Filtering the modular surface operad

0 Introduction

$\Sigma_{g,k}$. $\mathrm{Diff}^{\partial}(\Sigma_{g,k}) = \{\varphi : \Sigma_{g,k} \to \Sigma_{g,k} \mid \varphi|_{\partial\Sigma} = \mathrm{id}\}$

Goal Study $B\mathrm{Diff}^{\partial}(\Sigma_{g,k}) \overset{g\geq 2}{\simeq} B\Gamma_{g,k} \simeq \mathcal{M}_{g,k}$

Picture of $H_*(B\mathrm{Diff}(\Sigma_g);\mathbb{Q})$ mention:
- homological stability
- low genus computations
- Top weight comp.

Main Thm (in progress) There is a spectral sequence with $E^1_{g,k} = H^{5g-5-k}(\Gamma_g;\mathbb{Q})$ $(g\geq 2)$
converging to $H^{\varphi}_{g,k}(\Sigma MTSO_2;\mathbb{Q}) = \begin{cases} \mathbb{Q} & g+k \text{ odd} \\ 0 & \text{even} \end{cases}$

Idea $\mathcal{M}(k) := \coprod_{g\geq 0} B\mathrm{Diff}^{\partial}(\Sigma_{g,k})$ form a graded modular operad.

1 Modular operads

Def Graph is the category with:

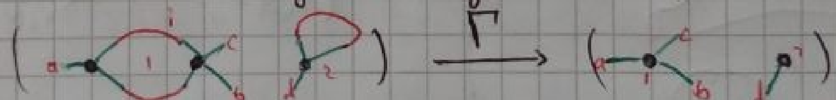
objects: "aggregates" $(H \xrightarrow{r} V)$ $\quad \begin{matrix} H \\ \downarrow \\ V \end{matrix}$

morphisms: "graphs" $\Gamma : \binom{H}{V} \longrightarrow \binom{H'}{V'}$

i.e. an involution $i: H \to H$ and bijections

$$\begin{array}{ccc} \{h\in H \mid i(h)=h\} & \xrightarrow{\cong} & H' \\ \downarrow & & \downarrow r \\ V/\sim & \xrightarrow{\cong} & V' \end{array}$$

where $\sim$ on V is generated by $r(h) \sim r(i(h))$

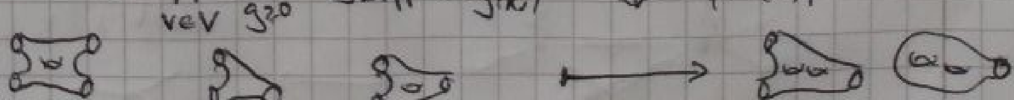


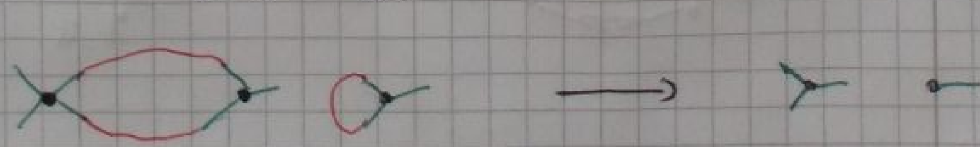
Warning: In reality I also need morphisms $\emptyset \longrightarrow \cancel{\emptyset}$, but they're tricky to define...

Def A modular operad is a sym. mon. functor $\mathcal{O}$: Graph $\longrightarrow \mathcal{S}$ = spaces = Top/sSet

Ex $\mathcal{M}\binom{H}{V}$ = $\left\{ \text{moduli of surfaces } W \text{ with } \begin{array}{ccc} \pi_0(\partial W) & \cong & H \\ \partial\downarrow & & \downarrow r \\ \pi_0(W) & \cong & V \end{array} \right\}$

$= \prod_{v\in V} \coprod_{g\geq 0} B\mathrm{Diff}^{\partial}(\Sigma_{g,k_v}) \quad k_v = |r^{-1}(v)|$





This is in fact a graded modular operad: $\mathcal{M}(k) := \mathcal{M}\left(\begin{smallmatrix}\{1,\dots,k\}\\ \downarrow \\ *\end{smallmatrix}\right) = \coprod_{g\geq 0} \mathcal{M}(g,k)$

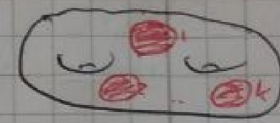
Def Graded modular operads are s.m. functors $\mathcal{O}:$ grGraph $\to \mathcal{S}$

where objects in grGraph are $\left(\begin{smallmatrix}H\\ \downarrow\\ V\end{smallmatrix}\right)$ with labelling $g: V \to \mathbb{N}$

and morphisms add genus.

Ex There also is the handlebody modular operad with

$$\mathcal{H}(g,k) = B\mathrm{Diff}\left((S^1 \times D^2)^{\#g} \text{ rel } \coprod_k D^2\right)$$



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2 Envelopes of modular operads

There is a construction

$$\mathrm{Env}: \mathrm{ModOp} \longrightarrow \mathrm{Cat}_\infty^{\otimes} = \begin{array}{c}\text{sym. mon.}\\ \text{top. enriched categories}\end{array}$$

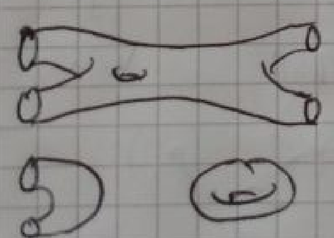
$$\text{s.t. } \mathcal{M} \longmapsto \mathrm{Bord}_2^{or} \text{ the surface category}$$

Thm (Galatius-Madsen-Tillmann-Weiss) $B(\mathrm{Bord}_2^{or}) \simeq \underbrace{\Omega^{\infty-1} MTSO_2}_{\text{"Pontryagin-Thom"}}$

Construction:

$\mathrm{Env}(\mathcal{O})$ has as objects finite sets $X, Y, \dots$ and as morphisms

$\mathcal{O}$-colored cospans i.e. $\left(X \to A \leftarrow Y,\ \alpha \in \mathcal{O}\left(\begin{smallmatrix}X \sqcup Y\\ \downarrow\\ A\end{smallmatrix}\right)\right)$



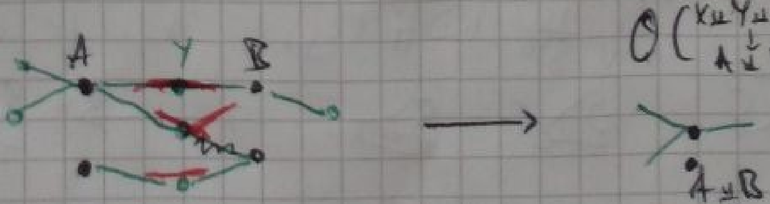
$\longmapsto \left(\pi_0 M \to \pi_0 W \leftarrow \pi_0 N,\ W \in \mathcal{M}\left(\begin{smallmatrix}\pi_0 M \sqcup \pi_0 N\\ \downarrow\\ \pi_0 W\end{smallmatrix}\right)\right)$

$M \xrightarrow{W} N$

Optional [Composition:

$$\begin{tikzcd}[column sep=small] & & A \sqcup_Y B & & \\ & A \arrow[ur] & & B \arrow[ul] & \\ X \arrow[ur] & & Y \arrow[ul] \arrow[ur] & & Z \arrow[ul] \end{tikzcd}$$

$$\begin{tikzcd} (\alpha,\beta) \in \mathcal{O}\left(\begin{smallmatrix}X \sqcup Y\\ \downarrow\\ A\end{smallmatrix}\right) \times \mathcal{O}\left(\begin{smallmatrix}Y \sqcup Z\\ \downarrow\\ B\end{smallmatrix}\right) \arrow[r] & \mathcal{O}\left(\begin{smallmatrix}X \sqcup Z\\ \downarrow\\ A \sqcup_Y B\end{smallmatrix}\right) \\ \mathcal{O}\left(\begin{smallmatrix}X \sqcup Y \sqcup Y \sqcup Z\\ \downarrow\\ A \sqcup B\end{smallmatrix}\right) \arrow[u, "\simeq"] \arrow[ur] & \end{tikzcd}$$



3 Restricting the genus

Defn A $(\leq g)$-graded modular operad is a s.m. functor $\mathcal{O}: \mathrm{grGraph}^{\leq g} \longrightarrow \mathcal{S}$ where $\mathrm{grGraph}^{\leq g} \subset \mathrm{grGraph}$ is the full subcategory on aggregates labelled in $\{0, \dots, g\}$.

There is an adjunction:

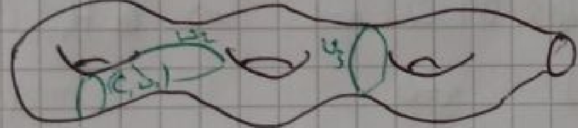
$$\begin{tikzcd} \mathrm{grModOp}^{\leq g} \arrow[r, bend left, "\mathrm{Ind}_g \text{ left Kan extend}"] \arrow[r, phantom, "\perp"] & \mathrm{grModOp} \arrow[l, bend left, "(-)_{\leq g} \text{ restrict}"] \end{tikzcd}$$

Define $\mathcal{O}^{(g)} := \mathrm{Ind}_g(\mathcal{O}_{\leq g}) \in \mathrm{grModOp}$ "reconstructing the modular operad from genus $\leq g$"

This gives: $\mathcal{O}^{(0)} \longrightarrow \mathcal{O}^{(1)} \longrightarrow \mathcal{O}^{(2)} \longrightarrow \dots \longrightarrow \mathcal{O}$ "filtration"

Ex $\mathcal{O} = \mathcal{M}$. To describe $\mathcal{M}^{(g)}$ define:

$$C^{(g)}(W) = \left\{ \begin{array}{l} \text{weighted systems of curves } \{(C_i, \omega_i)\}_{i \in I} \\ C_i \subset W \text{ simple closed curve}, \quad \omega_i \in [0,1] \\ \text{s.t. } (W \setminus \amalg C_i) \text{ has components of genus at most } g \end{array} \right\} \Big/ \sim \quad \begin{array}{l} \text{delete } (C_i, \omega_i) \\ \text{if } \omega_i = 0 \end{array}$$



$g = 1$ $\omega_1 = 0$ or $\omega_2 = 0$ or $\omega_3 = 0$ allowed, but not both

Prop $\mathcal{M}^{(g)}(g', k) \simeq C^{(g)}(\Sigma_{g',k}) /\!\!/ \mathrm{Diff}(\Sigma_{g',k})$

Note
- $C^{(g)}(\Sigma) \simeq *$ if Σ has genus $\leq g$, so $\mathcal{M}^{(g)}(g', k) \simeq \mathcal{M}(g', k)$ for $g' \leq g$
- $C^{(g-1)}(\Sigma_g)$ is the ordinary curve complex.

Thm (Giansiracusa) $\mathcal{M}^{(0)} \simeq \mathcal{H}$

4 The general spectral sequence

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For any graded modular operad $\mathcal{O}$ we have a filtration of infinite loop spaces

$$\mathbb{B}(\mathrm{Env}(\mathcal{O}^{(0)})) \longrightarrow \mathbb{B}(\mathrm{Env}(\mathcal{O}^{(1)})) \longrightarrow \dots \longrightarrow \mathbb{B}(\mathrm{Env}(\mathcal{O}))$$

always get a spectral sequence with E^1 = associated graded, converging to

$$\mathrm{BDiff}^{\partial}(D^2) \simeq \mathcal{O}(0,1)$$

<u>Prop</u> $\mathcal{O}: \mathrm{grGraph} \to \mathcal{S}$ a graded modular operad with $\mathcal{O}(0,1)$ contractible

then there is a pushout/pullback square of infinite loop spaces for all $g \geq 1$:

$$\begin{tikzcd}
\Omega^\infty \Sigma_+^{\infty+1}(\mathcal{O}^{(g-1)}(g,0)) \arrow[r] \arrow[d] & \Omega^\infty \Sigma_+^{\infty+1} \mathcal{O}(g,0) \arrow[d] \\
\mathrm{BEnd}(\mathcal{O}^{(g-1)}) \arrow[r] & \mathrm{BEnd}(\mathcal{O}^{(g)})
\end{tikzcd}$$

<u>Cor</u> $\exists$ first quadrant homological Serre grading spectral sequence

$$E^1_{g,k} = H_{k+g-1}(\mathcal{O}(g,0), \underbrace{\mathcal{O}^{(g-1)}(g,0)}_{\text{interpret as } * \text{ for } g=0}) \Rightarrow H^{Sp}_{g+k}(\mathrm{BEnd}(\mathcal{O}))$$

For $\mathcal{O} = \mathcal{M}$ have $\mathrm{BDiff}(\Sigma_g)$ rel $C(\Sigma_g)/\!/\mathrm{Diff}(\Sigma_g)$

<u>Thm</u> (Ivanov '87) For $g \geq 2$ $C(\Sigma_g)$ is a virtual dualizing complex of dimension $6g-6$ for Γ_g. Moreover its a bouquet of spheres of dim $2g-2$

So $H_i(B\Gamma_g, C(\Sigma_g)/\Gamma_g; \mathbb{Q}) \cong H^{6g-6-i}(\Gamma_g; \mathbb{Q})$

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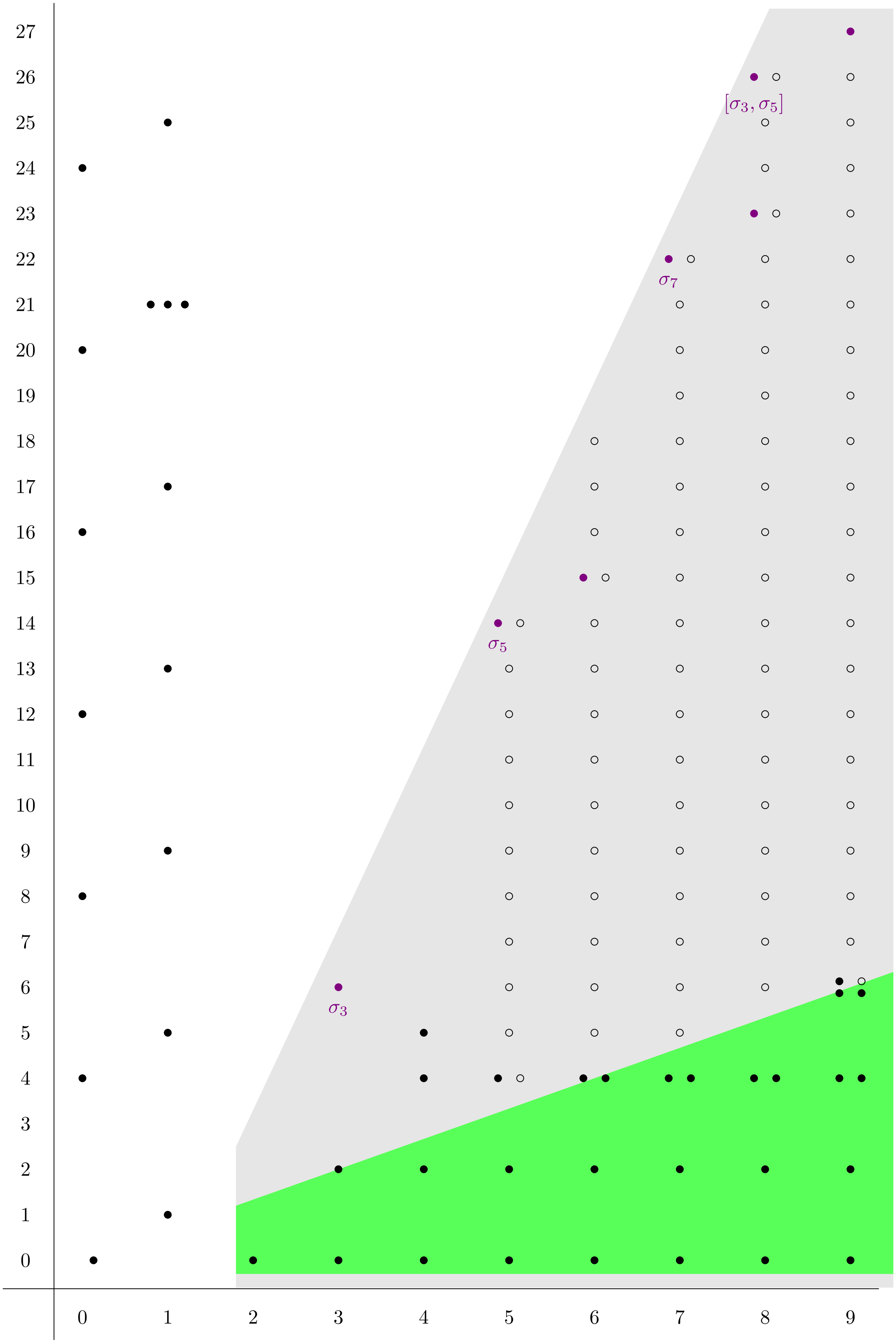
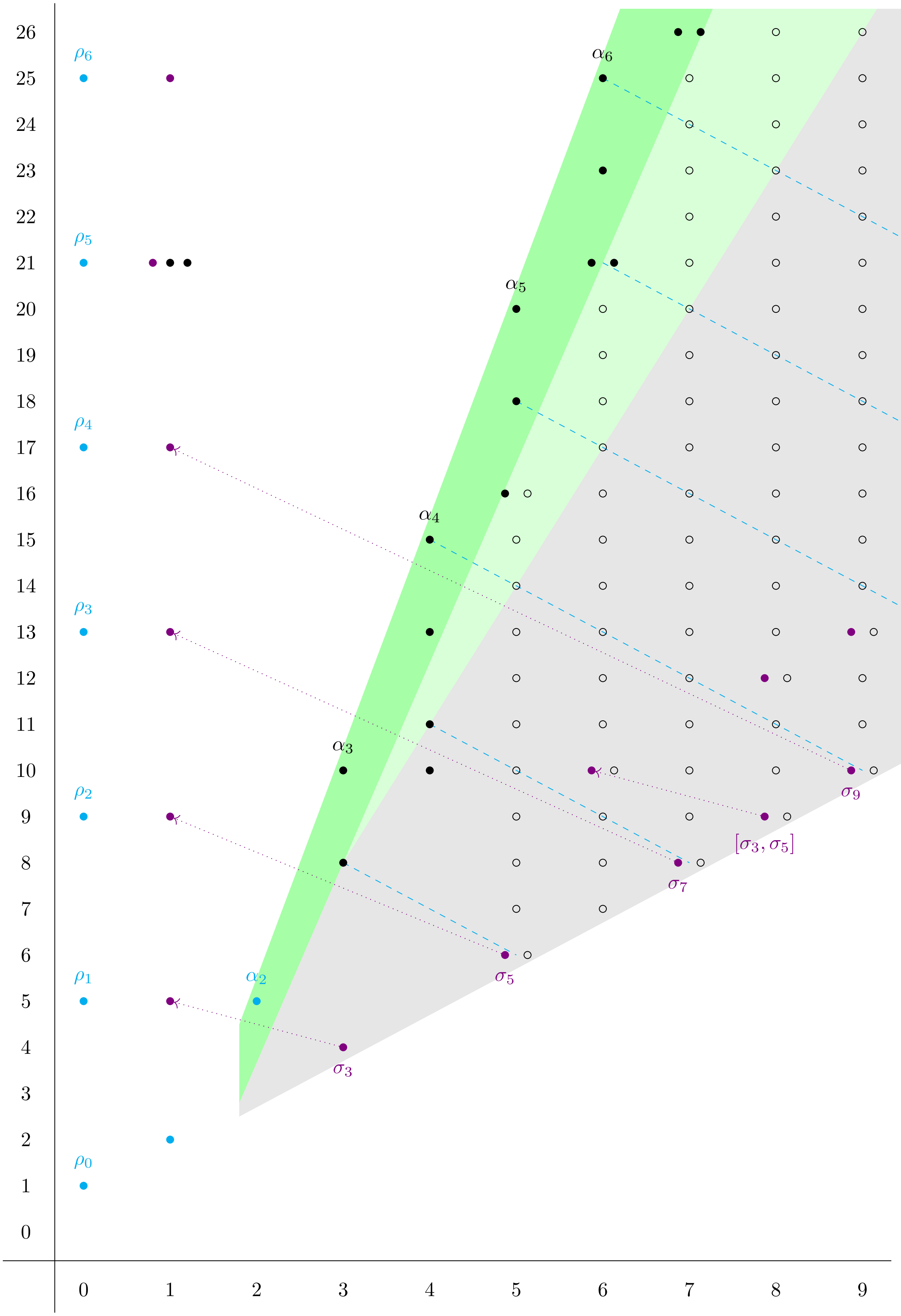
<u>Thm</u> (S. in progress) There is a spectral sequence

$$E^1_{g,k} = \begin{cases} H^{5g-5-k}(\Gamma_g; \mathbb{Q}) & g \geq 2 \\ H_k(\mathrm{BDiff}(S^1 \times D^2); \mathrm{BDiff}(S^1 \times D^2); \mathbb{Q}) & g=1 \\ H_{k-1}(\mathrm{BDiff}(S^2); \mathbb{Q}) & g=0 \end{cases} \Rightarrow H^{Sp}_{g+k}(\Sigma\, MTSO_2; \mathbb{Q}) = \begin{cases} \mathbb{Q} & g+k \text{ odd} \\ 0 & \text{even} \end{cases}$$

Known values of $H_k(B\mathrm{Diff}(\Sigma_g))$



The spectral sequence $E^1_{g,k} = H^{5g-5-k}(B\Gamma_g) \Rightarrow H^{\mathrm{Sp}}_{g+k}(\tau_{\geq 0}\Sigma MTSO_2)$



Proof that $H^{14}(B\Gamma_5) \cong \mathbb{Q}$ **or** $\mathbb{Q}^2$