* construction produces an oriented 3-diagram. By abuse of notation, we will still use Ψ to refer to this construction in the oriented case.

The following lemma seems worth recording, but is not needed in our exposition.

Lemma 10. *If D is a connected oriented 2-diagram with compatible *CCC*, then the orientation on $\Psi(D)$ is natural.*

Proof. Suppose that D is a connected oriented 2-diagram with compatible *CCC* and suppose C is a crossing circle for D . Consider two consecutive crossings c_1 and c_2 as one traverses C . Thinking of these as 3-crossings (two strands from D together with C), then if we orient C so that c_1 is Type A, then c_2 must also be of Type A. To see this, let R be the complementary region of D containing the arc of C connecting c_1 and c_2 . As we traverse the boundary of R from c_1 to c_2 , which is possible because D is connected, the orientation of ∂R reverses each time we pass the endpoint of a crossing circle inside R . Thus there are an even number of reversals between c_1 and c_2 and hence c_2 is also of Type A. Thus if one crossing on C is Type A, all the crossings on C are Type A. Now when we perform the over *CCC* construction, the orientation induced on C by detouring around C instead of following the over-crossing strand at the basepoint, is the one that makes all the crossings along C of Type A. \square

5. PROOF OF THEOREM 3

Lemma 11. *Let D be any 3-diagram with a piecewise natural orientation. Then there exists an oriented 2-diagram d with compatible *CCC* such that D and $\Psi(d)$ differ by 1-moves.*

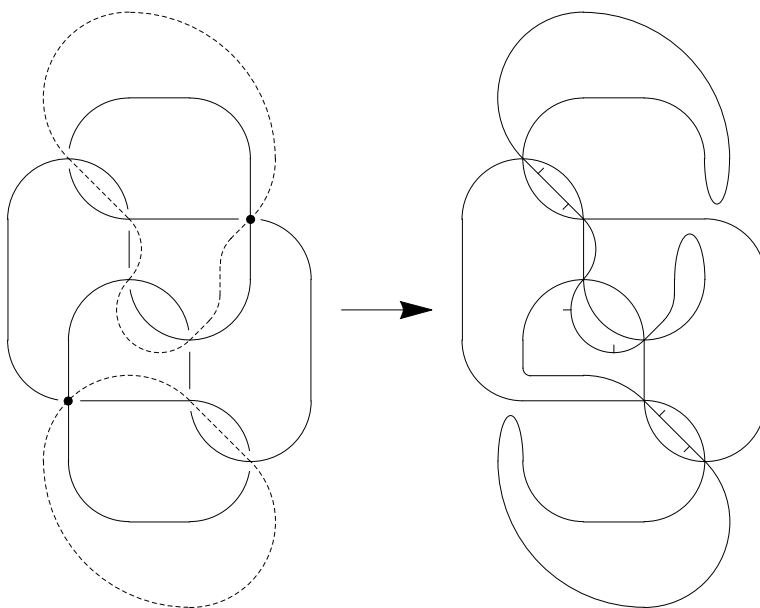


FIGURE 7. Using a CCC to turn a 2-diagram of a knot into a 3-diagram. Marks at each 3-crossing indicate the highest strand and point to the middle-level strand.

Proof. Let D be any 3-diagram of a link with a piecewise natural orientation. We may resolve each 3-crossing of D into three 2-crossings to obtain an oriented 2-diagram d of the same oriented link. Because of Lemma 2, a compatible CCC for d exists with one crossing circle for each such set of three crossings, as shown in Figure 8. The diagram $\Psi(d)$ and D are now related by a 1-move near each 3-crossing of D as shown in the figure. (A similar figure can be used in the case where the heights of the three strands in D are different than pictured.) \square

Proof of Theorem 3 and Corollaries 4 and 5: Suppose first that D_1 and D_2 are two 3-diagrams related by a sequence of the moves given in Theorem 3. It is not hard to see that the 1-move, 2-move, basepoint move and band move may be extended to 3-diagrams with checkerboard colorings. Thus 3-diagrams related by these moves have natural orientations that are either the same or differ by complete reversal. However, if D_1 is not connected and the trivial pass move is used to move a strand to the other side of a subdiagram, natural orientations of D_1 and D_2 may now differ by the reversal of a union of maximal

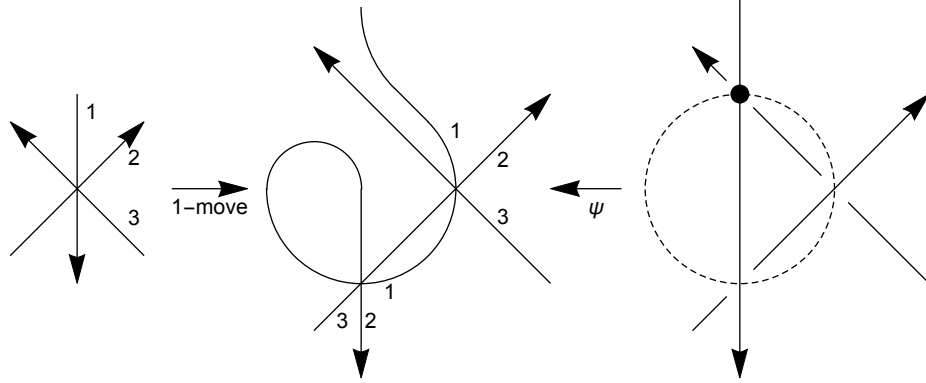


FIGURE 8. Changing the 3-diagram D on the left, with piecewise natural orientation, into a 3-diagram in the image of Ψ .

nonsplit sublinks of D_1 . This completes the proof of one direction of the theorem as well as one direction of both Corollaries 4 and 5.

Conversely, suppose that D_1 and D_2 are two 3-diagrams each having a natural orientation defining oriented links L_1 and L_2 , which are the same, up to reversal of maximal nonsplit sublinks. By Lemma 11, we may use 1-moves to change each 3-diagram D_i into a 3-diagram $D'_i = \Psi(d_i)$ where d_1 and d_2 are oriented 2-diagrams of L_1 and L_2 , respectively, each with compatible CCC . We may now change d_1 into d_2 by a sequence of the moves given in Theorem 9. Suppose this gives the sequence $d_1 = e_1, e_2, \dots, e_k = d_2$ where each e_i is an oriented 2-diagram equipped with a compatible CCC . It remains only to show that if e_i is taken to e_{i+1} by one of the moves given in Theorem 9, then $\Psi(e_i)$ can be taken to $\Psi(e_{i+1})$ by a sequence of the 3-diagram moves given in Theorem 3. We will consider each type of possible move.

If e_i and e_{i+1} are related by either the insertion or deletion of a trivial crossing circle, or $C1a$ or $C1b$, then $\Psi(e_i) = \Psi(e_{i+1})$. Now suppose e_i and e_{i+1} are related by $C2a$. If the crossing circle lying between the two parallel arcs on the left is trivial, then a basepoint must be located on one of the two crossings on the right. It now follows that $\Psi(e_i)$ and $\Psi(e_{i+1})$ are related by a 1-move and a trivial pass move. If instead, the crossing circle is not trivial, then $\Psi(e_i)$ and $\Psi(e_{i+1})$ are related by a 2-move. Now suppose e_i and e_{i+1} are related by $C3a$, then $\Psi(e_i)$ and $\Psi(e_{i+1})$ are related by a sequence of two 1-moves.

Next, if e_i and e_{i+1} are related by a CCC change of basepoint, then the reader can verify that $\Psi(e_i)$ and $\Psi(e_{i+1})$ are related by a 3-diagram

basepoint move. Finally, suppose e_i and e_{i+1} are related by a *CCC* band move. Now three cases can occur. Without loss of generality, we may assume that one crossing circle is being split into two crossing circles by the move. If a trivial crossing circle is split into two trivial crossing circles, then $\Psi(e_i)$ and $\Psi(e_{i+1})$ are equal. If a non-trivial crossing circle is split into a nontrivial crossing circle and a trivial crossing circle, then $\Psi(e_i)$ and $\Psi(e_{i+1})$ are either equal or related by a trivial pass move. Finally, if a nontrivial crossing circle is split into two nontrivial crossing circles, then $\Psi(e_i)$ and $\Psi(e_{i+1})$ are related by a band move.

Notice that the only time the trivial pass move is needed in the proof is when one of e_i or e_{i+1} is a disconnected diagram. Hence, if we assume from the beginning that D_1 and D_2 are two 3-diagrams each defining nonsplit links, then this cannot occur. Thus we obtain Corollaries 4 and 5. \square

6. TRIPLE AND QUINTUPLE CROSSING NUMBERS

In any link diagram with multicrossings, notice that an n -crossing can always be increased to an $(n + 2)$ -crossing. After passing through the crossing on any of the n strands, one can double back and pass through the crossing two more times before continuing on as before. This process is illustrated in Figure 9, where a 3-crossing is increased to a 5-crossing. Because every link has both a 2-diagram as well as a 3-diagram, it follows that every link has an n -diagram for every $n \geq 2$. This allows us to define the n -crossing number of a link, denoted $c_n(L)$, as the minimum number of n -crossings in any n -diagram of L .

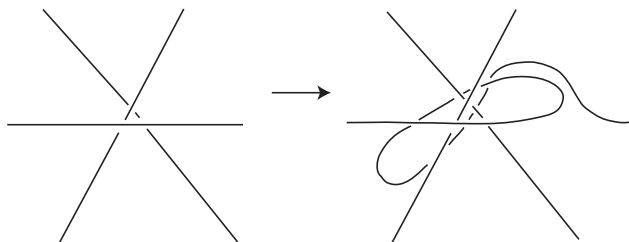


FIGURE 9. Turning a 3-crossing into a 5-crossing.

The process just described of turning any n -crossing into an $(n + 2)$ -crossing immediately proves that, for any link L ,

$$\begin{aligned} c_2(L) &\geq c_4(L) \geq c_6(L) \geq \dots \\ c_3(L) &\geq c_5(L) \geq c_7(L) \geq \dots \end{aligned}$$

However, the *CCC* construction shows that $c_2(L) > c_3(L)$ for every link L because a set of k 2-crossings lying on a crossing circle can be converted to a set of $k - 1$ 3-crossings. In this section, we prove that for any nontrivial nonsplit link L , other than the Hopf link, $c_3(L) > c_5(L)$. We begin with the following lemma.

Lemma 12. *If D is a connected 3-diagram with at least two 3-crossings and with at least one monogon among its complementary faces, then D can be converted into a 5-diagram of the same link with fewer crossings.*

Proof. Note first that if we can convert any subset of the 3-crossings of D into a smaller set of 5-crossings, then we can convert each of the remaining 3-crossings into a 5-crossing as in Figure 9 and hence obtain a 5-diagram of the same link with fewer crossings. Consider a complementary region, or face, of D that is a monogon and consider the 3-crossing B incident to the monogon. Because D is connected and has at least two 3-crossings, we are led to two possibilities, the first of which is shown in Figure 10. In this case we can eliminate the 3-crossing at A by producing a 5-crossing at B as shown in the figure. For example, suppose the two strands to be moved are the top and bottom strands at A . We first pick up the top strand and lay it down on top of B . We then move what was the bottom strand at A underneath the diagram, becoming the bottom strand at B . The other height arrangements at A are handled similarly.

The other possibility leads us to the case shown in Figure 11. If the strand heights at crossing B are not as shown in the figure (that is, the “vertical” strand is not the middle strand) then the link is split, crossing B can be eliminated, and the 5-crossing diagram obtained by simply changing every remaining 3-crossing to a 5-crossing has fewer crossings. If, instead, the heights at crossing B are as shown in the figure, then the link has a Hopf link summand and the diagram can be changed as shown in the figure. But now we are again in the situation of Figure 10. As before, this leads to a 5-diagram of the same link with fewer crossings. \square

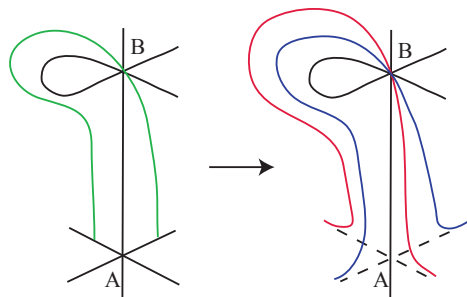


FIGURE 10. Using a monogon to replace two 3-crossings with one 5-crossing.

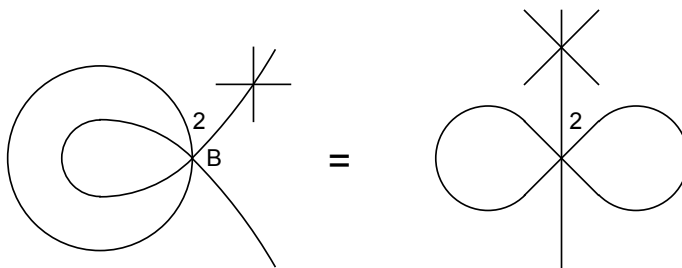


FIGURE 11. A monogon that gives a Hopf link summand.

We now consider 3-diagrams with no monogons. If a complementary region of the diagram has a certain pattern of adjacent bigons on its boundary, then again, we will be able to find a 5-diagram with fewer crossings.

Lemma 13. *Let D be a connected 3-diagram of a link L . Suppose there exists a complementary region F that is a polygon with $n > 1$ edges, and the following holds:*

- (1) *If n is even, at least every other edge of F is shared with a bigon.*
- (2) *If n is odd, there are at least enough bigons on the boundary of F such that, other than one pair of two adjacent edges, alternate edges are each shared with a bigon.*

Then the 3-diagram can be converted into a 5-diagram with one less crossing than D .

Proof. See Figures 12 and 13 to see how we can convert the n 3-crossings around the boundary of F into $n - 1$ 5-crossings by a process

similar to that given in the proof of Lemma 12. Each of the remaining 3-crossings can then individually be turned into a 5-crossing. \square

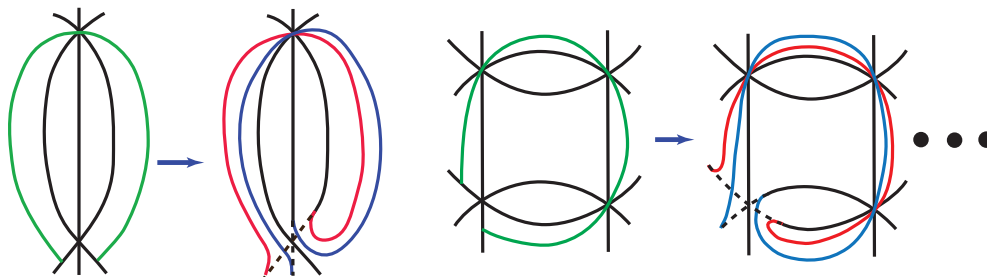


FIGURE 12. Converting 3-crossings into 5-crossings around an appropriate n -gon for n even.

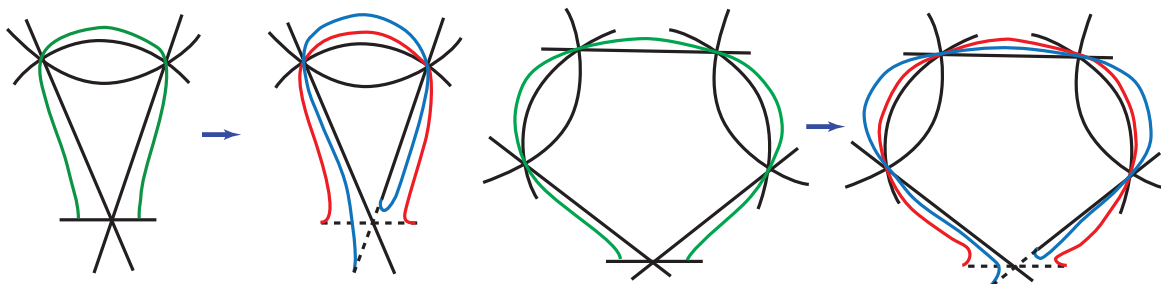


FIGURE 13. Converting 3-crossings into 5-crossings around an appropriate n -gon for n odd.

The goal now is to show that in any 3-diagram of a nontrivial nonsplit link that is not the Hopf link, a complementary region which is either a monogon or a polygon of the kind described in Lemma 13 must exist. We will need the following observation.

Lemma 14. *Let f_i be the number of faces with i edges in any 3-diagram, including the outer region. Then*

$$2f_1 + f_2 = 6 + f_4 + 2f_5 + 3f_6 + \dots$$

Proof. Considering the Euler characteristic gives $v - e + f = 2$. But $e = 6v/2$, so $v = e/3$. We now have $f = 2 + \frac{2e}{3}$, or equivalently

$3f = 6 + 2e$, where $f = f_1 + f_2 + f_3 + \dots$ and $e = \frac{f_1 + 2f_2 + 3f_3 + \dots}{2}$. Thus, $3(f_1 + f_2 + f_3 + \dots) = 6 + f_1 + 2f_2 + 3f_3 + \dots$ which yields the result. \square

Theorem 15. *If L is a nontrivial, nonsplit link, other than the Hopf link, then $c_3(L) > c_5(L)$.*

Proof. Let D be a 3-diagram of L that realizes the 3-crossing number $c_3(L)$. Because L is nonsplit, D must be connected. It is easy to check that the only connected 3-diagram with a single 3-crossing is either a trivial link or the Hopf link. Thus D must have at least two crossings. If D contains a monogon, then we are done by Lemma 12. Otherwise we have $f_1 = 0$ and

$$(1) \quad f_2 = 6 + f_4 + 2f_5 + 3f_6 + \dots$$

If any of the cases that occur in Lemma 13 appear in the 3-diagram D , we are done. So assume no such case occurs. In particular, this means that no two bigons share an edge and no bigon shares an edge with a triangle, etc. Now let's count how many bigons could be present. Each bigon can have no others on its boundary. Each triangle can have none on its boundary. Each quadrilateral can have at most two bigons on its boundary, where they are not opposite. Each pentagon can have at most two bigons on its boundary, for if it had three, two would be nonadjacent. More generally, for each n , if n is even, there can be at most $n - 2$ bigons on the boundary to avoid one of the cases from Lemma 13. If n is odd, there can be at most $n - 3$ bigons on the boundary.

But counting bigons this way, we have counted them twice. So the conclusion is that

$$f_2 \leq \frac{2f_4 + 2f_5 + 4f_6 + 4f_7 + 6f_8 + 6f_9 + \dots}{2}$$

So

$$f_2 \leq f_4 + f_5 + 2f_6 + 2f_7 + 3f_8 + 3f_9 + \dots$$

But this is too small, contradicting Equation 1. Thus D must contain a face as in Lemma 13, allowing us to convert D into a 5-diagram with fewer crossings. \square

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