

Link invariants from L^2 -Bureau maps of braids

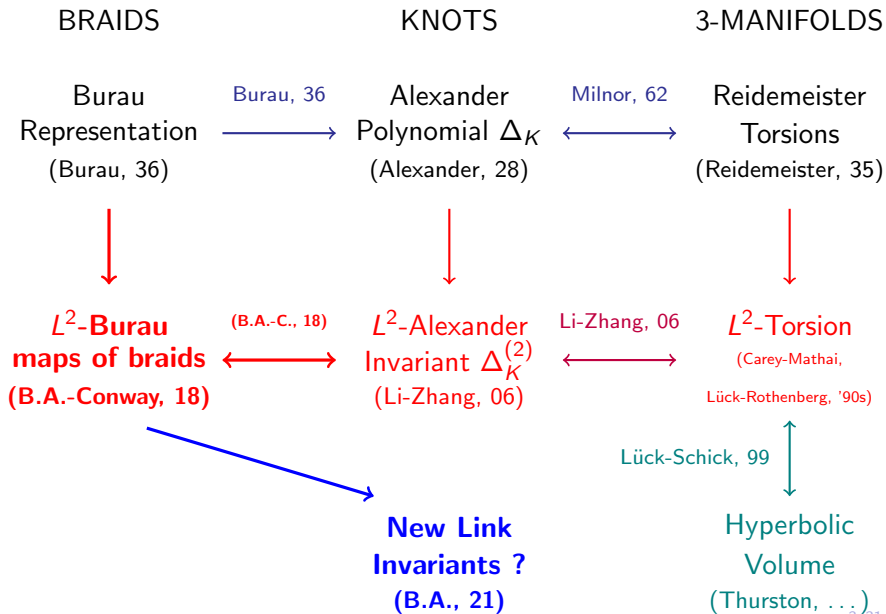
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Introduction

Nine decades of invariants



Context and main result

B.A., Conway 2018 : Construction of the L^2 -Burau maps $\mathcal{B}_{t,\gamma}^{(2)}$ of the braid groups B_n , with $t > 0$ and $\gamma: \mathbb{F}_n \twoheadrightarrow G$.

For $\gamma: \mathbb{F}_n \twoheadrightarrow G_K$, relation with $T_K^{(2)}(t)$ **the L^2 -Alexander torsion**.

Question

Which other *link invariants* can be built via $\det(\mathcal{B}_{t,\gamma}^{(2)} - \text{Id})$?

Theorem (B.A. 2021)

- 1 For γ lower than G_K : twisted L^2 -Alexander invariants.
- 2 For some other γ : no link invariance.

Method : Apply **Markov moves** of braids to $\mathcal{B}_{t,\gamma}^{(2)}$.

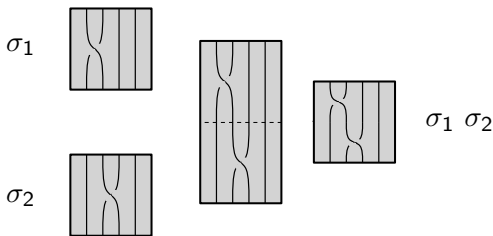
- 1 Burau representation and generalizations
- 2 L^2 -Burau maps
- 3 L^2 -Burau maps and L^2 -Alexander torsions
- 4 Finding new link invariants

Burau representation and generalizations

The **braid group** B_n is defined by the presentation

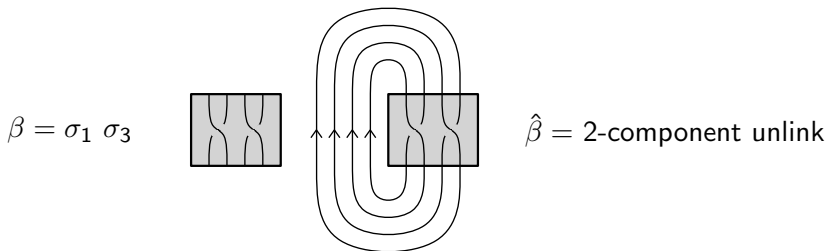
$$\langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2 \rangle$$

and a braid $\beta \in B_n$ also has a **topological description**, as a 1-submanifold of $\mathbb{D}^2 \times [0, 1]$ with no horizontal tangent.



Braids and links

The **closure** $\hat{\beta}$ of a braid $\beta \in B_n$ is a **link** in \mathbb{S}^3 .



Markov moves on braids $\sqcup_{n \geq 1} B_n$ are:

- Markov 1, the **conjugation**: $\beta \mapsto \alpha^{-1} \beta \alpha$, where $\alpha, \beta \in B_n$.
- Markov 2, the **stabilisations**: $B_n \ni \beta \mapsto \sigma_n^{\pm 1} \beta \in B_{n+1}$.

Theorem (Markov 1935)

$\hat{\beta}$ and $\hat{\beta}'$ are isotopic links iff β and β' are related by a finite number of Markov moves.

The Burau representations

Burau representation:

$$\mathcal{B}: B_n \ni \sigma_i \mapsto \text{Id}_{i-1} \oplus \begin{pmatrix} 1 - T & 1 \\ T & 0 \end{pmatrix} \oplus \text{Id}_{n-i-1} \in GL_n(\mathbb{Z}[T, T^{-1}])$$

Reduced Burau representation:

$$\overline{\mathcal{B}}: B_n \ni \sigma_i \mapsto \text{Id}_{i-2} \oplus \begin{pmatrix} 1 & T & 0 \\ 0 & -T & 0 \\ 0 & 1 & 1 \end{pmatrix} \oplus \text{Id}_{n-i-2} \in GL_{n-1}(\mathbb{Z}[T, T^{-1}])$$

Theorem (Burau 1936)

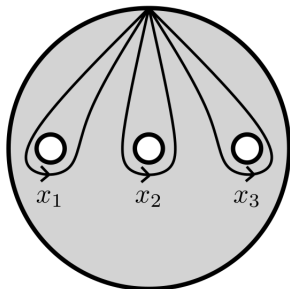
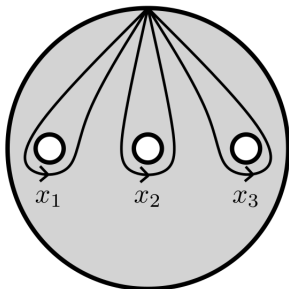
Let $\beta \in B_n$. Then the *Alexander polynomial* $\Delta_{\hat{\beta}}$ of the link $\hat{\beta}$ satisfies (up to product by a unit of $\mathbb{Z}[T, T^{-1}]$):

$$\Delta_{\hat{\beta}}(T) = \frac{1 - T}{1 - T^n} \det(\overline{\mathcal{B}}(\beta) - \text{Id}_{n-1}).$$

Braids and free groups

Free group $\mathbb{F}_n = \langle x_1, \dots, x_n \mid \rangle \cong \pi_1(D^2 \setminus \{p_1, \dots, p_n\})$.

$g_1 = x_1$, $g_2 = x_1 x_2$, \dots , $g_n = x_1 \dots x_n$ also generate \mathbb{F}_n .



Artin: A braid $\beta \in B_n$ induces an automorphism $h_\beta \in \text{Aut}(\mathbb{F}_n)$.

Example: For $\sigma_1 \in B_2$, $h_{\sigma_1}: x_1 \mapsto x_1 x_2 x_1^{-1}$, $x_2 \mapsto x_1$,
 $g_1 \mapsto g_2 g_1^{-1}$, $g_2 \mapsto g_2$.

$$B_n \hookrightarrow \text{Aut}(\mathbb{F}_n)$$

The **Fox derivatives** on \mathbb{F}_n are linear maps $\frac{\partial}{\partial x_i} : \mathbb{Z}\mathbb{F}_n \rightarrow \mathbb{Z}\mathbb{F}_n$ (where $i = 1, \dots, n$), inductively defined by:

$$\begin{aligned} \frac{\partial}{\partial x_i}(x_j) &= \delta_{i,j}, & \frac{\partial}{\partial x_i}(x_j^{-1}) &= -\delta_{i,j}x_j^{-1}, \\ \frac{\partial}{\partial x_i}(uv) &= \frac{\partial}{\partial x_i}(u) + u\frac{\partial}{\partial x_i}(v). \end{aligned}$$

$\left(\frac{\partial f(x_j)}{\partial x_i}\right)_{i,j} \in GL_n(\mathbb{Z}\mathbb{F}_n)$ is the **Fox jacobian** of $f \in \text{Aut}(\mathbb{F}_n)$.

Example: $h_{\sigma_1} : x_1 \mapsto x_1x_2x_1^{-1}, x_2 \mapsto x_1, \quad g_1 \mapsto g_2g_1^{-1}, g_2 \mapsto g_2.$

$$\left(\frac{\partial h_{\sigma_1}(x_j)}{\partial x_i}\right) = \begin{pmatrix} 1 - x_1x_2x_1^{-1} & 1 \\ x_1 & 0 \end{pmatrix}, \quad \left(\frac{\partial h_{\sigma_1}(g_j)}{\partial g_i}\right) = \begin{pmatrix} -g_2g_1^{-1} & 0 \\ 1 & 1 \end{pmatrix}.$$

$$\text{Aut}(\mathbb{F}_n) \hookrightarrow GL_n(\mathbb{Z}\mathbb{F}_n)$$

The change of coefficients $\kappa(\Phi_n, \gamma, t)$

We construct a **ring morphism** $\kappa(\Phi_n, \gamma, t)$ for

- $\Phi_n: \mathbb{F}_n \rightarrow \mathbb{Z}, x_i \mapsto 1$, the **augmentation** epimorphism,
- $\gamma: \mathbb{F}_n \rightarrow G$ such that Φ_n factors through γ ,
- $t > 0$.

We define $\kappa(\Phi_n, \gamma, t): \mathbb{Z}\mathbb{F}_n \rightarrow \mathbb{R}G$ by $g \mapsto t^{\phi(g)}\gamma(g)$.

Example: For $n = 2$ and $\gamma = T^{\Phi_2}: (\mathbb{F}_2 \rightarrow T^{\mathbb{Z}} = \{T^m, m \in \mathbb{Z}\})$,

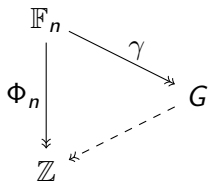
$$\kappa(\Phi_2, T^{\Phi_2}, t): \begin{pmatrix} 1 - x_1 x_2 x_1^{-1} & 1 \\ x_1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 - tT & 1 \\ tT & 0 \end{pmatrix} \text{ (Burau),}$$

$$\begin{pmatrix} -g_2 g_1^{-1} & 0 \\ 1 & 1 \end{pmatrix} \mapsto \begin{pmatrix} -tT & 0 \\ 1 & 1 \end{pmatrix} \text{ (reduced Burau).}$$

$\kappa(\Phi_n, \gamma, t)$ induces $GL_n(\mathbb{Z}\mathbb{F}_n) \rightarrow GL_n(\mathbb{R}G)$.

A family of maps between Artin and Burau

$$B_n \xrightarrow{\text{Artin}} \text{Aut}(\mathbb{F}_n) \xrightarrow{\text{Fox}} GL_n(\mathbb{Z}\mathbb{F}_n) \xrightarrow{\kappa(\Phi_n, \gamma, t)} GL_n(\mathbb{R}G)$$



We consider the maps

$$\beta \mapsto \kappa(\Phi_n, \gamma, t) \left(\frac{\partial h_\beta(x_j)}{\partial x_i} \right)_{1 \leq i, j \leq n}$$

for different $\gamma: \mathbb{F}_n \rightarrow G$.

(reduced: with g_j, g_i and $\leq n-1$ instead)

- $\gamma = \text{Id}_{\mathbb{F}_n}$: jacobian of the **Artin action**, always **injective**.

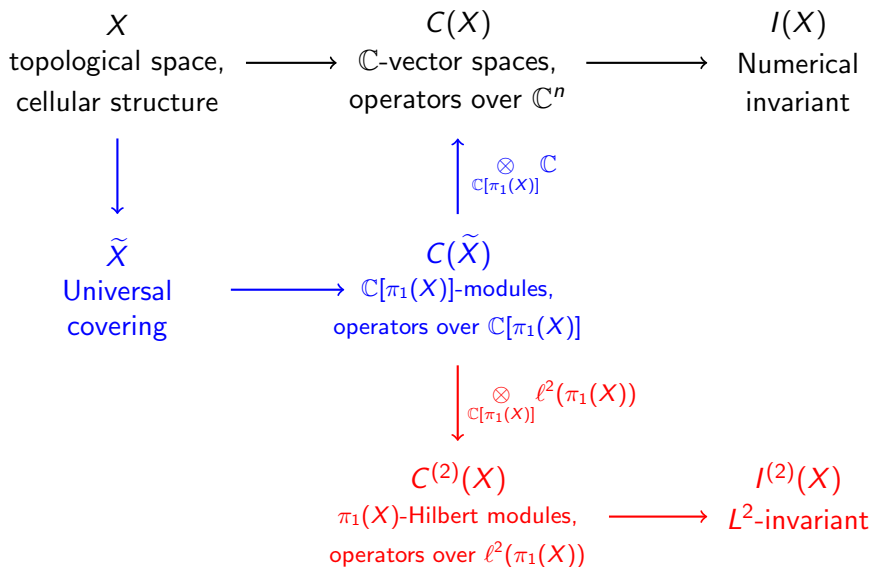
$$B_2 \ni \sigma_1 \mapsto \begin{pmatrix} 1 - t x_1 x_2 x_1^{-1} & 1 \\ t x_1 & 0 \end{pmatrix}, \quad B_2 \ni \sigma_1 \mapsto (-t g_2 g_1^{-1}).$$

- $\gamma = T^{\Phi_n}$: **Burau representation**, **non injective** for $n \geq 5$.

$$B_2 \ni \sigma_1 \mapsto \begin{pmatrix} 1 - tT & 1 \\ tT & 0 \end{pmatrix}, \quad B_2 \ni \sigma_1 \mapsto (-tT).$$

L^2 -Bourau maps

The general idea of L^2 -invariants



$$B_n \xrightarrow{\text{Artin}} \text{Aut}(\mathbb{F}_n) \xrightarrow{\text{Fox}} GL_n(\mathbb{Z}\mathbb{F}_n) \xrightarrow{\kappa(\Phi_n, \gamma, t)} GL_n(\mathbb{R}G)$$

Problem : Extracting info from $M_n(\mathbb{R}G)$ if G is not abelian.
 \rightsquigarrow On $\ell^2(G)$ there is the **Fuglede-Kadison determinant**.

We replace the algebra $\mathbb{R}G$ by the Hilbert space

$$\ell^2(G) := \left\{ \sum_{g \in G} \lambda_g g \mid \lambda_g \in \mathbb{C}, \sum_{g \in G} |\lambda_g|^2 < \infty \right\}.$$

Typical **bounded** G -equivariant operators:

$$R_h: \ell^2(G) \rightarrow \ell^2(G), g \mapsto gh,$$

the **right multiplication** operator by $h \in G$.

The operation R . induces $GL_n(\mathbb{R}G) \hookrightarrow B(\ell^2(G)^n)$.

L^2 -Burau maps of braids

$$B_n \xrightarrow{\text{Artin}} \text{Aut}(\mathbb{F}_n) \xrightarrow{\text{Fox}} GL_n(\mathbb{Z}\mathbb{F}_n) \xrightarrow{\kappa(\Phi_n, \gamma, t)} GL_n(\mathbb{R}G) \xrightarrow{R} B(\ell^2(G)^n),$$

The L^2 -**Burau map** associated to $t > 0$ and $\gamma: \mathbb{F}_n \rightarrow G$ is:

$$\mathcal{B}_{t, \gamma}^{(2)}: \begin{pmatrix} B_n & \rightarrow & B(\ell^2(G)^n) \\ \beta & \mapsto & R_A \end{pmatrix}, \text{ with } A = \kappa(\Phi_n, \gamma, t) \left(\frac{\partial h_\beta(x_j)}{\partial x_i} \right)_{1 \leq i, j \leq n}.$$

The **reduced L^2 -Burau map** associated to the same t, γ is:

$$\bar{\mathcal{B}}_{t, \gamma}^{(2)}: \begin{pmatrix} B_n & \rightarrow & B(\ell^2(G)^{n-1}) \\ \beta & \mapsto & R_{A'} \end{pmatrix}, \text{ with } A' = \kappa(\Phi_n, \gamma, t) \left(\frac{\partial h_\beta(g_j)}{\partial g_i} \right)_{1 \leq i, j \leq n-1},$$

where $g_1 = x_1$, $g_2 = x_1 x_2$, \dots , $g_n = x_1 \dots x_n$ also generate \mathbb{F}_n .

$$B_n \xrightarrow{\text{Artin}} \text{Aut}(\mathbb{F}_n) \xrightarrow{\text{Fox}} GL_n(\mathbb{Z}\mathbb{F}_n) \xrightarrow{\kappa(\Phi_n, \gamma, t)} GL_n(\mathbb{R}G) \xrightarrow{R} B(\ell^2(G)^n),$$

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The **reduced** L^2 -**Bureau map** associated to the same t, γ is:

$$\overline{\mathcal{B}}_{t, \gamma}^{(2)}: \left(\begin{array}{ccc} B_n & \rightarrow & B(\ell^2(G)^{n-1}) \\ \beta & \mapsto & R_{A'} \end{array} \right), \text{ with } A' = \kappa(\Phi_n, \gamma, t) \left(\frac{\partial h_\beta(\mathbf{g}_j)}{\partial \mathbf{g}_i} \right)_{1 \leq i, j \leq n-1},$$

where $\mathbf{g}_1 = x_1$, $\mathbf{g}_2 = x_1 x_2$, \dots , $\mathbf{g}_n = x_1 \dots x_n$ also generate \mathbb{F}_n .

Multiplicativity and computations

(Anti-)multiplication formula: $\forall \alpha, \beta \in B_n$,

$$\mathcal{B}_{t,\gamma}^{(2)}(\alpha\beta) = \mathcal{B}_{t,\gamma}^{(2)}(\beta) \circ \mathcal{B}_{t,\gamma \circ h_\beta}^{(2)}(\alpha).$$

Consequence : The more **injective** $\gamma: \mathbb{F}_n \twoheadrightarrow G$ is, the less $\mathcal{B}_{t,\gamma}^{(2)}$ looks like an (anti-)group representation.

$\mathcal{B}_{t,\gamma}^{(2)}$ is an (anti-)rep. of the **subgroup** $\{\beta \in B_n, \gamma \circ h_\beta = \gamma\}$.

We can still **compute** $\mathcal{B}_{t,\gamma}^{(2)}(\beta)$ via the images of the σ_i .

$$\mathcal{B}_{t,\gamma}^{(2)}: B_2 \ni \sigma_1 \mapsto \begin{pmatrix} \text{Id} - tR_{\gamma(x_1x_2x_1^{-1})} & \text{Id} \\ tR_{\gamma(x_1)} & 0 \end{pmatrix} \in B(\ell^2(G)^{\oplus 2}).$$

$$\overline{\mathcal{B}}_{t,\gamma}^{(2)}: B_2 \ni \sigma_1 \mapsto \left(-tR_{\gamma(g_2g_1^{-1})} \right) \in B(\ell^2(G)).$$

L^2 -Burau maps and L^2 -Alexander torsions

Fuglede-Kadison determinants of G -equivariant operators

The **von Neumann trace** $\text{tr}_G : (\lambda_1 \text{Id} + \dots + \lambda_g R_g) \mapsto \lambda_1 \in \mathbb{C}$.

The **Fuglede-Kadison determinant** \det_G is :

$$\det_G : R_{M_n(\mathbb{Z}G)} (\subset B(\ell^2(G)^n)) \rightarrow \mathbb{R}_{\geq 0}$$
$$\det_G(A) := \lim_{\epsilon \rightarrow 0^+} \left(\exp \circ \left(\frac{1}{2} \text{tr}_G \right) \circ \ln \right) (A^* A + \epsilon \text{Id}).$$

\leftrightarrow counting loops on the **Cayley graph** of G . (hard to compute !)

Ex : If $g \in G$ has ∞ order, then $\det_G(\text{Id} + tR_g) = \max\{1, |t|\}$.

Ex : For $G = \mathbb{Z}/3\mathbb{Z}$: $\ell^2(G) = \mathbb{C}^3$, R_g is a permutation matrix,
 $\text{tr}_G = \frac{1}{3} \text{tr}_{\mathbb{C}}$, $\det_G = |\det|^{1/3}$.

Theorem (B.A. 21)

For $G = \mathbb{F}_2 = \langle x, y \rangle$, $\det_G(1 + x + y) = \frac{2}{\sqrt{3}}$.

(Dubois-Friedl-Lück 14) L^2 -**Alexander torsion** of the link L :

$$T_L^{(2)} : \begin{pmatrix} \mathbb{R}_{>0} & \longrightarrow & \mathbb{R}_{\geq 0} \\ t & \longmapsto & T_L^{(2)}(t) \end{pmatrix}$$

defined with $\det_{\pi_1(S^3 \setminus L)}$ and **Fox calculus**.

Some properties:

(Lück-Schick 99) $T_L^{(2)}(1)$ contains the **volume** of $S^3 \setminus L$.

(Friedl-Lück, Liu 15) $T_L^{(2)}(0^+)$, $T_L^{(2)}(+\infty)$ yield the **genus** $g(L)$.

(Miscellaneous) $T_L^{(2)}$ is the **zero map** iff L is **split**.

L^2 -Bureau map at the link group

For $\gamma = \gamma_\beta: \mathbb{F}_n \rightarrow G_\beta \cong \pi_1(S^3 \setminus \hat{\beta})$ quotient by $h_\beta(x_i) = x_i$, the **reduced** L^2 -Bureau map $\bar{\mathcal{B}}_{t, \gamma_\beta}^{(2)}$ gives the L^2 -Alexander torsion $T_{\hat{\beta}}^{(2)}$:

Theorem (B.A.-Conway 18)

Let $\beta \in B_n$, $L = \hat{\beta}$ its closure and $t > 0$. Then:

$$T_L^{(2)}(t) \cdot \max(1, t)^n = \det_{G_L} \left(\bar{\mathcal{B}}_{t, \gamma_\beta}^{(2)}(\beta) - \text{Id}^{\oplus(n-1)} \right).$$

Finding new link invariants

Markov moves on braids $\sqcup_{n \geq 1} B_n$ are:

- Markov 1, the **conjugation**: $\beta \mapsto \alpha^{-1}\beta\alpha$, where $\alpha, \beta \in B_n$.
- Markov 2, the **stabilisations**: $B_n \ni \beta \mapsto \sigma_n^{\pm 1}\beta \in B_{n+1}$.

Markov : $\hat{\beta}$ and $\hat{\beta}'$ are **isotopic links** iff β and β' are related by a finite number of **Markov moves**.

A function F on $\sqcup_{n \geq 1} B_n$ is a **Markov function** if it is invariant under all the Markov moves.

\rightsquigarrow F provides a **invariant of knots and links**.

Question

For which $\gamma: \mathbb{F}_n \rightarrow G$ is $\beta \mapsto \det_G \left(\overline{\mathcal{B}}_{t,\gamma}^{(2)}(\beta) - \text{Id}^{\oplus(n-1)} \right)$ a Markov function?

Markov-admissibility of a family of epimorphisms

A family \mathcal{Q} of **epimorphisms of free groups**

$$\mathcal{Q} = \{ Q_\beta : \mathbb{F}_{n(\beta)} \twoheadrightarrow G_{Q_\beta} \mid \beta \in \sqcup_{n \geq 1} B_n \}$$

is **Markov-admissible** if for all pairs of braids β, β' related by a Markov move, Q_β and $Q_{\beta'}$ are “comparable”.

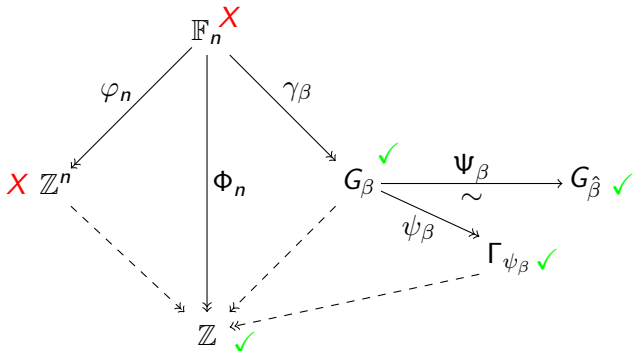
$$\begin{array}{ccc} \mathbb{F}_n & \xrightarrow[\sim]{h_\alpha} & \mathbb{F}_n \\ \downarrow Q_\beta & & \downarrow Q_{\alpha^{-1}\beta\alpha} \\ G_{Q_\beta} & \xrightarrow[\sim]{\exists \chi_{\beta,\alpha}^Q} & G_{Q_{\alpha^{-1}\beta\alpha}} \end{array}$$

$$\begin{array}{ccc} \mathbb{F}_n & \xleftarrow{l_n} & \mathbb{F}_{n+1} \\ \downarrow Q_\beta & & \downarrow Q_{\sigma_n^\varepsilon \beta} \\ G_{Q_\beta} & \xleftarrow[\sim]{\exists \sigma_{\beta,\varepsilon}^Q} & G_{Q_{\sigma_n^\varepsilon \beta}} \end{array}$$

Goal : Constructing a **Markov function** on $\sqcup_{n \geq 1} B_n$ via \mathcal{Q} .

Several Markov-admissible families of epimorphisms

- $Id_{\mathbb{F}_n} :=$ the **identity** on \mathbb{F}_n .
- $\varphi_n :=$ the **abelianisation** of \mathbb{F}_n , onto \mathbb{Z}^n .
- $\gamma_\beta :=$ the quotient by $h_\beta(x_i) = x_i$ (\iff **closure** of β).
- $\Phi_n :=$ the **augmentation** of \mathbb{F}_n , onto \mathbb{Z} .



Is $\det(\text{Burau}^{(2)} - \text{Id})$ a **Markov function**? \checkmark : Yes. \times : No.

Obtaining a Markov function

For $\mathcal{Q} = \{Q_\beta\}_{\beta \in \sqcup_{n \geq 1} B_n}$ Markov-admissible and $t > 0$, we define:

$$F_{\mathcal{Q}} := \left(\begin{array}{l} \sqcup_{n \geq 1} B_n \rightarrow \mathcal{F}(\mathbb{R}_{>0}, \mathbb{R}_{>0}) / \{t \mapsto t^m, m \in \mathbb{Z}\} \\ \beta \mapsto \left[t \mapsto \frac{\det_{G_{Q_\beta}} \left(\overline{B}_{t, Q_\beta}^{(2)}(\beta) - \text{Id}^{\oplus(n-1)} \right)}{\max(1, t)^n} \right] \end{array} \right)$$

Theorem (B.A. 2021)

- 1 If $\{Q_\beta\} = \{\psi_\beta \circ \gamma_\beta\}$, then $F_{\mathcal{Q}}$ is a Markov function, and the associated invariant is a L^2 -Alexander torsion twisted by ψ_β .
- 2 If the Q_β are the identity maps $\text{Id}_{\mathbb{F}_{n(\beta)}}$ or the abelianisations $\varphi_{n(\beta)}$, then $F_{\mathcal{Q}}$ is not a Markov function.

Sketch of proof for Theorem 1

Conjugation : For $\gamma: \mathbb{F}_n \twoheadrightarrow G$, the product formula gives:

$$\overline{\mathcal{B}}_{t,\gamma}^{(2)}(\alpha^{-1}\beta\alpha) = \overline{\mathcal{B}}_{t,\gamma}^{(2)}(\alpha) \circ \overline{\mathcal{B}}_{t,\gamma \circ h_\alpha}^{(2)}(\beta) \circ \left(\overline{\mathcal{B}}_{t,\gamma \circ h_{\alpha^{-1}\beta\alpha}}^{(2)}(\alpha) \right)^{-1},$$

a **conjugation** if γ goes deep enough so that $\gamma \circ h_{\alpha^{-1}\beta\alpha} = \gamma$.

✓ for $\gamma = \psi_\beta \circ \gamma_\beta$.

Stabilisations : Via elementary operations, we get

$$\overline{\mathcal{B}}_{t,\gamma}^{(2)}(\sigma_n^{\pm 1}\beta) \approx \begin{pmatrix} \overline{\mathcal{B}}_{t,\gamma}^{(2)}(\beta) & * \\ (\gamma \circ h_\beta - \gamma)(*) & * \end{pmatrix},$$

where $*$ has a known Fuglede-Kadison determinant.

✓ for $\gamma = \psi_\beta \circ \gamma_\beta$.

In both cases we prove that $F_{\mathbb{Q}}(\sigma_1^{-1}) \neq F_{\mathbb{Q}}(\sigma_2\sigma_1^{-1})$.

Difficulty: In order to prove that two Fuglede-Kadison determinants are **different**, we must **compute** them.

- For the abelianisations $\varphi_n: \mathbb{F}_n \rightarrow \mathbb{Z}^n$,

$$F_{\mathbb{Q}}(\sigma_2\sigma_1^{-1}) = \det_{\mathbb{Z}^2} (\text{Id} + R_x + R_y) = 1.38135\dots \neq 1 = F_{\mathbb{Q}}(\sigma_1^{-1}).$$

(Boyd, computations of **Mahler measures**)

- For the identity maps $\text{Id}_{\mathbb{F}_n}$,

$$F_{\mathbb{Q}}(\sigma_2\sigma_1^{-1}) = \det_{\mathbb{F}_2} (\text{Id} + R_x + R_y) = \frac{2}{\sqrt{3}} = 1.15\dots \neq 1 = F_{\mathbb{Q}}(\sigma_1^{-1})$$

(Bartholdi, Dasbach-Lalin, B.A. 21, **counting paths on trees**)

- Finding larger classes of \mathcal{Q} such that $F_{\mathcal{Q}}$ is not a Markov function.
- Trying to obtain link invariants from L^2 -Burau maps via **other formulas** than $\det(\cdot - \text{Id})$.
- (with C. Anghel-Palmer) Adapting these techniques to future L^2 versions of **Lawrence representations** of the braid groups, which are known to yield **quantum invariants**.