Exotic stabilization maps for configuration spaces

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Configuration spaces

Definition

Let $\text{Conf}_n(M) = \{(x_1, \ldots, x_n) | x_i \in M, x_i \neq x_j \text{ for } i \neq j\}/S_n$.

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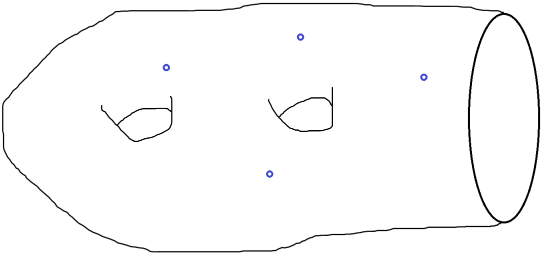
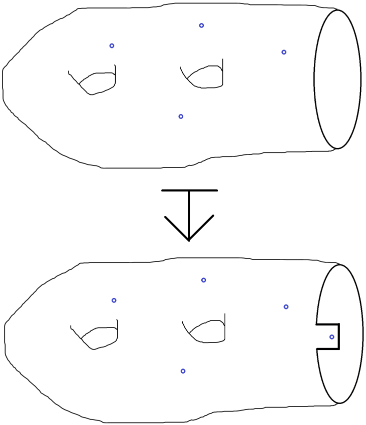


Stabilization maps for open manifolds

There is a map $t : \mathrm{Conf}_n(M) \to \mathrm{Conf}_{n+1}(M)$.



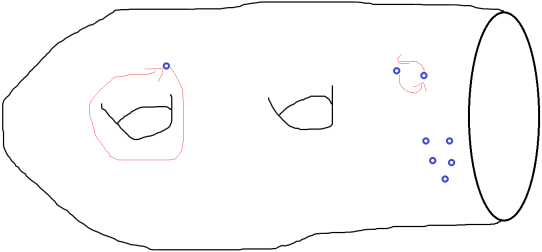
Homological stability

Theorem (McDuff)

For M connected, non-compact, and $n \gg i$, the map $t : H_i(\mathrm{Conf}_n(M)) \to H_i(\mathrm{Conf}_{n+1}(M))$ is an isomorphism.

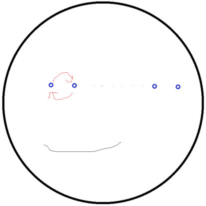
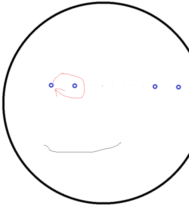
Homological stability (example)

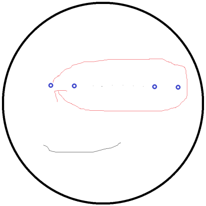
For M a non-compact surface, $H_1(\mathrm{Conf}_n(M)) = \begin{cases} 0 & \text{if } n = 0 \\ H_1(M) & \text{if } n = 1 \\ H_1(M) \oplus \mathbb{Z} & \text{if } n \geq 2 \end{cases}$



Failure of homological stability (example)

$$H_1(\mathrm{Conf}_n(S^2)) = \begin{cases} 0 & \text{if } n \leq 2 \\ \mathbb{Z}/(2n-2) & \text{if } n \geq 2 \end{cases}$$

(2n-2)  =(n-1) 

=  =0

Stable periodicity

Theorem (Cantero–Palmer, Nagpal, Kupers–M.)

For M connected and $n \gg i$, $H_i(\mathrm{Conf}_n(M);\mathbb{F}_p) \cong H_i(\mathrm{Conf}_{n+p}(M);\mathbb{F}_p)$

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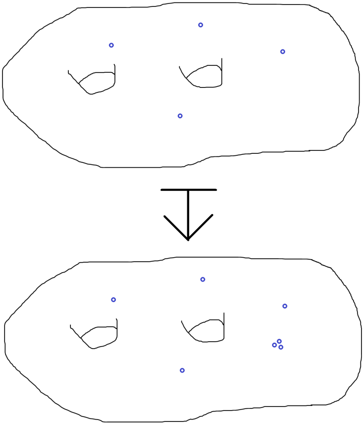
Example

For $n \geq 2$, $H_i(\mathrm{Conf}_n(S^2); \mathbb{F}_3) = \begin{cases} \mathbb{F}_3 & \text{if } n = 1 \bmod 3 \\ 0 & \text{otherwise.} \end{cases}$

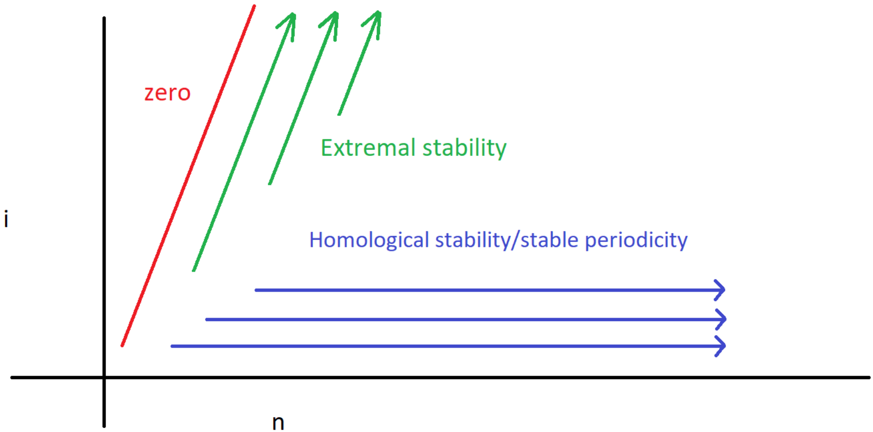
What are the maps inducing stable periodicity?

Goal: Find a map $H_i(\mathrm{Conf}_n(M);\mathbb{F}_p) \otimes H_0(\mathrm{Conf}_p(M);\mathbb{F}_p) \to H_i(\mathrm{Conf}_{n+p}(M);\mathbb{F}_p)$ inducing stable periodicity.



Stability in high dimensions

Let $d = \dim M$. Then $H_i(\mathrm{Conf}_n(M)) = 0$ for $i > (d-1)n + 1$.



Extremal stability

Theorem (Knudsen–M.–Tosteson)

Let $d = \dim M$. *Let* $\nu_n = (d-1)n + 1$. *There are polynomials* p_i^{odd} *and* p_i^{even} *of degree* $\leq \dim_{\mathbb{Q}} H_{d-1}(M;\mathbb{Q})$ *such that, for* $n \gg i$,

$$\dim_{\mathbb{Q}} H_{\nu_n - i}(\mathrm{Conf}_n(M);\mathbb{Q}) = \begin{cases} p^{even}(n) & \text{if } n \text{ is even} \\ p^{odd}(n) & \text{if } n \text{ is odd.} \end{cases}$$

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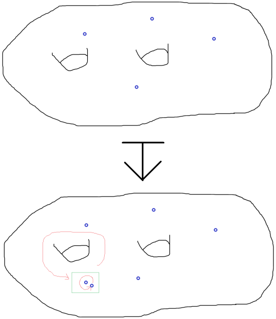
Example

$$\dim_{\mathbb{Q}} H_{n+1}(\mathrm{Conf}_n(\Sigma_2);\mathbb{Q}) = \begin{cases} \frac{n^3+n^2+16}{16} & \text{if } n \text{ is even} \\ \frac{n^3+n^2-9n-9}{16} & \text{if } n \text{ is odd.} \end{cases}$$

What are the maps extremal stability?

Goal: Find a map
$H_i(\mathrm{Conf}_n(M);\mathbb{Q}) \otimes H_{2d-2}(\mathrm{Conf}_2(M);\mathbb{Q}) \to H_{i+2d-2}(\mathrm{Conf}_{n+2}(M);\mathbb{Q})$
inducing stable periodicity.



Algebraic reformation of classical stability

Let $W_0, W_1, \ldots$ be a sequence of finitely generated abelian groups.

- The data of maps $W_n \to W_{n+1}$ is the same as the data of a $\mathbb{Z}[x]$-module structure on the graded module $W = \bigoplus_n W_n$. $|x| = 1$.
- The maps $W_n \to W_{n+1}$ are eventually isomorphisms if and only if W is a finitely generated $\mathbb{Z}[x]$-module.

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Example

Take $W_n = H_i(Conf_n(M))$ for M non-compact.

Algebraic reformation of perodicity

Let $W_0, W_1, \ldots$ be a sequence of finitely generated abelian groups.

- The data of maps $W_n \to W_{n+k}$ is the same as the data of a $\mathbb{Z}[x]$-module structure on the graded module $W = \bigoplus_n W_n$. $|x| = k$.
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Example

Take $W_n = H_i(Conf_n(M); \mathbb{F}_p)$ for M compact and $p = k$.

Reformulation of polynomial growth

Let $W_0, W_1, \ldots$ be a sequence of finitely generated abelian groups and $W = \bigoplus_n W_n$. Let $V = \mathbb{Z}^m =< x_1, \ldots, x_m >$ with $|x_i| = 1$ for all i.

- The data of maps $V \otimes W_n \to W_{n+1}$ (satisfying a commutativity condition) is the same data as a $\mathbb{Z}[x_1, \ldots, x_m]$-module structure on M.
- If W is a finitely generated $\mathbb{Z}[x_1, \ldots, x_m]$-module, then for each field $\mathbb{F}$, there is a polynomial p of degree $m-1$ such that for large n, $\dim_{\mathbb{F}} W_n \otimes \mathbb{F} = p(n)$.

Reformulation of quasi-polynomial growth

Let $W_0, W_1, \ldots$ be a sequence of finitely generated abelian groups and $W = \bigoplus_n W_n$. Let $V = \mathbb{Z}^m =< x_1, \ldots, x_m >$ with $|x_i| = k$ for all i.

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Example

Take $W_n = H^{n(d-1)-i}(Conf_n(M); \mathbb{Q})$ for $\dim W = d$, $k = 2$, $m = \dim_{\mathbb{Q}} H_{d-1}(M; \mathbb{Q})$.

Strategy overview

- Try to build a map $C_*(\mathrm{Conf}(M)) \otimes C_*(\mathrm{Conf}(M)) \to C_*(\mathrm{Conf}(M))$.
- Using factorization homology, just need to build a map $C_*(\mathrm{Conf}(\mathbb{R}^d)) \otimes C_*(\mathrm{Conf}(\mathbb{R}^d)) \to C_*(\mathrm{Conf}(\mathbb{R}^d))$.
- Use operadic cells to find the obstruction to building such a map.

Algebra over the little disks operad

Definition

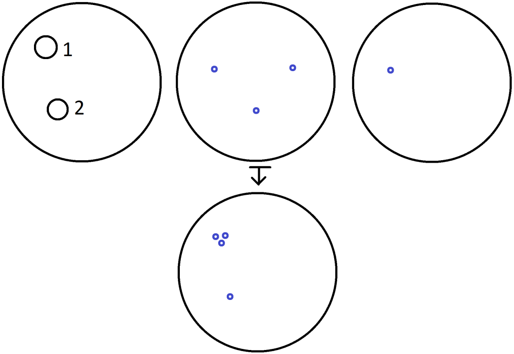
Let $E_d(n)$ be the space of n d-dimensional disks in an n-dimensional disk.

Definition

An E_d-algebra is a space A and maps $E_d(n) \times A^n \to A$ such that ...

Examples of E_d-algebras

$\mathrm{Conf}(\mathbb{R}^d)$ is an E_d-algebra.



$\mathbb{N}$, $\Omega^d X$, $\mathrm{Conf}(\mathbb{R}^d) \times \mathrm{Conf}(\mathbb{R}^d)$ are also E_d-algebras.

Factorization homology

Input of factorization homology: An E_d-algebra A and a d-manifold M.
Output of factorization homology: A space $\int_M A$.

Example

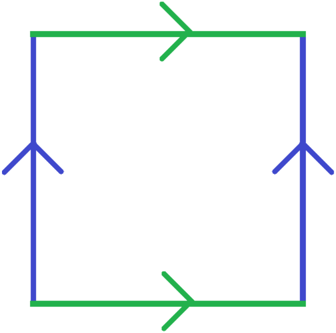
- $\int_M \mathrm{Conf}(\mathbb{R}^d) \simeq \mathrm{Conf}(M)$.
- $\int_M \mathbb{N} \simeq \mathrm{Sym}(M)$.
- $\int_M \Omega^d X \simeq Map^c(M, \Sigma^d X)$
- $\int_M \mathrm{Conf}(\mathbb{R}^d) \times \mathrm{Conf}(\mathbb{R}^d) \simeq \mathrm{Conf}(M) \times \mathrm{Conf}(M)$.

Warning: I am sporadically assuming the manifolds are parallelizable.

Question: What is the data of a based map $S^1 \times S^1 \to X$?

Answer: Two maps $f, g : S^1 \to X$ and a choice of null homotopy of $[f, g]$.

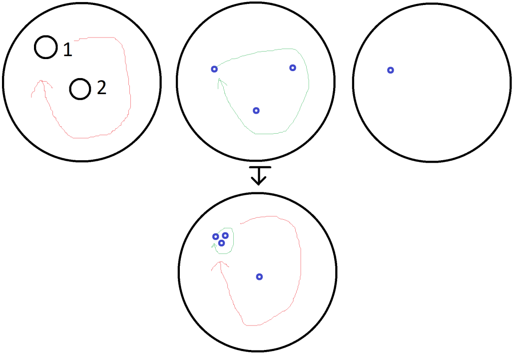
$S^1 \times S^1 = S^1 \vee S^1$ with a cell attached along the commutator.



Commutator for E_d-algebras

$E_d(2) \simeq S^{d-1}$. Plugging in the fundamental class into $E_2(2) \times A \times A \to A$ gives a Lie bracket

$$[\ ,\ ] : H_i(A) \otimes H_j(A) \to H_{i+j+d-1}(A).$$



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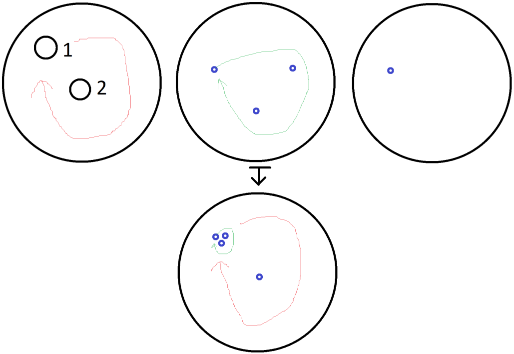


Class of a point gives a product $\bullet : H_i(A) \otimes H_j(A) \to H_{i+j}(A)$.

Classical cell attachments

Let X be a space. Let $f : S^{N-1} \to X$ be a map. X with a cell attached along f is the pushout:

$$\begin{array}{ccc} S^{N-1} & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ D^N & \longrightarrow & X \cup_f D^N \end{array}$$

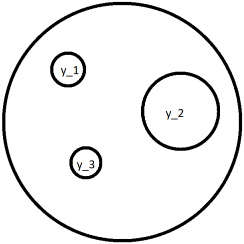
E_d-cell attachments

Let E_d denote the free E_d-algebra functor. Let A be an E_d-algebra. Let $f : S^{N-1} \to A$ be a map. X with an E_d-cell attached along f is the pushout (in the category of E_d-algebras):

$$\begin{array}{ccc} E_d S^{N-1} & \xrightarrow{f} & A \\ \downarrow & & \downarrow c \\ E_d D^N & \longrightarrow & A \cup_f^{E_d} D^N \end{array}$$

Free E_d-algebras

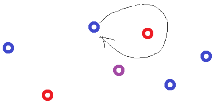
For Y a space, $E_d Y$ is the configuration space of disks with labels in Y.



- $E_d\{pt\} \simeq \mathrm{Conf}(\mathbb{R}^d)$.
- $E_d\{red, blue\}$ is homotopy equivalent to the configuration of red and blue points in $\mathbb{R}^d$.

Examples of E_d-cell structures

- $E_d\{pt\} \simeq \mathrm{Conf}(\mathbb{R}^d)$ is obtained from the trivial E_d-algebra by attaching one 0-cell.
- $E_d\{red, blue\}$ is obtained from the trivial E_d-algebra by attaching two 0-cells.
- $\mathrm{Conf}(\mathbb{R}^d) \times \mathrm{Conf}(\mathbb{R}^d) \simeq E_d\{red\} \times E_d\{blue\}$ is equivalent to $E_d\{red, blue\}$ with a cell attached along $[red, blue]$.



Will think of $red \in H_0(\mathrm{Conf}_1(\mathbb{R}^d))$ and $blue \in H_0(\mathrm{Conf}_1(\mathbb{R}^d))$.

Universal mapping property of E_d-cell attachments

- Let $f : S^{N-1} \to A$. A map $A \cup_f^{E_d} D^N \to B$ is the data of a map of E_d-algebras $A \to B$ and a null homotopy (in spaces) of the map $S^{N-1} \to A \to B$.
- A map $f : \text{Conf}(\mathbb{R}^d) \times \text{Conf}(\mathbb{R}^d) \to A$ is a choice of $f(red) \in A$ and $f(blue) \in A$ and a choice of null homotopy of $[f(red), f(blue)]$.

$red \in H_0(\text{Conf}_1(\mathbb{R}^d))$, $blue \in H_0(\text{Conf}_1(\mathbb{R}^d))$.

Warning: I will be cavalier about spaces vs chain complexes vs homology.

Applying factorization homology

- A choice of $f(red) \in \mathrm{Conf}(\mathbb{R}^d)$ and $f(blue) \in \mathrm{Conf}(\mathbb{R}^d)$ and a choice of null homotopy of $[f(red), f(blue)]$ gives a map

$$f : \mathrm{Conf}(\mathbb{R}^d) \times \mathrm{Conf}(\mathbb{R}^d) \to \mathrm{Conf}(\mathbb{R}^d).$$

- $\int_M \mathrm{Conf}(\mathbb{R}^d) \simeq \mathrm{Conf}(M)$.
- A choice of $f(red) \in \mathrm{Conf}(\mathbb{R}^d)$ and $f(blue) \in \mathrm{Conf}(\mathbb{R}^d)$ and a choice of null homotopy of $[f(red), f(blue)]$ gives a map

$$f : \mathrm{Conf}(M) \times \mathrm{Conf}(M) \to \mathrm{Conf}(M).$$

Constructing the periodic stability map

- Let *teal* be the generator of $H_0(\mathrm{Conf}_1(\mathbb{R}^d))$.
  $[teal^p, teal] = p(teal)^{p-1}[teal, teal] = 0 \in H_{d-1}(\mathrm{Conf}_{p+1}(\mathbb{R}^d); \mathbb{F}_p)$.
- Letting $f(red) = teal^p$ and $f(blue) = teal$ gives a map:

$$f : H_i(\mathrm{Conf}_n(M); \mathbb{F}_p) \otimes H_j(\mathrm{Conf}_m(M); \mathbb{F}_p)$$

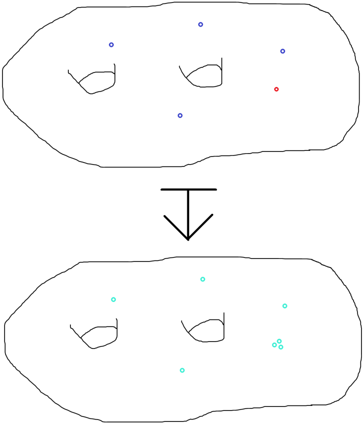
$$\to H_{i+j}(\mathrm{Conf}_{pn+m}(M); \mathbb{F}_p).$$

- This restricts to a map

$$H_0(\mathrm{Conf}_1(M); \mathbb{F}_p) \otimes H_j(\mathrm{Conf}_m(M); \mathbb{F}_p) \to H_j(\mathrm{Conf}_{m+p}(M); \mathbb{F}_p).$$

Picture of periodict stability map

$$H_0(\mathrm{Conf}_1(M);\mathbb{F}_p) \otimes H_j(\mathrm{Conf}_m(M);\mathbb{F}_p) \to H_j(\mathrm{Conf}_{m+p}(M);\mathbb{F}_p).$$



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Constructing the extremal stability map

- $[[teal, teal], teal] = 0$.
- Letting $f(red) = [teal, teal] \in H_{d-1}(\mathrm{Conf}_1(\mathbb{R}^d))$ and $f(blue) = teal$ gives a map:

$$f : H_i(\mathrm{Conf}_n(M)) \otimes H_j(\mathrm{Conf}_m(M)) \to H_{i+(d-1)n+j}(\mathrm{Conf}_{2n+m}(M)).$$

- This restricts to a map

$$H_{d-1}(\mathrm{Conf}_1(M)) \otimes H_j(\mathrm{Conf}_m(M)) \to H_{j+2d-2}(\mathrm{Conf}_{m+2}(M)).$$

Picture of extremal stability map

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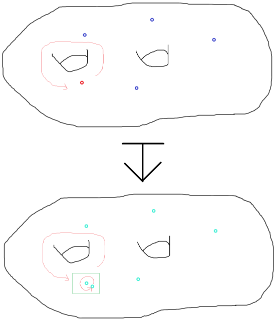

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The end