Homological stability for the ribbon Higman–Thompson groups

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History of Thompson groups

- [1965] The groups F, T, and V were first -defined by Richard Thompson. T and V are the first known examples of finitely presented infinite simple groups.

- [1974] Higman introduced what we call nowdays the Higman–Thompson groups.

- [70s] Fred–Heller, independtly Dydak, rediscovered F as the universal group encoding a free homotopy idempotent.

History of braided Thompson groups

- [2006] Brin and Dehornoy independently defined braided V.

- [2017] Thumann introduced ribbon Thompson groups.

- [$\approx$ 2020] Braided and ribbon version of Higman–Thompson groups was first studied by Aroca–Cumplido and Skipper-W.

History of braided Thompson groups

- [2004] Funar–Kapoudjian introduced the asymptotic mapping class group on an infinite type surface.

$\mathrm{Map}(S \setminus e)$

$\mathrm{Map}(D \setminus e)$

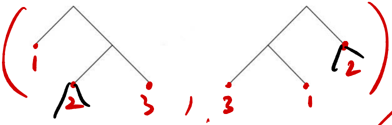
- [2006] Brin and Dehornoy independently defined braided V.

- [2017] Thumann introduced the ribbon Thompson groups.

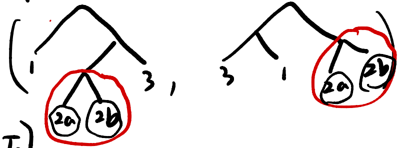
- [$\approx$ 2020] Braided and ribbon version of Higman–Thompson groups was first studied by Aroca–Cumplido and Skipper-W.

Define Thompson's group V using paired tree diagram

- An element in V is a paired tree diagram $[T_1, \sigma, T_2]$ where T_1 and T_2 are two finite rooted binary trees with the same number of leaves, and σ is a bijection from the leaves of T_1 to T_2.



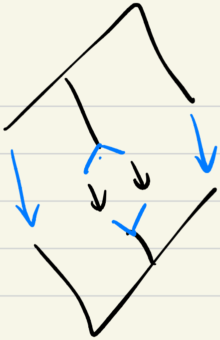
- Equivalence relation:

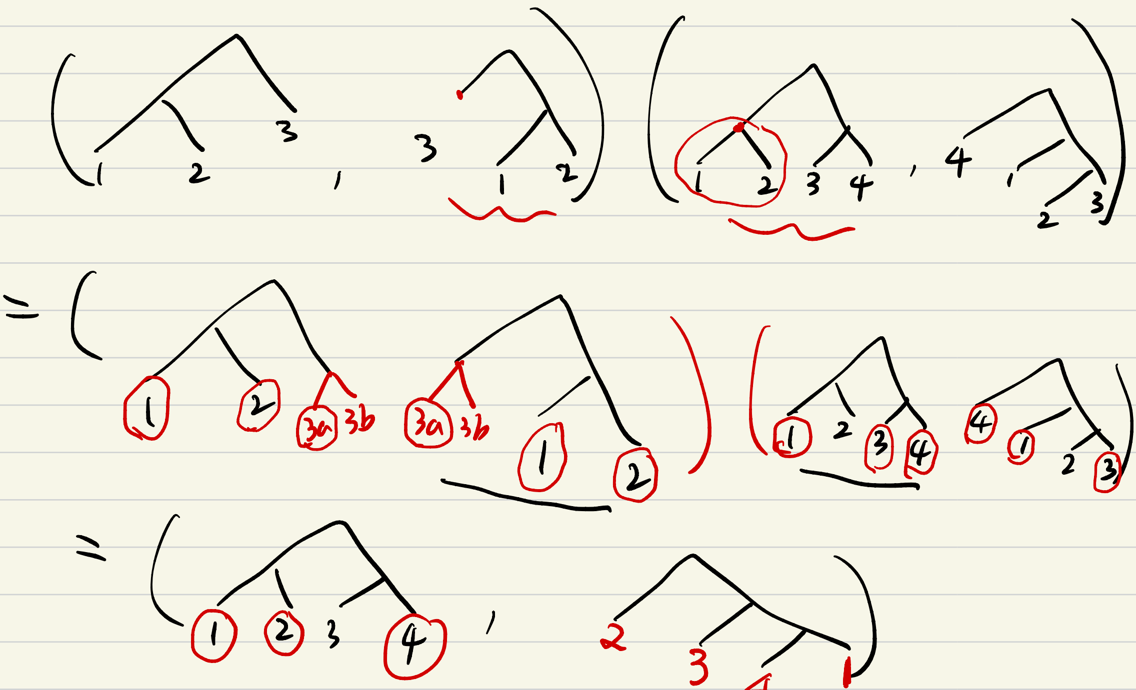


- Multiplication:

$[T_1, \sigma, T_2][T_2, \sigma', T_3) = [T_1, \sigma\sigma', T_3]$

diff??





1 2 3 , 3 1 2
1 2 3 4 , 4 1 2 3
=
1 2 3a 3b 3a 3b 1 2
1 2 3 4 4 1 2 3
=
1 2 3 4 , 2 3 4 1

F and T as subgroups of V

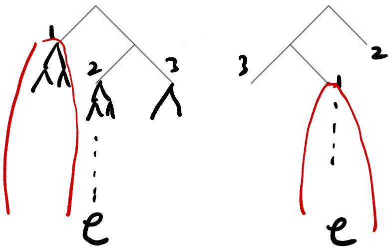
V: bijection between leaves of T_1 and T_2

T: cyclic permutation

F: "id"

$F \leq T \leq V$.

Thompson's group V as a subgroup of homeomorphisms of the Cantor set



Higman–Thompson groups

$V_{d,r}$

degree

# of roots

$d=2$

d

$[T_1, \sigma, T_2]$

tree , forests with the same # of components.

$V_{d,r} = [F_1, \sigma, F_2] / \sim$

$[F_1, \sigma, F_2][F_3, \sigma', F_2]$

F_1, F_2 are finite rooted d-ary trees with r components

Why Thompson's groups

- V contains all finite groups. $[T_0, \sigma, T_0)$

- [Brown–Geoghegan 1984] F is of type F_∞. This provides the first example of a torsion-free group of type F_∞ that has infinite cohomological dimension.
A grp G is of type F_∞ if there is a model $K(G,1)$ s.t. it has finitely many cells in each dim.

- [Brown 1987] T and V are also of type F_∞.

- V is dense in $\mathrm{Homeo}(\mathcal{C})$.

- [Szymik–Wahl 2019] V is acyclic. $\hat{H}_*(V, \mathbb{Z}) = 0.$

How to show V is acyclic

<u>Step 1</u>: $V_{2,1} \le V_{2,2} \le \cdots \quad V_{2,n} \le \cdots$

$$[T_1, \sigma, T_2) \to [T_1 \cup *_0, \bar{\sigma}, T_2 \cup *_1]$$

σ

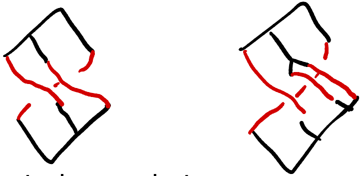
$H_*(V_{2,n}) \to H_*(V_{2,n+1})$ is iso for any n, and $*$.

$$\downarrow \qquad \updownarrow \mathrm{id}$$

<u>Step 2</u>: identify the stable homology.

Braided V [Brin 2006, Dehornoy 2006]

- An element in bV is a braided paired tree diagram $[T_1, b, T_2]$ where T_1 and T_2 are two finite rooted trees with the same number of leaves, b is a braid.



- Equivalence relation:
- multiplication:

Similarly, we define braided Higman–Thompson groups $bV_{d,r}$.

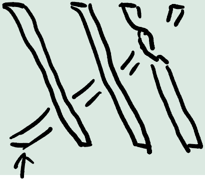
Ribbon braid groups

Definition

An element in the ribbon braid group RB_n is

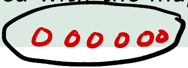
$RB_n \cong B_n \ltimes \mathbb{Z}^n$

S_n



Fact

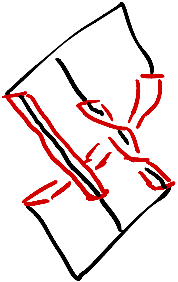
- *B_n can be identified with the mapping class group of a disk with n punctures.*
- *PRB_n can be identified with the mapping class group of a n-holed disk.*



Ribbon Higman–Thompson groups

- An element in $RV_{d,r}$ is a braided paired tree diagram $[F_1, \mathfrak{r}, F_2]$ where F_1 and F_2 are two finite rooted forests with the same number of leaves, $\mathfrak{r}$ is a ribbon braid. $d=2$

- Equivalence relation:



$$\Delta: G \to G \times G$$
$$g \to (g,g)$$
$$B_n(G) \hookrightarrow B_{n+1}(G)$$

Question

Question

Is bV also acyclic? How about RV?

Normal subgroups of braided V

$PB_1 \subseteq PB_2 \subseteq PB_4 \subseteq \cdots PB_{2^n}$

$1 \to PB_\infty \to bV \to V$

$[T_1, b, T_2] \to [T_1, \sigma, T_2]$

Theorem (Zaremsky 18)

Any proper normal subgroup of bV is contained in PB_∞.

$\Downarrow$ **Corollary**

$H_1(bV) = 0$. $\quad$ [V is simple]

$H_1(\dot{R}V) = 0$ ✓

How one might prove bV is acyclic?

Step 1 $\quad \underline{bV_{2,1} \leq bV_{2,2} \leq \cdots}$

Step 2 $\quad$ identify stable homology!

Our theorem

Theorem (Skipper–W)

Suppose $d \geq 2$. Then the inclusion maps induce isomorphisms

$$\iota_* : H_i(RV_{d,r}, \mathbb{Z}) \to H_i(RV_{d,r+1}, \mathbb{Z})$$

in homology in all dimensions $i \geq 0$, for all $r \geq 1$.

A quick review on Homogeneous category

$G_1 \leq \cdots G_n \cdots \cdots C$

Definition

$\mathrm{Aut}(X) \leq \mathrm{Aut}(X \oplus X) \cdots \mathrm{Aut}(X^n)$

A monoidal category $(\mathcal{C}, \oplus, 0)$ *is called homogeneous*

- 0 *is initial in* $\mathcal{C}$*;*
- $\mathrm{Hom}(A, B)$ *is a transitive* $\mathrm{Aut}(B)$*-set under postcomposition;*
- *The map* $\mathrm{Aut}(A) \to \mathrm{Aut}(A \oplus B)$ *taking* f *to* $f \oplus \mathrm{id}_B$ *is injective with image*

$$\mathrm{Fix}(B) := \{\phi \in \mathrm{Aut}(A \oplus B) \mid \phi \circ (\imath_A \oplus \mathrm{id}_B) = \imath_A \oplus \mathrm{id}_B\}$$

where $\imath_A \colon 0 \to A$ *is the unique map.*

The space $S_n(X)$

$\mathrm{Aut}(X^{\oplus n})$

Let X be a object of the homogeneous category $(\mathcal{C}, \oplus, 0)$, then $S_n(X)$ denote the simplicial complex

- Vertices: morphisms $f \colon \underline{X \to X^{\oplus n}}$;
- p-simplices: $(p+1)$-sets $\{f_0, \dots, f_p\}$ such that there exists a morphism $f \colon \underline{X^{\oplus p+1} \to X^{\oplus n}}$ with $f \circ i_j = f_j$ for some order on the set, where

$$i_j = \imath_{X^{\oplus j}} \oplus \mathrm{id}_X \oplus \imath_{X^{\oplus p-j}} \colon X = 0 \oplus X \oplus 0 \longrightarrow X^{\oplus p+1}.$$

How to prove homological stability?

Theorem (Randal-Williams–Wahl)

Let $(\mathcal{C}, \oplus, 0)$ be a homogeneous category such that the space $S_n(X)$ is highly connected, then

$$H_i(\mathrm{Aut}(X^{\oplus n})) \longrightarrow H_i(\mathrm{Aut}(X^{\oplus n+1}))$$

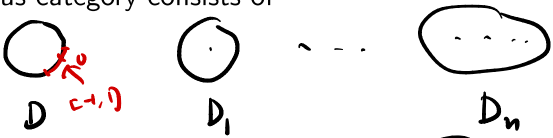
induced by the natural inclusion map is an isomorphism if $n >> i$.

An example: braid groups I

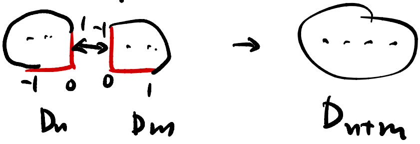
Key fact: B_n can be identified with the mapping class group of a disk with n punctures.

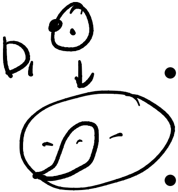
The homogeneous category consists of

- Objects:



- monoid operation:

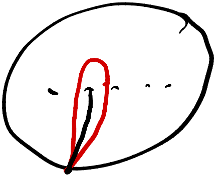




- morphisms:

$\mathrm{Hom}(D_n, D_m) =:$ isotopy class of embeddings $(D_n, 0) \to (D_m, 0)$

An example: braid groups II



The complex:

- vertices: $(D, 0) \hookrightarrow (D_n, D)$
- p-simplices: $f_0, \dots, f_p$
  image disjoint outside base pt

Arc complex : Vertices: arc connecting base pt to a marked pt

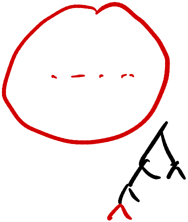
Key difficulty

Build a geometric model for the Ribbon Higman–Thompson groups.

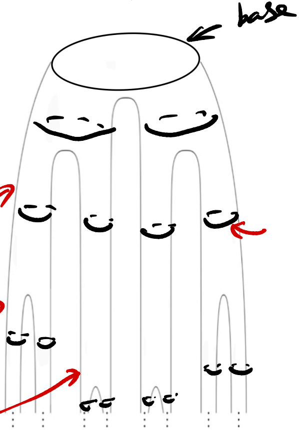
braided

Asymptotic mapping class group I: rigid structure

Mapc D(e)



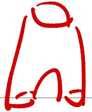
- Reconstruction of the surface:



- Suited subsurface:

finite connected union of pants and base

- Suited loops and parameterization of the pieces



Asymptotic mapping class group II: paired surface diagram

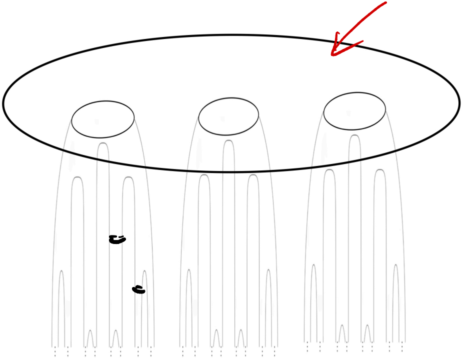
An element in $\mathcal{B}V = \mathcal{B}V_{2,1}$ is a paired surface diagram

$$[\Sigma_1, \phi, \Sigma_2]$$

such that

- Σ_1 and Σ_2 are suited subsurface with the same number of suited loops.
- ϕ is a homeomorphism from Σ_1 to Σ_2 which coincides with the parametrization of the suited loops.
- Equivalence relation: $\sim$ isotopy , $\sim$
- Composition:

$\mathcal{B}_{d,r}$ for any d and r



$r = 3$

$d \geq 2$

$\mathcal{B}V_{d,r}$ as a subgroup of $\mathrm{Map}(D \setminus \mathcal{C})$

- An element $[\Sigma_1, \phi, \Sigma_2]$ of $\mathcal{B}V_{d,r}$ naturally represents an element of $\mathrm{Map}(D \setminus \mathcal{C})$.

- We shall call an element f in $\mathrm{Map}(D \setminus \mathcal{C})$ **asymptotic rigid** if it can be written in the form $[\Sigma_1, \phi, \Sigma_2]$. Σ_1 is called the support of f.

- More generally, let S_1 and S_2 be two surface equipped with rigid structures, then a homeomorphism $f : S_1 \to S_2$ is called **asymptotic rigid** if it is "rigid" outside a suited subsurface of S_1.

Isomorphism results

$bV_{d,r}$

Theorem (Skipper-W)

The groups $\mathcal{B}V_{d,r}$ and $RV_{d,r}$ are isomorphic in a canonical way.

Remark

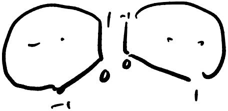
We also proved that $\mathcal{B}V_{d,r}$ is dense inside the big mapping class group $\mathrm{Map}(D \setminus \mathcal{C})$.

Homogenous category for asymptotic mapping class groups

- Objects:

$X_0 \quad X_1 \quad \cdots \quad X_n$

- monoid operation:



$X_n \oplus X_m = X_{n+m}$.

- Morphisms:

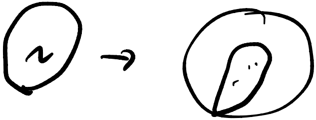
$$\mathrm{Hom}(X_m, X_n) = \begin{cases} \phi & m > n \\ (X_{n-m}, f)/\sim & m \le n \end{cases}$$

$f: X_{n-m} \oplus X_m \to X_n$

The complex S_n

- vertices:

Hom(X_1, X_n) :



$f: X_1 \to X_n$ s.t. $f(X_1) \to f(X_1)$ is ARH.

- simplices:

Complent of $f(X_1)$ in X_n is ARH to X_{n-1}.

$f_0 \cdots f_p$ give a p-simplex

① disjunt outside base pt

② there is some left over $p < n-1$

no left over $p = n-1$

ϕ $p > n-1$

First reduction

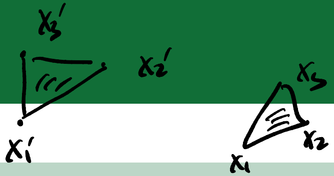
Let U_n be the following complex:

- vertices: $f|_{\partial X_1}$
- simplices:

Theorem

The forgetful map $\pi : S_n \to U_n$ is a complete join. In particular, U_n is highly connected iff S_n is.

Complete join



Definition

A surjective simplicial map $\pi : Y \to X$ is called a complete join if:

- *π is injective on simplices.*
- *For each p-simplex $\sigma = \langle v_0, \cdots, v_p \rangle$ of X, $\pi^{-1}(\sigma)$ is the join $\pi^{-1}(v_0) * \pi^{-1}(v_1) \cdots * \pi^{-1}(v_p)$.*

Theorem (Hatcher–Wahl)

Let $\pi : Y \to X$ be a complete join. Then X is highly connected if and only if Y is.

Second reduction

Let U_n^∞ be the following complex:

- vertices: Same
- p-simplices: den'g always left over

Remark

U_n^∞ and U_n share the same $(n-2)$-skeleton.

Third reduction

Definition

An almost admissible loop is a loop $\alpha : (I, \partial I) \to (D \setminus \mathcal{C}, 0)$ *which is freely isotopic to one of the nonbased suited loops.*

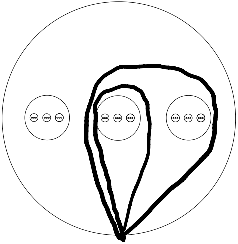
Let T_n^∞ be the following complex:

- vertices:

- p-simplices:

U_n^∞

$d=3$
$r=3$



The inclusion map is highly connected

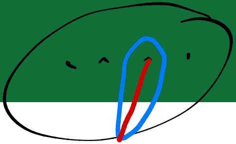
Theorem

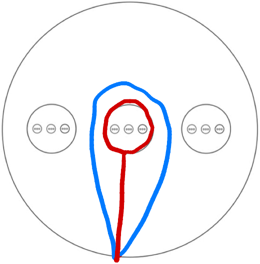
The inclusion map $T_n^\infty \leq U_n^\infty$ is highly connected.

Summary

$$
\begin{tikzcd}
 & & T_n^{\infty} \arrow[d] \\
S_n \arrow[r, "\pi"] & U_n & U_n^{\infty} \\
 & U_n^{(n-2)} \arrow[u, "\subseteq"] \arrow[r, "\cong"] & U_n^{\infty(n-2)} \arrow[u]
\end{tikzcd}
$$

Summary continued



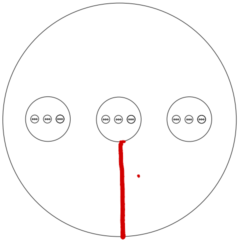


$$\begin{tikzcd}
 & & T_n^{\infty} \arrow[r, "\cong"] \arrow[d] & L_n^{\infty} \\
S_n \arrow[r, "\pi"] & U_n & U_n^{\infty} & \\
 & U_n^{(n-2)} \arrow[u, "\subseteq"] \arrow[r, "\cong"] & U_n^{\infty(n-2)} \arrow[u] &
\end{tikzcd}$$

Key gadget: Lollipop

- Let $A = [0,2]/1 \sim 2$.

- $L : (A, 0) \to (D \setminus \mathcal{C}, 0)$ is a lollipop if
  - L is an embedding.
  - $L\,|_{[1,2]}$ is isotopic to a suited loop in $D \setminus \mathcal{C}$,
  - $L\,|_{[0,1]}$ is an arc connecting the base point 0 to $L([1,2])$.



The Lollipop complex L_n^∞

- vertices: isotopy classes of lollipops;

- $L_0, L_1, \cdots, L_p$, form a p-simplex if they are pairwise disjoint outside the base point 0 and there exists at least one suited loop which does not lie inside the disks bounded by the L_is.

The Lollipop complex is contractible

Theorem (Skipper-W 2021)

The Lollipop complex is contractible.

Connectivity of the lollipop complex