Steenrod squares on Khovanov homology

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Introduction

L a link in $\mathbb{R}^3$.

$$\mathbf{Top} \xrightarrow{\quad \Sigma^\infty \quad} \mathbf{Sp} \xrightarrow{\quad H^*(-;R) \quad} gr\text{-}R\text{-Mod} \xrightarrow{\quad \text{Euler char.} \quad} \mathbb{Z}$$

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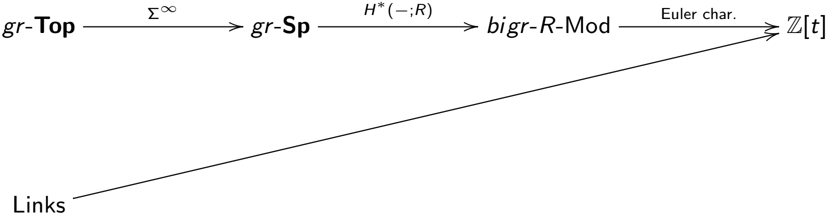

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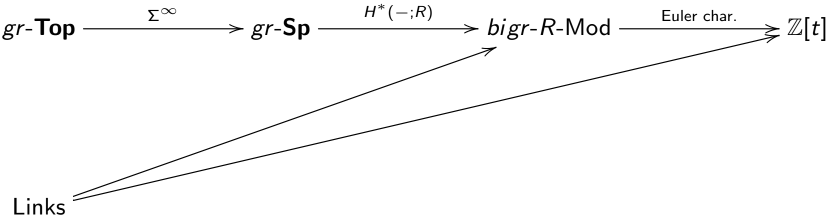

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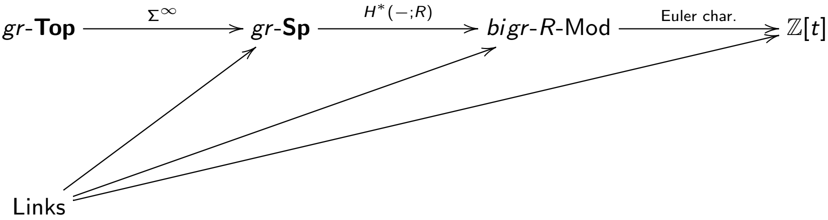

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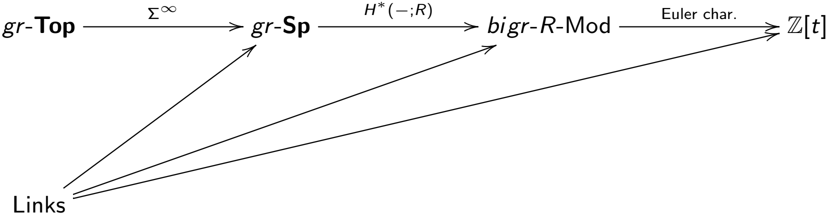

Corollary

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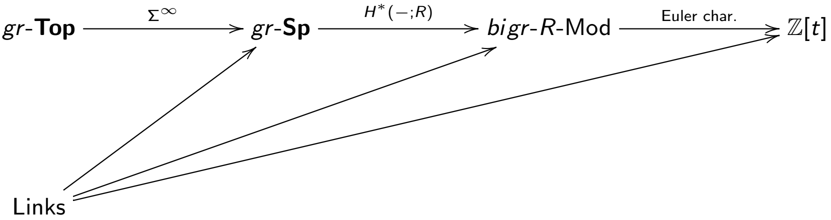

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Question

How to compute this action?

Khovanov homology (Khovanov '00)

In 2000, Khovanov assigned to each knot diagram D, a $\mathbb{Z}$-bigraded family of cochain complexes and proved that it was invariant under Reidemeister moves.

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Trefoil (neg.)

j, i	-3	-2	-1	0
-1				$\mathbb{Z}$
-3				$\mathbb{Z}$
-5		$\mathbb{Z}$		
-7		$\mathbb{Z}_2$		
-9	$\mathbb{Z}$			

Hopf link

j, i	-2	-1	0
0			$\mathbb{Z}$
-2			$\mathbb{Z}$
-4	$\mathbb{Z}$		
-6	$\mathbb{Z}$		

Unknot

j, i	0
1	$\mathbb{Z}$
-1	$\mathbb{Z}$

Semi-simplicial objects, cubes and their realisations

Semi-simplicial objects

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- The *relative realisation* of an augmented semi-simplicial set $X_\bullet$:

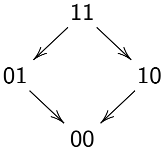
$$|X_\bullet| := \mathrm{hocofib}\left(|X_{\geqslant 0}| \to X_{-1}\right)$$

Cubes and semi-simplicial objects

Let $\mathbf{2^c}$ be the cube poset $\{0 \to 1\}^c$.

$(\mathbf{2^1})^{\text{op}}$

$(\mathbf{2^2})^{\text{op}}$



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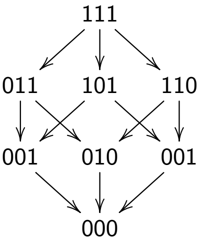
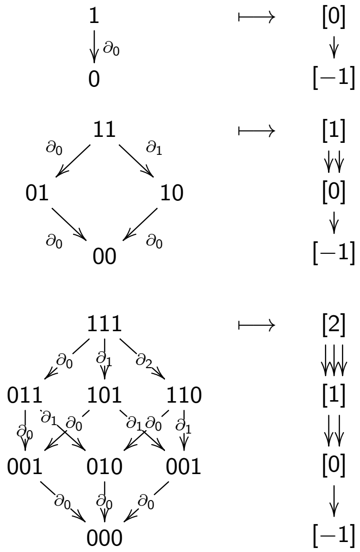


Let $\mathbf{2^c}$ be the cube poset $\{1 \to 0\}^c$.



There is a functor $(\mathbf{2^c})^{\mathrm{op}} \to \tilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}}$ given by

$$u_1 \dots u_c \mapsto \left[-1 + \sum_{i=1}^{c} u_i\right]$$

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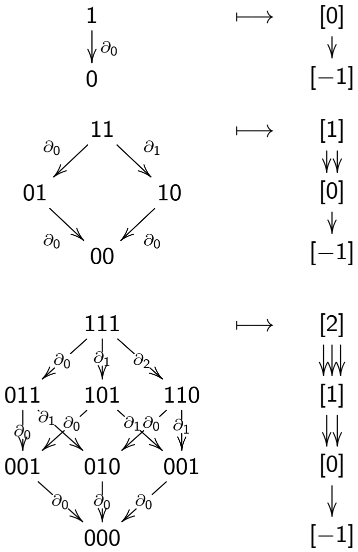

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Defines a functor

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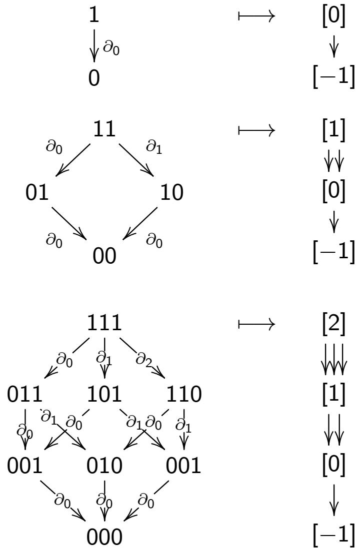

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$$\mathcal{C}^{(\mathbf{2^c})^{\mathrm{op}}} \longrightarrow \mathcal{C}^{\tilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}}}, \qquad \mathcal{C}^{(\mathbf{2^c})^{\mathrm{op}}} \xrightarrow{\mathrm{Tot}} \mathcal{C}, \qquad \mathcal{C}^{\tilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}}} \xrightarrow{|-|} \mathcal{C}, \qquad F \mapsto \hat{F}$$

$$\mathrm{Tot} F := \mathrm{hocofib}\left(\mathrm{hocolim}\left(F|_{\mathbf{2^n} \setminus \{\vec{0}\}}\right) \to F(\vec{0})\right) \in \mathcal{C}$$

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- D a knot diagram with n negative crossings.

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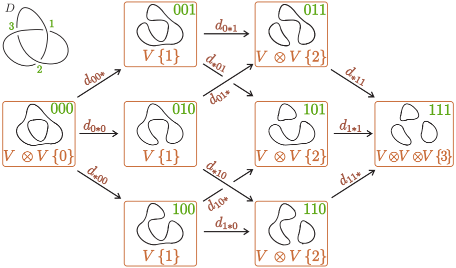

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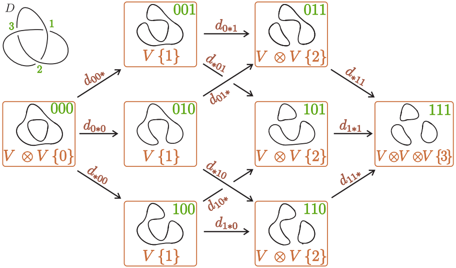

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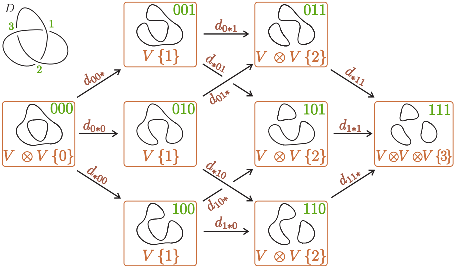

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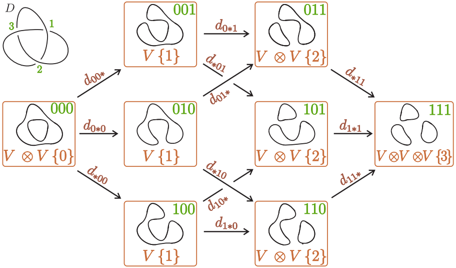
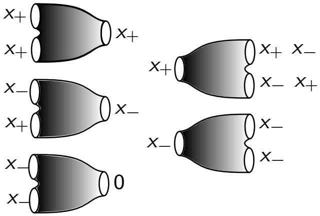


$TQFT(\text{circle}) = V := \mathbb{Z}\langle x_+, x_- \rangle$



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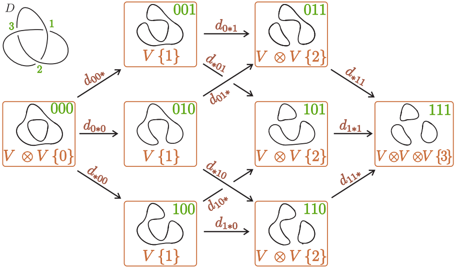

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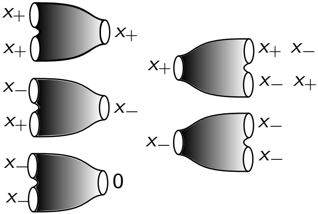

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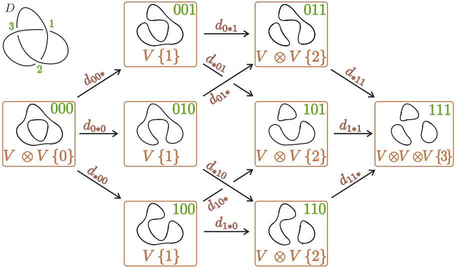

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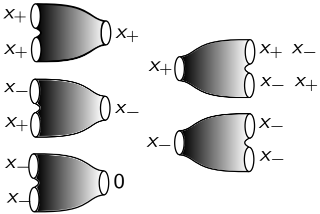

$Kh^i(D) := H^i(\Sigma^{-n}\mathrm{Tot}F_D)$. It splits along the *quantum grading* $\mathrm{Kh}^i(D) = \bigoplus_{j\in\mathbb{Z}} Kh^{i,j}(D)$. It is homotopy invariant under Reidemeister moves.

Lipshitz and Sarkar (and Lawson) spectrum (2014)

How can we assign to each link L a topological space with $H^*(X) \cong \bigoplus_q \mathrm{Kh}^{*,q}(L)$?

- Khovanov homology is computed as the cohomology of a chain complex obtained as the cohomology of the totalisation of a functor

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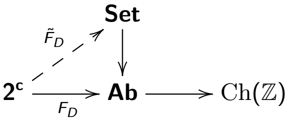

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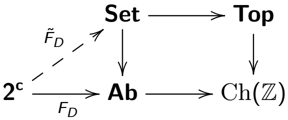

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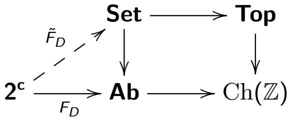

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we would obtain a space whose cohomology coincides with the n-suspension of Khovanov homology $\mathrm{Tot}(F_D)$.

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Problem: Afterwards we would have to desuspend n times this space!!!

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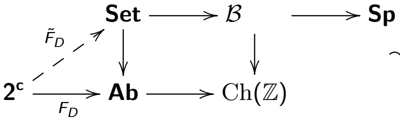

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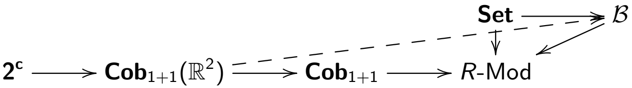

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The Burnside category

Definition

Let $\mathcal{B}$ be the Burnside 2-category for the trivial group whose objects are finite sets, whose morphisms are spans

$$X \xleftarrow{s} Q \xrightarrow{t} Y$$

with composition

$$\begin{array}{ccccccc} & & & Q'' & & & \\ & & \swarrow & & \searrow & & \\ & Q & & & & Q' & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ X & & & Y & & & Z \end{array}$$

and whose 2-morphisms are bijections between spans.

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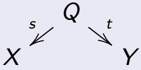

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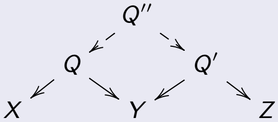

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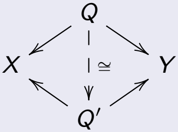

A morphism f in the Burnside category can be interpreted as a linear map between sets with coefficients in sets:

$$f(x) = \sum_{y \in Y} s^{-1}(x) \cap t^{-1}(y) \cdot y$$

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Steenrod squares in Khovanov homology

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Theorem (Cantero, 2021)

The Khovanov chain complex admits explicit stable $\smile_i$-products, which give rise to formulae for the Steenrod squares.

Stable $\smile_i$-products

- (C^*, d) a cochain complex of $\mathbb{Z}/2$-modules,
- $T\colon C^* \otimes C^* \to C^* \otimes C^*$ the twist homomorphism $T(a \otimes b) = b \otimes a$,
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$\smile_i$-products

A *unstable* $\smile_i$*-product* on (C^*, d) is a family of homomorphisms

$$\smile_i\colon C^* \otimes C^* \longrightarrow C^*$$

with $i \in \mathbb{Z}$, such that $\smile_i$ has degree i and

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$\smile_i$-products

A *unstable* $\smile_i$*-product* on (C^*, d) is a family of homomorphisms

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with $i \in \mathbb{Z}$, such that $\smile_i$ has degree i and

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Steenrod squares

$\mathrm{Sq}^i([x]) = [x \smile_{n-i} x]$ if $x \in C^n$ is a cocycle.