Filtrations on the Mapping Class Group of the Punctured Sphere

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Introduction : statement of the main result

Context: Artin action

The group $G = PMCG_{\bullet}(\mathbb{S}^2, n)$ acts on $\pi_1(\mathbb{S}^2 - \{n \text{ pts}\}) \cong F_{n-1}$.

Main theorem (D., 2020)

Let $\beta \in G$ and $k \geq 1$.

$$\beta \in \Gamma_k(G) \quad \Leftrightarrow \quad \forall w \in F_{n-1},\ \beta(w) \equiv w\ [\Gamma_{k+1}(F_{n-1})].$$

- $(\Rightarrow)$ is easy;
- $(\Leftarrow)$ is difficult;
- Right side: computable.

Computations: Milnor invariants

Artin action

$\beta \in G = PMCG_\bullet(\mathbb{S}^2, n) \rightsquigarrow$ a basis-conjugating automorphism of $F_{n-1} = \pi_1(\mathbb{S}^2 - \{n \text{ pts}\})$: $x_i \mapsto w_i x_i w_i^{-1}$ $(i = 1, 2, ..., n-1)$.

Magnus expansions

Each $w_i \in F_{n-1}$ is a word in the $x_j^{\pm 1}$. Get a formal power series $\Phi(w_i)$ in the (non-commuting) X_j by $x_j \mapsto 1 + X_j$.

E.g.: $\Phi(x_1 x_2^{-1} x_3) = (1 + X_1)(1 - X_2 + X_2^2 - ...)(1 + X_3)$.

Milnor invariants

Coefficients of monomials of degree d in the $\Phi(w_i) - 1$ are integers, called *Milnor invariants of degree d* of β.

$\forall w \in F_{n-1},\ \beta(w) \equiv w\ [\Gamma_{k+1}(F_{n-1})]$

$\Leftrightarrow$ Milnor invariants of degree $< k$ are trivial.

Outline

The Mapping Class Group of the punctured sphere

$MCG_\bullet(\mathbb{S}^2, n)$

Isotopy classes of self-homeomorphisms of the sphere, permuting n points and fixing a basepoint.

$$\begin{array}{ccccccc}
MCG_\bullet(\mathbb{S}^2, n) & \twoheadleftarrow & B_n \cong MCG_\partial(\mathbb{D}^2, n) & \hookleftarrow & ? & \hookleftarrow & \mathbb{Z} \\
\uparrow & & \uparrow & & \uparrow & & = \uparrow \\
PMCG_\bullet(\mathbb{S}^2, n) & \twoheadleftarrow & P_n \cong PMCG_\partial(\mathbb{D}^2, n) & \hookleftarrow & ? & \hookleftarrow & \mathbb{Z}
\end{array}$$

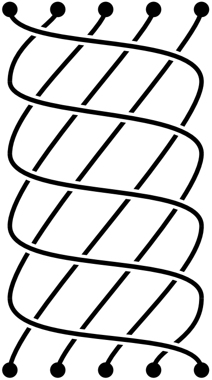
Kernels generated by ξ_n "full twist" (Dehn twist around $\partial\mathbb{D}^2$), and $\langle \xi_n \rangle = \mathcal{Z}(B_n) = \mathcal{Z}(P_n)$.

$$PMCG_\bullet(\mathbb{S}^2, n) \cong P_n/\mathcal{Z} =: P_n^*.$$

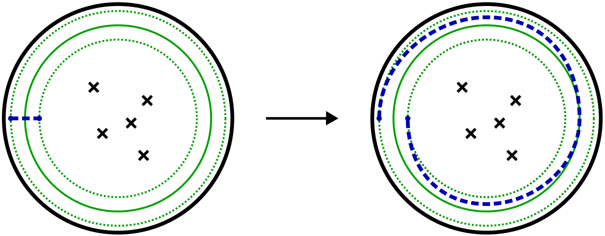
The center of the braid group

$$\mathcal{Z}(B_n) = \mathcal{Z}(P_n) = \langle \xi_n \rangle.$$

ξ_5 as a braid

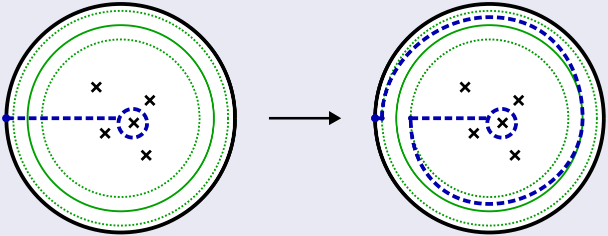


ξ_5 as a mapping class



The center of the braid group

ξ_5 as a mapping class



ξ_5 as an automorphism of $\pi_1(\mathbb{D}^2 - \{5 \text{ pts}\}) = F_5$

Its acts by conjugation by $\partial_5 = x_1x_2x_3x_4x_5$.

Artin actions

The group $B_n = MCG_\partial(\mathbb{D}^2, n)$ acts on $\pi_1(\mathbb{D}^2 - \{n \text{ pts}\}) \cong F_n$.

Theorem (Artin, 1925)

This action is faithful and $B_n \cong \mathrm{Aut}_C^\partial(F_n)$, *where* $\partial = x_1 \cdots x_n$.

The group $B_n^* = MCG_\bullet(\mathbb{S}^2, n)$ acts on $\pi_1(\mathbb{S}^2 - \{n \text{ pts}\}) \cong F_{n-1}$.

Theorem (Magnus, 1934)

This action is faithful.

The latter can be seen as a quotient of the former.

Outline

Group filtrations

Notations

In a group G:

- $[x, y] = xyx^{-1}y^{-1}$,
- $x^y := y^{-1}xy$ and ${}^y x := yxy^{-1}$,
- $[A, B] = \langle [a, b] \rangle_{(a,b) \in A \times B}$.

Filtration on a group G

Nested sequence of subgroups $G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots$ such that:

$$\forall i, j \geq 1, \quad [G_i, G_j] \subseteq G_{i+j}.$$

The *lower central series* $\Gamma_*(G)$: the *minimal* filtration on G

$$\begin{cases} \Gamma_1(G) = G, \\ \Gamma_{k+1}(G) = [G, \Gamma_k(G)]. \end{cases}$$

Lie rings associated to group filtrations

$$\mathcal{L}(G_*) := \bigoplus_{i\geq 1} G_i/G_{i+1} \text{ is a graded abelian group.}$$

Commutators in G $\rightsquigarrow$ $[-,-] : \mathcal{L}(G_*) \times \mathcal{L}(G_*) \longmapsto \mathcal{L}(G_*)$.

In $G = G_1$

- $[x, y]^{-1} = [y, x]$,
- $[x, yz] = [x, y] \cdot ({}^{y}[x, z])$,
- $[[x, y], {}^{y}z] \cdot [[y, z], {}^{z}x] \cdot [[z, x], {}^{x}y] = 1$.

In $\mathcal{L}(G_*)$

- $[y, x] = -[x, y]$,
- $[x, y + z] = [x, y] + [x, z]$,
- $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

Consequence

$\mathcal{L}(G_*)$ is a graded Lie ring ($=$ a graded Lie algebra over $\mathbb{Z}$).

Lie ring of a group

Notation

$\mathcal{L}(G) := \mathcal{L}(\Gamma_* G)$.

Theorem (Magnus, 1931)

$\mathcal{L}(F_n) = \mathfrak{L}_n$ is the free Lie ring on n generators.

In general, $\mathcal{L}(G)$ can be difficult to compute.

Outline

The Andreadakis filtration

Two filtrations on IA_n

$$\begin{array}{ccccc} & & \mathrm{Aut}(F_n) & & \\ & & \cup & & \\ & & IA_n = \mathcal{A}_1 & & \\ & \swarrow & & \searrow & \\ \Gamma_2(IA_n) & & \subset & & \mathcal{A}_2 \\ \cup & & & & \cup \\ \Gamma_3(IA_n) & & \subset & & \mathcal{A}_3 \\ \cup & & & & \cup \\ \cdots & & & & \cdots \end{array}$$

Definitions

$$IA_n = \ker\left(\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n^{ab})\right)$$

(automophisms acting trivially on the abelianization $F_n^{ab} \cong \mathbb{Z}^n$)

$$\mathcal{A}_k = \ker\left(\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n/\Gamma_{k+1})\right)$$

(automophisms acting trivially modulo $\Gamma_{k+1}(F_n)$)

Residual nilpotence

$$\bigcap_k \Gamma_k(F_n) = \{1\} \quad \Rightarrow \quad \bigcap_k \mathcal{A}_k = \{id\} \quad \Rightarrow \quad \bigcap_k \Gamma_k(IA_n) = \{id\}.$$

The Andreadakis problem

Conjecture (Andreadakis - 1965)

Do we always have $\Gamma_k(IA_n) = \mathcal{A}_k$?

Bartholdi (2013)

No.

Problem (Bartholdi - 2013)

Are they the same up to finite index ?

Bartholdi (2017)

Not even !

This uses computer calculations, notably by Day and Putman ('17).

Restriction to subgroups

Let $G \subseteq IA_n$ be a subgroup.

3 filtrations on G

$$\Gamma_*(G) \;\subseteq\; G \cap \Gamma_*(IA_n) \;\subseteq\; G \cap \mathcal{A}_*.$$

Problem (Andreadakis for subgroups)

When are these the same ?

Obviously, not always. When it is the case, we say that G satisfies the *Andreadakis equality*.

Theorem (Habegger-Mausbaum '00, Mostovoy-Willerton '02)

For $G = P_n = PMCG_\partial(\mathbb{D}^2, n)$ acting (faithfully) on $F_n = \pi_1(\mathbb{D}^2 - \{n \text{ pts}\})$, the Andreadakis equality holds:

$$\Gamma_*(P_n) = \Gamma_*(IA_n) \cap P_n = \mathcal{A}_* \cap P_n.$$

Our goal: $G = P_n^* \subseteq IA_{n-1}$.

Another set of Milnor invariants

A direct product decomposition

$$P_n \underset{\overset{s}{\dashleftarrow}}{\longrightarrow} P_n^* = P_n/\mathcal{Z} \quad \Rightarrow \quad P_n \cong \mathbb{Z} \times P_n^*.$$

$$\Gamma_k(P_n) \cong \Gamma_k(\mathbb{Z}) \times \Gamma_k(P_n^*).$$

$$\beta \in \Gamma_k(P_n^*) \quad \Leftrightarrow \quad s(\beta) \in \Gamma_k(P_n)$$

$\Leftrightarrow$ Milnor invariants of $s(\beta)$ of degree $< k$ vanish.

Comparison with our Milnor invariants

Milnor invariants of $s(\beta) \circlearrowleft F_n$ [Artin action]
$\neq$ Milnor invariants of $\beta \circlearrowleft F_{n-1}$ [Magnus action].

- The first ones depend on the choice of s.
- The second ones distinguish elements of P_n^* if and only if the Magnus action is faithful.

Translation in terms of Lie algebras

The Johnson morphism

$IA_n \circlearrowright F_n \ \rightsquigarrow\ \mathcal{L}(\mathcal{A}_*) \circlearrowright \mathcal{L}(F_n) \cong \mathfrak{L}_n$ (action on the free Lie ring).

$\leftrightsquigarrow\ \tau : \mathcal{L}(\mathcal{A}_*) \to \mathrm{Der}(\mathfrak{L}_n)$, injective by definition of $\mathcal{A}_*$.

Explicitly: $\tau(\overline{\sigma}) : \overline{x} \mapsto \overline{\sigma(x)x^{-1}}$ (for $\sigma \in IA_n$, $x \in F_n$).

The Andreadakis equality for $G \subseteq IA_n$

$$\begin{aligned}
\Gamma_*(G) = G \cap \mathcal{A}_* \ ?& \\
&\Leftrightarrow\ i_\# : \mathcal{L}(G) \to \mathcal{L}(\mathcal{A}_*) \ \text{ is injective.}\\
&\Leftrightarrow\ \tau \circ i_\# : \mathcal{L}(G) \to \mathcal{L}(\mathcal{A}_*) \hookrightarrow \mathrm{Der}(\mathfrak{L}_n) \ \text{ is injective.}\\
&\Leftrightarrow\ \tau : \mathcal{L}(G) \to \mathrm{Der}(\mathfrak{L}_n) \ \text{ is injective.}
\end{aligned}$$

Translation in terms of Lie algebras

The Andreadakis equality for $G \subseteq IA_n$

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$$\Leftrightarrow \ \tau \circ i_\# : \mathcal{L}(G) \to \mathcal{L}(\mathcal{A}_*) \hookrightarrow \mathrm{Der}(\mathfrak{L}_n) \ \text{ is injective.}$$

$$\Leftrightarrow \ \tau : \mathcal{L}(G) \to \mathrm{Der}(\mathfrak{L}_n) \ \text{ is injective.}$$

Proof of ($\Leftarrow$): An element $x \in \mathcal{A}_k \cap G - \Gamma_k(G)$ must be in $\Gamma_j(G)$ but not in $\Gamma_{j+1}(G)$ for some $j < k$. It would then give a non-trivial class $\overline{x}$ in $\Gamma_j(G)/\Gamma_{j+1}(G)$. Since $x \in \mathcal{A}_k \subseteq \mathcal{A}_{j+1}$, the image of $\overline{x}$ in $\mathcal{A}_j/\mathcal{A}_{j+1}$ would be trivial, hence $\overline{x} \in \ker(i_\#) - \{0\}$.

Outline

Main result

Theorem (D., 2020)

For $G = P_n^* = PMCG_\bullet(\mathbb{S}^2, n)$ *acting (faithfully) on* $F_{n-1} = \pi_1\left(\mathbb{S}^2 - \{n \text{ pts}\}\right)$, *the Andreadakis equality holds:*

$$\Gamma_*(P_n^*) = \mathcal{A}_* \cap P_n^*.$$

Otherwise said, for all $k \geq 1$ *and all* $\beta \in P_n^*$,

$$\beta \in \Gamma_k(P_n^*) \quad \Leftrightarrow \quad \forall w \in F_{n-1},\ \beta(w) \equiv w\ [\Gamma_{k+1}(F_{n-1})].$$

Equivalently, the Johnson morphism $\tau : \mathcal{L}(P_n^*) \to \mathrm{Der}(\mathfrak{L}_{n-1})$ *is injective.*

Thus, $\mathcal{L}(P_n^*)$ identifies to a *sub-Lie ring* of $\mathrm{Der}(\mathfrak{L}_{n-1})$ (and generators are known). In particular, it is *without torsion.*

Decomposition of P_n^*

$P_n \cong \mathrm{Aut}_C^{\partial}(F_n)$

$$\begin{cases} x_i \mapsto w_i x_i w_i^{-1} \ (i \leq n), \\ \partial_n = x_1 \cdots x_n \text{ is fixed.} \end{cases}$$

$P_n^* \cong \mathrm{Inn}(F_{n-1})P_{n-1}$

$$\begin{cases} x_i \mapsto w_i x_i w_i^{-1} \ (i \leq n-1), \\ \partial_{n-1} = x_1 \cdots x_{n-1} \ \mapsto w \partial_{n-1} w^{-1}. \end{cases}$$

A quotient

P_n acts on $F_n/\partial_n = F_n/(x_n = \partial_{n-1}^{-1}) \cong F_{n-1}$:

$$x_1 \cdots x_n = 1 \ \Leftrightarrow \ x_n = (x_1 \cdots x_{n-1})^{-1}.$$

This factors through $P_n/\xi_n = P_n^*$:

ξ_n acts on F_n by conjugation by ∂_n.

Decomposition of filtrations

Theorem (D., 2020)

K subgroup of IA_n ($\rightsquigarrow \tau : \mathcal{L}(K) \to \mathrm{Der}(\mathfrak{L}_n)$). Suppose that:

- *K satisfies the Andreadakis equality $\mathcal{A}_* \cap K = \Gamma_*(K)$ (τ is injective),*
- *Every element of $\tau(\mathcal{L}(K)) \cap \mathrm{ad}(\mathfrak{L}_n)$ equals $\tau(\overline{x})$ for some $x \in K \cap \mathrm{Inn}(F_n)$.*

Then $G = \mathrm{Inn}(F_n)K$ also satisfies the Andreadakis equality:

$$\mathcal{A}_* \cap G = \Gamma_*(G).$$

This is a rare result.

Decomposition of filtrations on P_n^*

Application to $P_n^* \subset IA_{n-1}$

Decomposition: $P_n^* = \mathrm{Inn}(F_{n-1})P_{n-1}$.

- $K = P_{n-1}$ satisfies the Andreadakis equality. ✓
- $\tau(\mathcal{L}(P_{n-1})) \cap \mathrm{ad}(\mathfrak{L}_{n-1}) = \mathbb{Z} \cdot \tau\left(\overline{\xi}_{n-1}\right)$ and $\xi_{n-1} \in P_{n-1} \cap \mathrm{Inn}(F_{n-1})$. ✓

Thus $P_n^* \subset IA_{n-1}$ satisfies the Andreadakis equality.

A derivation d in $\tau(\mathcal{L}(P_{n-1}))$ must send $\overline{\partial}_{n-1} = X_1 + \cdots + X_{n-1}$ to 0. If $d = [w, -]$ for some $w \in \mathfrak{L}_n$, then w must be in $\mathbb{Z} \cdot \overline{\partial}_{n-1}$. But $[\overline{\partial}_{n-1}, -] = \tau(\overline{\xi}_{n-1})$, which acts on F_{n-1} by conjugation by $x_1 \cdots x_{n-1}$.