Stable homology of mapping class groups with some twisted contravariant coefficients

(Joint work with Nariya Kawazumi)

Moduli and Friends seminar

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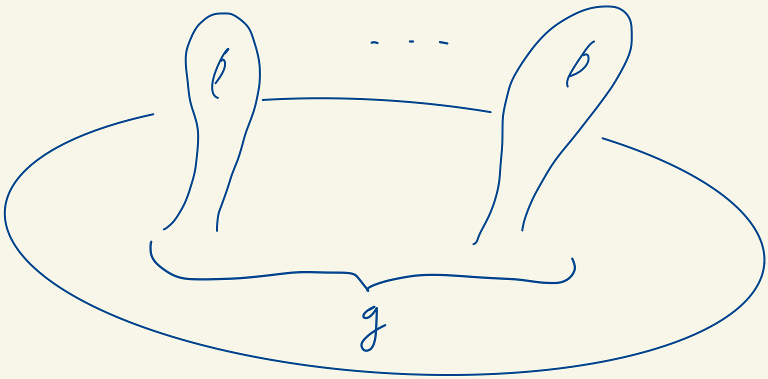
Context and motivations

- **Mapping class groups of surfaces:**

$\Sigma_{g,1} :=$ 

$$\Gamma_{g,1} := \pi_0 \operatorname{Homeo}(\Sigma_{g,1}, \partial\Sigma_{g,1}) \cong \pi_0 \operatorname{Diff}(\Sigma_{g,1}, \partial\Sigma_{g,1}).$$

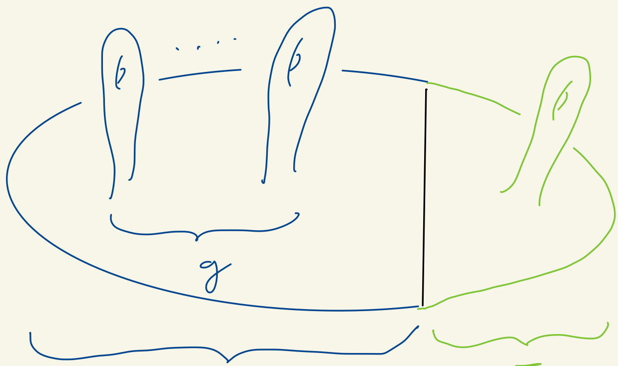
Key fundamental information:

$H_*(\Gamma_{g,1}; M) = ?$ $\rightsquigarrow$ Hard in general!

for M a $\Gamma_{g,1}$-module

- Homological stability:

(1)



$$\Sigma_{g+1,1} \simeq \Sigma_{g,1} \sqcup \Sigma_{1,1}$$

$$\Sigma_{g,1} \supset \Gamma_{g,1}$$

$$\Gamma_{g,1} \xhookrightarrow{\gamma_g} \Gamma_{g+1,1}$$

$$\varphi \longmapsto \varphi \sqcup \mathrm{id}_{\Sigma_{1,1}}$$

Sequence of groups:

$$\{0\} = \Gamma_{0,1} \xhookrightarrow{\gamma_0} \Gamma_{1,1} \xhookrightarrow{\gamma_1} \cdots \xhookrightarrow{\gamma_{g-1}} \Gamma_{g,1} \xhookrightarrow{\gamma_g} \Gamma_{g+1,1} \xhookrightarrow{\gamma_{g+1}} \cdots$$

(2) Sequence of $\Gamma_{g,1}$-modules $\{M_g\}_{g\in\mathbb{N}}$

with $\Gamma_{g,1}$-equivariant $M_g \xrightarrow{m_g} M_{g+1}$

$\longrightarrow$ Define: $H_i(\Gamma_{g,1}; M_g) \xrightarrow{\Psi_{i,g}} H_i(\Gamma_{g+1,1}; M_{g+1})$

③ THEN:

"Homological Stability"   functoriality + polynomiality.   "trivial"
Ex: $\forall g,\ M_g = \mathbb{Z}$

Property: Under some (technical) conditions on $\{M_g\}_{g\in\mathbb{N}}$

$$\Psi_{i,g} : H_i(\Gamma_{g,1}; M_g) \xrightarrow{\cong} H_i(\Gamma_{g+1,1}; M_{g+1})$$

is an isomorphism for $g \geq N(i, M_g) \in \mathbb{N}$.

Homological stability has been proven:

- [Harer; 1985]: for M_g trivial (and some non-trivial M_g).
- [Ivanov; 1991]: for general M_g.
- [Randal-Williams, Wahl; 2017]: optimal framework.

- Stable homology:

$$H_i(\Gamma_{\infty,1};M_\infty) := \operatorname{colim}_{g\in\mathbb{N}} \left(H_i(\Gamma_{g,1};M_g) \right)$$

If there is homological stability, this is the stable value.

ⓐ For constant coefficients:

- [Madsen, Weiss ; 2007]:

$$H^*(\Gamma_{\infty,1};\mathbb{Q}) \cong \mathbb{Q}[\{e_i, i\geq 1\}] =: \mathcal{E}$$

Mumford–Morita–Miller classes $e_i \in H^{2i}(\Gamma_{\infty,1};\mathbb{Q})$

- [Galatius ; 2004] computes $H_*(\Gamma_{\infty,1};\mathbb{F}_p)$ for p a prime number.

(b) For twisted coefficients:

- Symplectic representation:

$$H^1(\Sigma_{g,1};\mathbb{Z}) \underset{\text{as } \Gamma_{g,1}\text{-modules.}}{\cong}$$

Natural action $\Gamma_{g,1} \curvearrowright H_1(\Sigma_{g,1};\mathbb{Z})$

preserves the algebraic intersection pairing (which is a symplectic form by P.D.), inducing

$$\begin{tikzcd}
\Gamma_{g,1} \arrow[rr, two heads] \arrow[dr] & & Sp_{2g}(\mathbb{Z}) \\
& \mathrm{Aut}(H_1(\Sigma_{g,1};\mathbb{Z})) \arrow[ur, two heads] &
\end{tikzcd}$$

- [Kawazumi; 2008] computes for all $d \geq 1$:

$$H_*(\Gamma_{\infty,1};\Lambda^d H_1(\Sigma_{g,1};\mathbb{Q}))$$

$$H_*(\Gamma_{\infty,1};T^d H_1(\Sigma_{g,1};\mathbb{Q})) \qquad T^d H_1(\Sigma_{g,1};\mathbb{Q}) = H_1(\Sigma_{g,1};\mathbb{Q})^{\otimes d}.$$

See also [Ebert, Randal-Williams; 2012] for generalisations to $\Gamma_{g,r}^s$ with $r+s \geq 1$ and coefficients $\Lambda^d(H_1(\Sigma_{g,r}^s;\mathbb{Q}))$. $\quad r+s \geq 1.$

- [Looijenga; 1996] computes for closed surfaces Γ_g.

$$H^*(\Gamma_\infty;T^d H_1(\Sigma_g;\mathbb{Q})),\ H^*(\Gamma_\infty;\Lambda^d H_1(\Sigma_g;\mathbb{Q})) \ \ldots$$

Let $UT\Sigma_{g,1}$ be the unit tangent bundle of $\Sigma_{g,1}$

$$\Gamma_{g,1} \curvearrowright H_1(UT\Sigma_{g,1};\mathbb{Z}) \quad \text{[Trapp; 1992].}$$

$$\not\cong$$

$$\Gamma_{g,1} \curvearrowright H^1(UT\Sigma_{g,1};\mathbb{Z})$$

Aim: Compute $H_*(\Gamma_{\infty,1}; M_\infty)$

with M_∞ given by $H_1(UT\Sigma_{\infty,1};\mathbb{Q})$; $H^1(UT\Sigma_{\infty,1};\mathbb{Q})$

$\Lambda^d H_1(UT\Sigma_{\infty,1};\mathbb{Q})$; $\Lambda^d H^1(UT\Sigma_{\infty,1};\mathbb{Q})$

$\vdots \qquad \vdots$

Remark: Some of these are outside the classical framework for homological stability.

$\boxed{\text{PLAN}}$

I) Backgrounds

II) Stable cohomology with coeff in $H_1(\mathcal{U}T\Sigma_{g,1})$

III) Stable cohomology with coeff in $\Lambda^d H_1(\mathcal{U}T\Sigma_{g,1})$

IV) Further cases and perspectives

Rmk: Ground ring is $\mathbb{Q}$ (although some things remain true over $\mathbb{Z}$).

I) Backgrounds

① Unit tangent bundle representations

- Notation : $H := H_1(\Sigma_{g,1})$

Locally trivial fibration :

$$S^1 \longrightarrow UT\Sigma_{g,1} \longrightarrow \Sigma_{g,1} .$$

provides the SES :

$$0 \longrightarrow \mathbb{Q} \longrightarrow \underset{\substack{\| \\ \tilde{H}}}{H_1(UT\Sigma_{g,1})} \longrightarrow H \longrightarrow 0 .$$

As a $\mathbb{Q}$-module, $\tilde{H} \cong \mathbb{Q} \oplus H \cong \mathbb{Q}^{2g+1}$.

But not as a $\Gamma_{g,1}$-module ;

$$\forall \varphi \in \Gamma_{g,1}, \quad \varphi \cdot \tilde{H} = \begin{array}{cc} \begin{array}{cc} \mathbb{Q} & H \end{array} & \\ \left[\begin{array}{c|c} 1 & * \neq 0 \\ \hline 0 & \underset{\text{Symplectic rep.}}{\varphi \cdot H} \end{array}\right] & \begin{array}{c} \mathbb{Q} \\ H \end{array} \end{array}$$

- <u>Notation</u>: $M^\vee := \mathrm{Hom}_{\mathbb{Q}}(M, \mathbb{Q})$.

SES:

$$0 \longrightarrow H \longrightarrow \underset{\substack{\cong \\ H^1(UT\Sigma_{g,1})}}{\tilde{H}^\vee} \longrightarrow \mathbb{Q} \longrightarrow 0.$$

<u>NB:</u> As $\Gamma_{g,1}$-modules, $H \cong H^\vee$ and $\tilde{H}^\vee \not\cong \tilde{H}$.

$\Gamma_{g,1} \curvearrowright H$ symplectic

but $\cong$ as $\mathbb{Q}$-mod

② Cohomology classes

- **Earle class**: [Earle ; 1978] ; [Morita ; 1989] ; [Trapp ; 1992]

$g \geq 2$: $H^1(\Gamma_{g,1} ; \overbrace{H_1(\Sigma_{g,1} ; \mathbb{Z})}^{H}) \cong \mathbb{Z} =: \langle m_{1,1} \rangle$

Earle class.

NB: $\Gamma_{g,1} \curvearrowright \tilde{H}$ induced by $m_{1,1}$.

$[\bar{m}_{1,1}] = m_{1,1}$

$$\tilde{H} = \left[\begin{array}{c|c} 1 & \bar{m}_{1,1} \\ \hline 0 & H \end{array}\right]$$

a.k.a Trapp representation.

- **Twisted Mumford–Morita–Miller**: [Kawazumi ; 1998]

$i \geq 0$

$j \geq 0$

$2i+j \geq 2$

$$m_{i,j} \in H^{2i+j-2}(\Gamma_{\infty,1} ; \Lambda^j H_\infty)$$

$|e_i| = 2i$

Ex: $m_{i+1,0} = e_i$ ($\mathcal{E} \cong \mathbb{Q}[\{e_i ; i \geq 1\}]$)

concentrated in even degrees

$m_{1,1}$: the Earle class

③ Categorical framework and functor homology

Consider a family of groups:

$$\Gamma_{0,1} \overset{\gamma_0}{\hookrightarrow} \Gamma_{1,1} \overset{\gamma_1}{\hookrightarrow} \cdots \hookrightarrow \Gamma_{g,1} \overset{\gamma_g}{\hookrightarrow} \cdots$$

Compatible representations:

$$\begin{tikzcd}
\Gamma_{g,1} \arrow[r, "\rho_g"] \arrow[d, hook, "\gamma_g"'] & \mathrm{Aut}_R(V_g) \arrow[dl, Rightarrow, "v_g"] \\
\Gamma_{g+1,1} \arrow[r, "\rho_{g+1}"] & \mathrm{Aut}_R(V_{g+1})
\end{tikzcd}$$

$$\forall \varphi \in \Gamma_{g,1} \qquad \begin{tikzcd}
V_g \arrow[r, "\exists v_g"] \arrow[d, "\rho_g(\varphi)"'] \arrow[dr, phantom, "\circlearrowright"] & V_{g+1} \arrow[d, "\rho_{g+1}(\gamma_g(\varphi))"] \\
V_g \arrow[r, "\exists v_g"] & V_{g+1}
\end{tikzcd}$$

Question: How can we encode these compatibilities?

$\rightsquigarrow$ Use categories and functors.

- The category $\mathcal{M}_2$:

  - Objects: $\Sigma_{g,1}$, $g \geq 0$.

  - Morphisms: $\mathrm{Hom}_{\mathcal{M}_2}(\Sigma_{g,1}, \Sigma_{h,1}) = \begin{cases} \Gamma_{g,1} & \text{if } g = h \\ \emptyset & \text{if } g \neq h \end{cases}$

  - Braided monoidal structure: $(\mathcal{M}_2, \natural, \Sigma_{0,1})$

    monoidal: boundary connected sum on objects and induced maps on morphisms

    Braided: Isomorphisms

$$b^{\mathcal{M}_2}_{n,m} : \Sigma_{n,1} \natural \Sigma_{m,1} \longrightarrow \Sigma_{m,1} \natural \Sigma_{n,1}$$

Notation: • $\mathcal{C}$ a small category

• R_Mod category of $R_$modules

• $Fct(\mathcal{C}, R\_Mod)$ category of functors $\mathcal{C} \to R\text{-}Mod$.

Observation: $\left(\text{An object of } Fct(\mathcal{M}_2, \mathbb{Q}\text{-}Mod) \right) \Longleftrightarrow \left(\text{A collection of "independent" representations } \{\Gamma_{g,1} \to Aut_{\mathbb{Q}}(M_g)\}_{g \in \mathbb{N}} \right)$

No compatibilities $\gamma_g : \Gamma_{g,1} \longrightarrow \Gamma_{g+1,1}$

Quillen's construction $\mathcal{U}\mathcal{M}_2$:

* <u>Objects</u>: $Obj(\mathcal{U}\mathcal{M}_2) = Obj(\mathcal{M}_2)$

* <u>Morphisms</u>: $Hom_{\mathcal{U}\mathcal{M}_2}(\Sigma_{g,1}, \Sigma_{h,1}) = \begin{cases} \emptyset \text{ if } h < g \\ \Gamma_{h-g,1} \backslash \Gamma_{h,1} \text{ if } h \geq g \end{cases}$

precomposition by

$\Gamma_{h-g,1} \curvearrowright \Sigma_{h-g,1} \natural \Sigma_{g,1}$

$\cong$

$\Sigma_{h,1}$

<u>Solution</u>: $\left(\begin{array}{l} \text{An object of} \\ Fct(\mathcal{U}\mathcal{M}_2, \mathbb{Q}\text{-Mod}) \end{array} \right) \Longleftrightarrow \left(\begin{array}{l} \text{representations } \{\Gamma_{g,1} \to Aut_{\mathbb{Q}}(M_g)\}_{g \in \mathbb{N}} \\ \text{with } \Gamma_{g,1}\text{-equivariant } M_g \xrightarrow{m_g} M_{g+1} \end{array} \right)$

- <u>Homological stability</u>:

[Ivanov; 1991]; [Randal-Williams, Wahl; 2017]

If a family of $\Gamma_{g,1}$-modules $\{M_g\}_{g\in\mathbb{N}}$ define a functor

$$M : \mathcal{UM}_2 \longrightarrow \mathbb{Q}\text{-Mod}$$

satisfying some polynomiality conditions, then:

$$H_i(\Gamma_{g,1}; M_g) \xrightarrow{\cong} H_i(\Gamma_{g+1,1}; M_{g+1})$$

for $g \geq N(g,d)$
└ polynomial degree.

<u>Ex:</u> ① $H = H_1(\Sigma_{g,1})$ define $H : \mathcal{UM}_2 \longrightarrow \mathbb{Q}\text{-Mod}$.

② $\tilde{H} = H_1(UT\Sigma_{g,1})$ define $\tilde{H} : \mathcal{UM}_2 \longrightarrow \mathbb{Q}\text{-Mod}$,

⚠ $\tilde{H}^\vee = H^1(UT\Sigma_{g,1})$ define $\tilde{H}^\vee : \mathcal{UM}_2^{op} \longrightarrow \mathbb{Q}\text{-Mod}$.

$$H \hookrightarrow \tilde{H} \twoheadrightarrow \mathbb{Q}.$$

- Stable homology: [Djament, Vespa; 2013] for $Aut(F_n)$, $\mathfrak{S}_n$.

Theorem: [S.; 2020] $M : \mathcal{UM}_2 \longrightarrow \mathbb{Q}\text{-Mod}$ functor

No assumption.

$$H_*(\Gamma_{\infty,1}; M_\infty) \underset{\text{graded}}{\cong} \underbrace{H_*(\Gamma_{\infty,1}; \mathbb{Q})}_{\mathcal{E}} \otimes_{\mathbb{Q}} H_*(\mathcal{UM}_2; M)$$

$\mathcal{E}$ — [Madsen-Weiss].

Corollary: For any $M : \mathcal{UM}_2 \to \mathbb{Q}\text{Mod}$,

$H_*(\Gamma_{\infty,1}; M_\infty)$ is a free $\mathcal{E}$-module.

* [Kawazumi; 2008]: $d \geq 1$

Def: Let $\mathcal{Q}_d$ be the set of weighted partitions of d.

$\{(i_1, j_1), \ldots, (i_\nu, j_\nu)\}$ where
- $i_1, \ldots, i_\nu \in \mathbb{N}$
- $j_1 \geq j_2 \geq \cdots \geq j_\nu \geq 0$ s.t. $j_1 + \cdots + j_\nu = d$.
- $i_a + j_a \geq 2$ and $i_a \geq i_{a+1}$ if $j_a = j_{a+1}$.

$$H_*(\Gamma_{\infty,1}; \Lambda^d H_\infty) \cong \bigoplus_{Q \in \mathcal{Q}_d} \mathcal{E}\, m_Q$$

$\rightsquigarrow$ concentrated in degrees of the same parity as d.

II) Stable cohomology with coeff in $\tilde{H}$ and $\tilde{H}^\vee$

Preliminaries:

- Universal coefficient theorem: M a $\mathbb{Q}[\Gamma_{g,1}]$-module. $M^\vee$ its dual $\mathrm{Hom}_R(M, R)$.

Then $H^*(\Gamma_{g,1}; M^\vee) \cong H_*(\Gamma_{g,1}; M)$.

work with cohomology: use cup product structure.
$\rightsquigarrow$ dual results!

Notation: $H^i(M) := H^i(\Gamma_{\infty,1}; M_\infty)$

- If $H^\bullet(M)$ has a $\mathcal{E}$-module structure, then there is a natural $\mathcal{E}$-module decomposition.

$$H^\bullet(M) \cong H^{even}(M) \oplus H^{odd}(M)$$

$\mathrm{Tor}_0^{\mathcal{E}}(\mathbb{Q}; H^\bullet(M))$ measures the number of generators of $H^\bullet(M)$
$\mathrm{Tor}_j^{\mathcal{E}}(\mathbb{Q}; H^\bullet(M))$, $j>1$ measures the number of the relations in $H^\bullet(M)$.
complexity

① $\tilde{H}^\vee$ coefficient:

$$0 \longrightarrow H \longrightarrow \tilde{H}^\vee \longrightarrow Q \longrightarrow 0.$$

Cohomology LES:

$$\cdots \to \underbrace{\cancel{H^{2i}(H)}}_{=0} \to H^{2i}(\tilde{H}^\vee) \to \overbrace{H^{2i}(Q)}^{\text{known [MW].}} \xrightarrow{\delta^{2i}} \underbrace{H^{2i+1}(H)}_{\text{known [K]}} \to H^{2i+1}(\tilde{H}^\vee) \to \underbrace{\cancel{H^{2i+1}(Q)}}_{=0} \to \cdots$$

Prop: [Kawazumi, S.] $\delta^{2i} = m_{1,1} \cdot -$

Corollary: $H^*(\tilde{H}^\vee) \cong H^{odd}(\tilde{H}^\vee)$
$\cong \xi\{m_{a,1} \,;\, a \geq 2\}.$

② $\widetilde{H}$ coefficient:

SES: $0 \longrightarrow \mathbb{Q} \longrightarrow H_1(UT\Sigma_{g,1}) \longrightarrow H \longrightarrow 0$.

Cohomology LES:

$$\cdots \longrightarrow \underbrace{\cancel{H^{2i+1}(\mathbb{Q})}}_{=0} \longrightarrow H^{2i+1}(\widetilde{H}) \longrightarrow \overbrace{H^{2i+1}(H)}^{\text{Known}} \xrightarrow{\delta^{2i+1}} \underbrace{H^{2i+2}(\mathbb{Q})}_{\text{Known}} \longrightarrow H^{2i+2}(\widetilde{H}) \longrightarrow \underbrace{\cancel{H^{2i}(H)}}_{=0} \longrightarrow \cdots$$

Prop: [Kawazumi, S.]

$$\delta^{2i+1} = \mu_*(m_{1,1}, -)$$

where $\mu : H \otimes H \longrightarrow \mathbb{Q}$

Ex: $\mu_*(m_{1,1}, m_{\alpha,1}) = -e_\alpha$ for $\alpha \geq 1$.

Def: $i, j \geq 1$, $M_{i,j} = e_i m_{j,1} - e_j m_{i,1}$

Theorem: [Kawazumi, S.]

$$H^*(\tilde{H}) \cong \mathbb{Q} \oplus \mathbb{Q}\{M_{i,j} \,;\, i,j \geq 1\} / \langle R_{ijk} \rangle$$

where $R_{ijk} = e_i M_{j,k} + e_j M_{k,i} + e_k M_{i,j}$.

III) Stable cohomology with coeff in $\Lambda^d \tilde{H}$ and $\Lambda^d \tilde{H}^\vee$

<u>Notation:</u> $p(d)$ = parity of d.

① $\Lambda^d \tilde{H}^\vee$ coefficient:

- $\Gamma_{g,1}$-module SES:

$$\text{(E2d)} \quad 0 \longrightarrow \Lambda^d H \longrightarrow \Lambda^d \tilde{H}^\vee \longrightarrow \Lambda^{d-1} H \longrightarrow 0.$$

- Cohomology LES: $\quad p(i) = p(d)$.

$$\cdots \longrightarrow H^{i-1}(\Lambda^d H) \longrightarrow H^{i-1}(\Lambda^d \tilde{H}^\vee) \longrightarrow H^{i-1}(\Lambda^{d-1} H) \xrightarrow{\delta^{i+1}} H^i(\Lambda^d H) \longrightarrow H^i(\Lambda^d \tilde{H}^\vee) \longrightarrow H^i(\Lambda^{d-1} H) \longrightarrow \cdots$$

<u>Prop:</u> [Kawazumi, S.] $\delta^{i+1} = (id_{\Lambda^{d-1}\tilde{H}^\vee} \wedge m_{1,1})$.

<u>Corollary:</u> $H^*(\Lambda^d \tilde{H}^\vee) \simeq H^{p(d)}(\Lambda^d \tilde{H}^\vee)$

$$\simeq \mathcal{E}\{m_{i,j};\ i+j \geq 3\}.$$

② $\Lambda^d \tilde{H}$ coefficient:

- $\Gamma_{g,1}$ - module SES:

$$0 \longrightarrow \Lambda^{d-1} H \longrightarrow \Lambda^d \tilde{H} \longrightarrow \Lambda^d H \longrightarrow 0 .$$

- Cohomology LES:

$p(i) := p(d)$

$$\cdots \longrightarrow H^i(\Lambda^{d-1} H) \longrightarrow H^i(\Lambda^d \tilde{H}) \longrightarrow H^i(\Lambda^d H) \xrightarrow{\delta^i}$$

$$\longrightarrow H^{i+1}(\Lambda^{d-1} H) \longrightarrow H^{i+1}(\Lambda^d \tilde{H}) \longrightarrow H^{i+1}(\Lambda^d H) \longrightarrow \cdots$$

Prop: [Kawazumi, S.]

$$\delta^i = \left(id_{\Lambda^{d-1} H} \wedge \mu(m_{1,1}, -)\right)_* =: \mu_d(m_{1,1}, -)$$

Ex: $\mu_d(m_{1,1}, m_{i,j}) = -j\, m_{i+1, j-1}$ for $i, j \geq 1$

$\boxed{d=2}$

$$\mu_2(m_{1,1}, -) : m_{a,2} \longmapsto -2\, m_{a+1,1}$$
$$m_{i,1} \cdot m_{j,1} \longmapsto -e_1 m_{j,1} - e_j \cdot m_{i,1}$$

Section: $m_{j,1} \mapsto -\frac{1}{2} m_{j-1,2}$ for $j \geq 2$.

Theorem: [Kawazumi, S.]

$$H^*(\Lambda^2 \tilde{H}) \simeq \mathcal{E}\{ m_{i,1} \cdot m_{j,1} - \tfrac{1}{2}(e_j \cdot m_{i,2} + e_i \cdot m_{j,2}) ;\ i,j \geq 1 \}.$$

$d \geq 3$

Theorem: [Kawazumi, S.]

The $\mathcal{E}$-module $H^*(\Lambda^d \tilde{H})$ is not free for $d \geq 3$.

Proof:

$$\begin{tikzcd}
H^{p(d)}(\Lambda^d \tilde{H}) \arrow[r, hook] & H^{p(d)}(\Lambda^d H) \arrow[d, "\mu_d(m_{1,1}, -)"] & \\
& H^{p(d)+1}(\Lambda^{d-1} H) \arrow[r, two heads] & H^{p(d)+1}(\Lambda^d \tilde{H})
\end{tikzcd}$$

The class $m_{0,d-1} \in H^{p(d)+1}(\Lambda^d \tilde{H})$

But $\exists \alpha$ s.t $e_\alpha m_{0,d-1} = 0$ in $H^{p(d)+1}(\Lambda^d \tilde{H})$.

$$d = 3$$

$$m_{a,3} \mapsto -3\, m_{a+1,2}$$

$$m_{a,2} \cdot m_{b,1} \mapsto -e_b\, m_{a,2} - 2\, m_{a+1,1} \cdot m_{b,1}$$

$$m_{a,1} \cdot m_{b,1} \cdot m_{c,1} \mapsto -e_a\, m_{b,1} \cdot m_{c,1} - e_b\, m_{a,1} \cdot m_{c,1} - e_c\, m_{a,1} \cdot m_{b,1}.$$

$$E := \bigoplus_i \mathbb{Q} e_i \; ; \quad E_\ell := E \Big/ \bigoplus_{i<\ell} \mathbb{Q} e_i .$$

Proposition: [Kawazumi, S.]

- $H^{even}(\Lambda^3 \tilde{H}) \cong \mathcal{E} \big/ (e_1^2, e_\alpha ; \alpha \geq 2) \{ m_{0,2} \}$.
- $\mathrm{Tor}_j^{\mathcal{E}}(\mathbb{Q}, H^*(\Lambda^3 \tilde{H})) \cong \Lambda^{j-1} E_2 \oplus \Lambda^j E_2 \oplus \Lambda^{j+1} E_2 \oplus \Lambda^{j+1} E_2$
  $j > 1$
- $\mathrm{Tor}_j^{\mathcal{E}}(\mathbb{Q}, H^*(\Lambda^3 \tilde{H})) \cong E \oplus \Lambda^2 E_2 \oplus \mathbb{Q}\{ [m_{\alpha-1,2}\, m_{\beta,1} - m_{\beta-1,2}\, m_{\alpha,1}] ; [m_{1,1}\, m_{1,1}\, m_{\gamma,1}] \}$.

$d = 4$: same complexity as $d = 3$.

Proposition: [Kawazumi, S.]

- $H^{odd}(\Lambda^4 \tilde{H}) \cong \mathcal{E} / (e_1^3, e_\alpha ; \alpha \geq 2)\{m_{0,3}\}$.

- $$\mathrm{Tor}_j^{\mathcal{E}}(\mathbb{Q}, H^*(\Lambda^4 \tilde{H})) \cong \Lambda^{j-1} F_2 \oplus \Lambda^j F_2 \oplus \Lambda^{j+1} F_2 \oplus \Lambda^{j+2} F_2.$$

- $$\mathrm{Tor}_0^{\mathcal{E}}(\mathbb{Q}, H^*(\Lambda^4 \tilde{H})) \cong F_2 \oplus \Lambda^2 F_2$$

$$\oplus\, \mathbb{Q}\{[m_{\alpha,2} m_{\beta,2} - m_{\alpha-1,3} m_{\beta+1,1} - m_{\beta-1,3} m_{\alpha+1,1}], [e_8 m_{1,1} m_{1,1} m_{1,1}]\}.$$

$d=5$: much more difficult.

Proposition: [Kawazumi, S.]

- $H^{even}(\Lambda^5 \tilde{H}) \cong \mathcal{E}/(e_1^3, e_2^2, e_3^2, e_\alpha e_\beta e_\gamma ; \alpha, \beta \geq 1, \gamma \geq 4)\{m_{0,4}\}$

$$\oplus\ \mathcal{E}/(e_1^2, e_2^2, e_3^2, e_\alpha ; \alpha \geq 2)\{m_{0,3}\, m_{0,1}\}$$

$$\oplus\ \mathcal{E}/(e_1^3, e_\alpha ; \alpha \geq 2)\{m_{0,3}\, m_{2,1}\}$$

$$\oplus\ \mathcal{E}/(e_1^2, e_2^2, e_3^2, e_\alpha ; \alpha \geq 2)\{m_{0,2}\, m_{0,2}\}$$

- $\mathrm{Tor}_j^{\mathcal{E}}(\mathbb{Q}, H^*(\Lambda^5 \tilde{H})) \cong$

$$(\Lambda^{j-1} E_2 \oplus \Lambda^j E_2)^{\oplus 3} \oplus \Lambda^{j-1} E_4 \oplus \Lambda^j E_4$$

$$\oplus\ (\Lambda^{j+1} E_2 \oplus \Lambda^{j+2} E_2)^{\oplus 3} \oplus \Lambda^{j+1} E_4 \oplus \Lambda^{j+2} E_4 .$$

IV) Further cases and perspectives

① Tensor powers:

- Twisted Mumford-Morita-Miller: [Kawazumi ; 1998]

$i \geq 0$

S set

$$m_{i,S} \in H^{2i+|S|-2}(\Gamma_{\infty,1} ; T^{|S|} H_\infty)$$

* [Kawazumi ; 2008]

$$H^*(\Gamma_{\infty,1} ; T^d H_\infty) \simeq \mathcal{C}\{m_{i,S} ; S \in \mathcal{P}_d\}$$

set of weighted partitions of $\{1, \dots, d\}$.

- Compute $H^*(\Gamma_{\infty,1}; T^d \tilde{H}_\infty)$ and $H^*(\Gamma_{\infty,1}; T^d \tilde{H}^\vee_\infty)$ via d SES's of type

(a)

$$T^{d-k}H \otimes T^{k-1}\tilde{H} \hookrightarrow T^{d-k}H \otimes T^{k}\tilde{H} \twoheadrightarrow T^{d-k}H \otimes T^{k-1}\tilde{H} \otimes H$$

(b)

$$T^{d-k}H \otimes T^{k-1}\tilde{H}^\vee \otimes H \hookrightarrow T^{d-k}H \otimes T^{k}\tilde{H}^\vee \twoheadrightarrow T^{d-k}H \otimes T^{k-1}\tilde{H}^\vee$$

Prop: [Kawazumi, S.]

The connecting morphisms are of the form:

(a) $\mathrm{id} \otimes \mu(m_{1,1}, -)$

(b) $(\mathrm{id} \otimes m_{1,1}) \otimes -$

② Coefficients induced by Schur functors:

$\lambda \vdash d$; $S^\lambda :=$ irreducible Specht module over $\mathbb{Q}[\mathfrak{S}_d]$.

$$S_\lambda := \mathrm{Hom}_{\mathfrak{S}_d}(S^\lambda, -).$$

Fact: $H^*(\Gamma_{\infty,1}; S_\lambda(\tilde{H})) \cong \mathrm{Hom}_{\mathfrak{S}_d}(S^\lambda, H^*(\Gamma_{\infty,1}; T^d\tilde{H}))$

Ex: • $\lambda = (d)$, $S_{(d)}(\tilde{H}) = \mathrm{Sym}^d(\tilde{H})$

$\rightsquigarrow$ multiplicity of the trivial rep.

• $\lambda = (1,\dots,1)$, $S_{(1,\dots,1)}(\tilde{H}) = \Lambda^d(\tilde{H})$.

$\rightsquigarrow$ multiplicity of the sign rep.