Genus bounds from quantum invariants

Daniel López Neumann

(joint w/ Roland van der Veen)

May 9 2022

Outline

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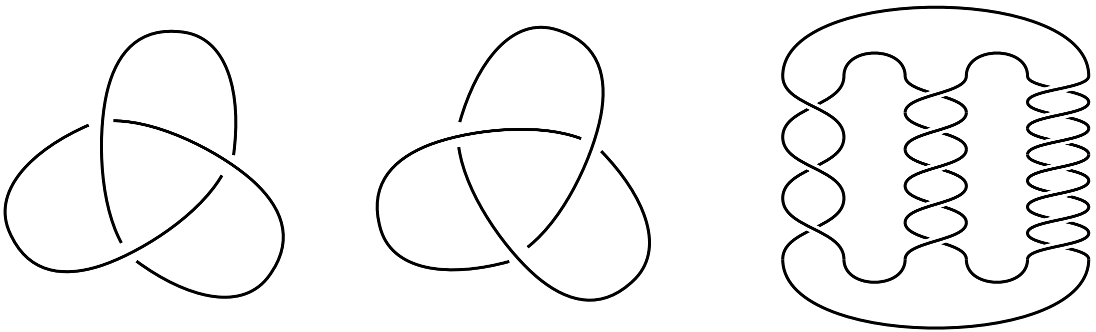

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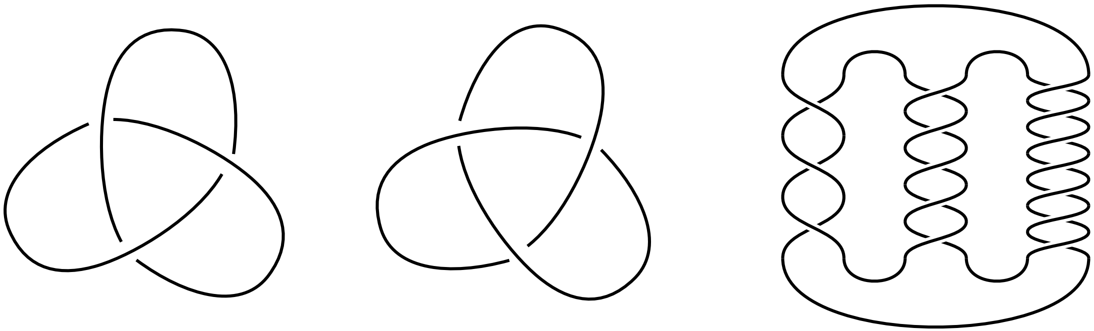

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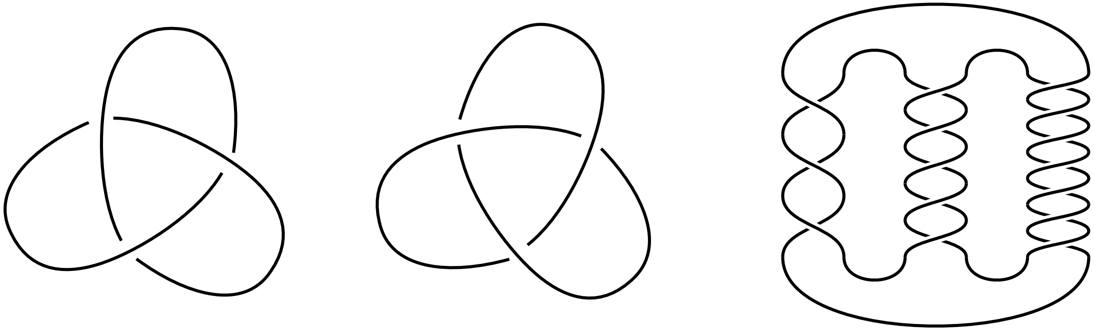

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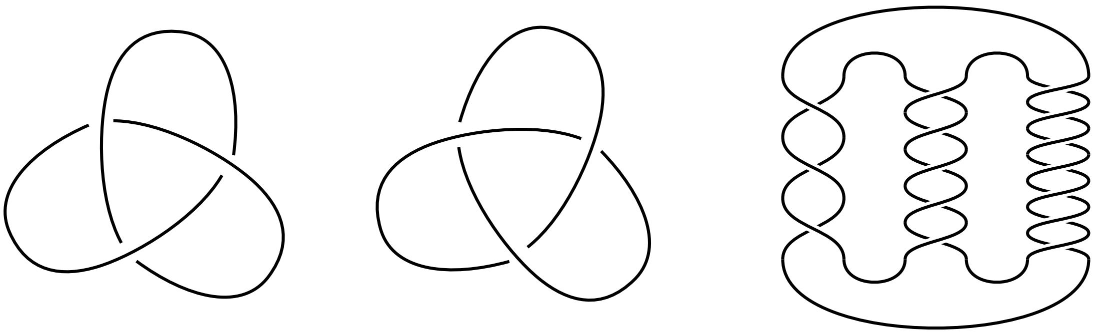

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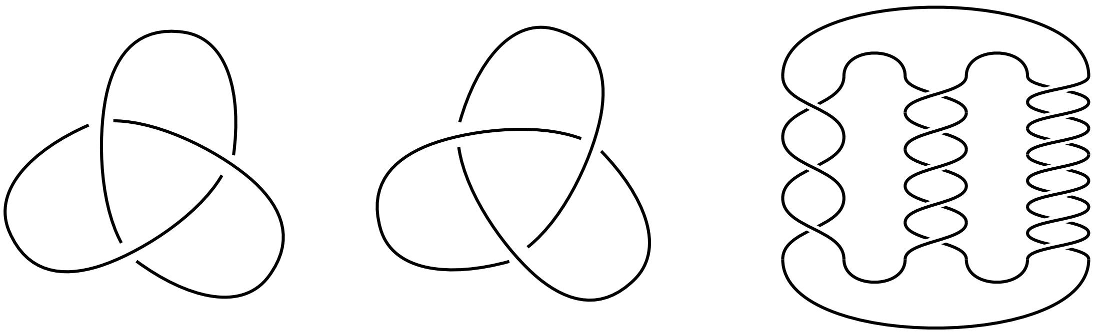

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Quantum invariants: physics, rep. theory (Jones, Witten, Reshetikhin-Turaev, 80s-90s). Jones polynomial $J_K(q) \in \mathbb{Z}[q^{\pm 1}]$, HOMFLY, etc.

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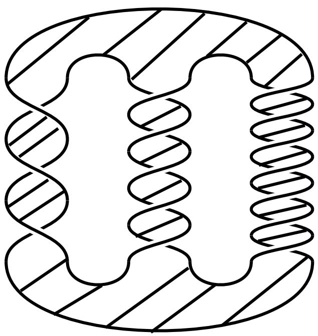

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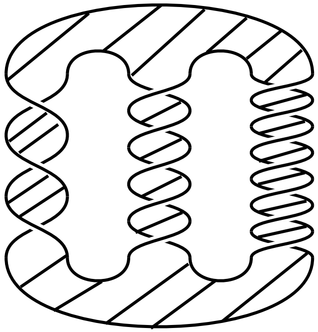

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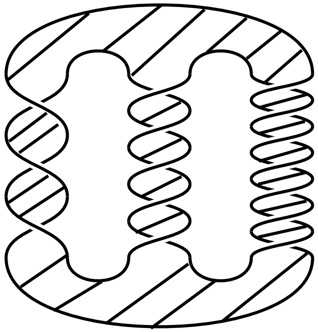

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Friedl-Kim, 2006: $\deg \frac{\tau^{\rho \otimes h}(X_K)}{n} \leq 2g - 1$, $\rho : \pi_1 \to GL(n, \mathbb{C})$.

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More precisely:

Theorem (LN-van der Veen, '22)

For any f.d. $\mathbb{Z}$-graded Hopf algebra H there is a polynomial invariant $P_H(K,t) \in \mathbb{C}[t^{\pm 1}]$ of knots $K \subset S^3$ such that

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For $H = \Lambda(\mathbb{C}^n)$ our thm recovers Friedl-Kim's (in a twisted version $P_H^{\rho}(K,t)$ of our knot polynomial, where $\rho : \pi_1(S^3 \setminus K) \to \mathrm{Aut}(H)$).

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This can be computed easily via **Fox calculus** given a presentation of π!

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$\pi_1 M_K \to H_1 = \mathbb{Z}$

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- In all the above cases, the target of ρ is infinite ($\mathbb{Z}$ for Alex. poly).

Hopf algebras

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- Algebra $(H, m, 1)$ over $\mathbb{K}$.
- Coproduct $\Delta : H \to H \otimes H$ and counit $\epsilon : H \to \mathbb{K}$
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3. If $q^{2p} = 1$, the quotient $B_q/(E^p = 0, K^{2p}) = 1)$ is a Hopf algebra.

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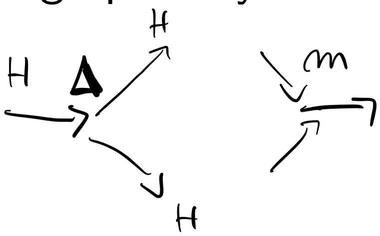
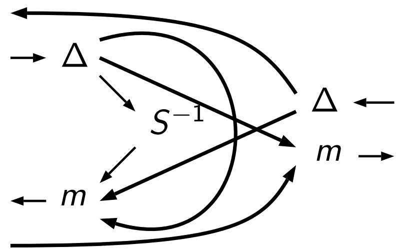
$p \in H^*$, $a \in H$; $\in H^{\otimes 2}$

or graphically





$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$

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$\Delta : H^* \longrightarrow H^* \otimes H^*$

$H^* = (m_H)^*$

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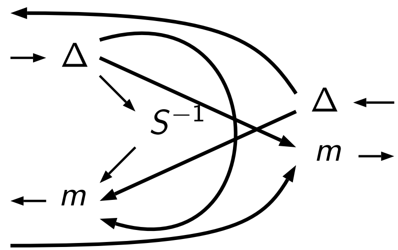

Why is $D(H)$ interesting? Because $R \in D(H)^{\otimes 2}$ defined by

$$R = \sum (\epsilon \otimes h_i) \otimes (h^i \otimes 1) = \sum h_i \otimes h^i \in D(H)^{\otimes 2}$$

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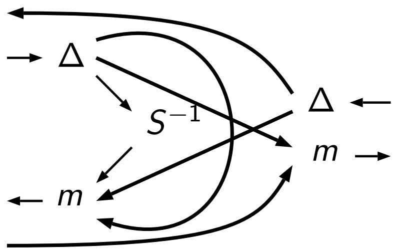

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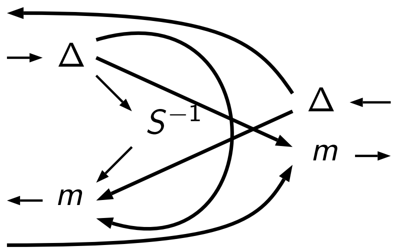

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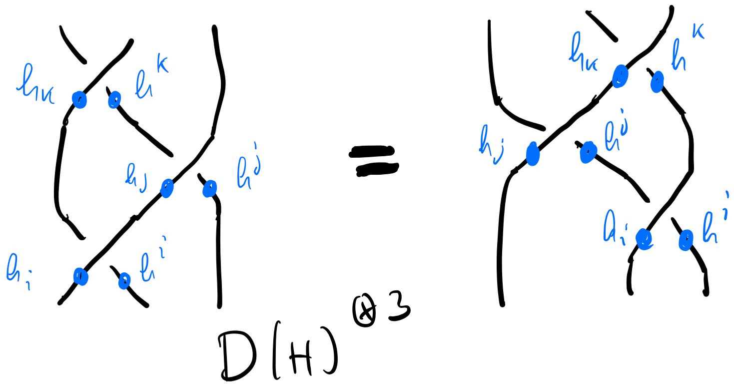
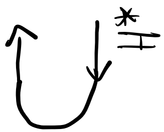


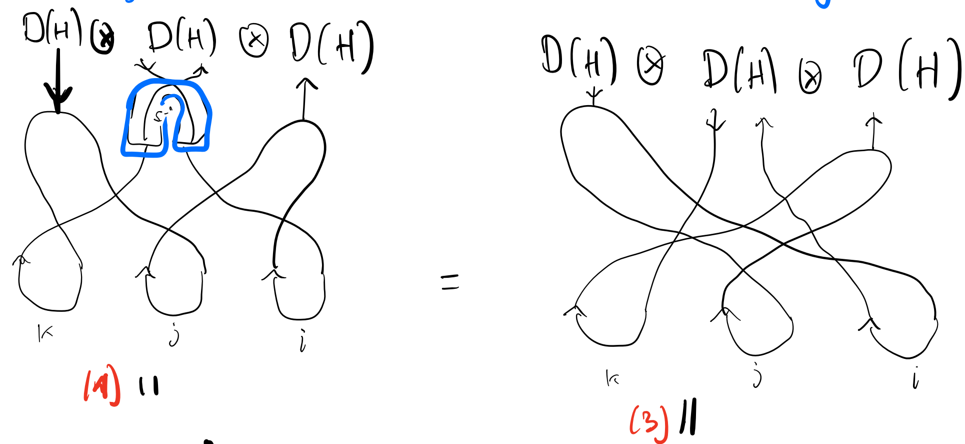
$$h^k h^j \otimes h_k h^i \otimes h_j h_i = h^j h^i \otimes h^k h_i \otimes h_k h_j$$

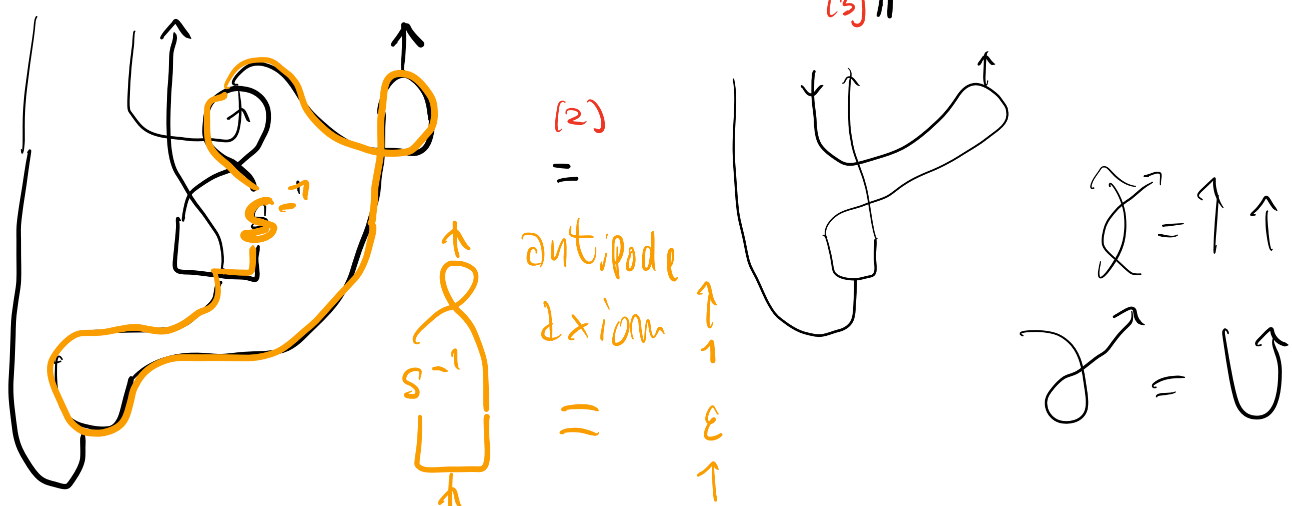
Here is the proof of this:

$$h^k h^j \otimes h_k h^i \otimes h_j h_i = h^j h^i \otimes h^k h_i \otimes h_k h_j$$

$$R = \sum_H h_i \otimes h^i$$







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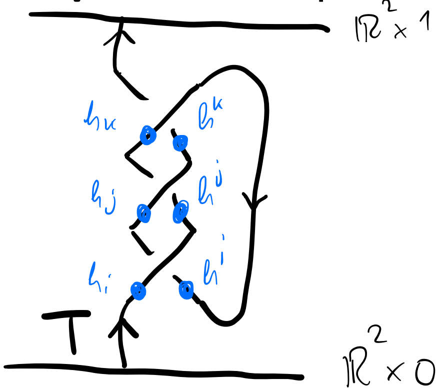

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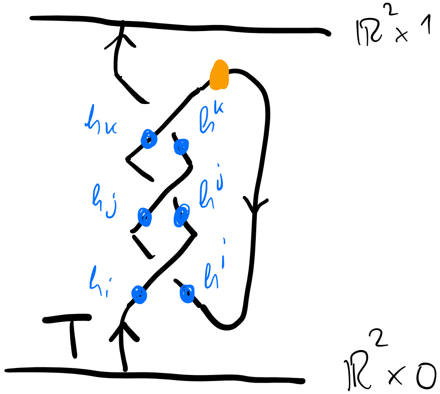

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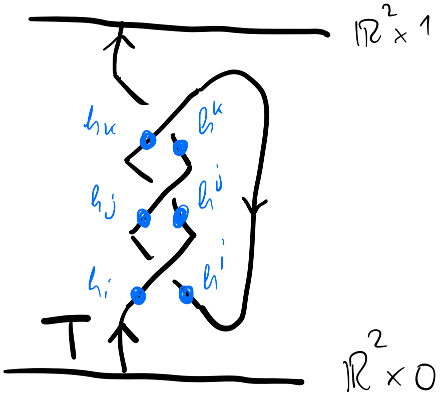

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$\mathrm{Rep}(A_\alpha)$

$G \circlearrowright$

Rmk: Recall that we want infinite groups as target. By a theorem of Etingof-~~Gelaki~~-Ostrik we are forced to consider non-semisimple Hopf algebras (or monoidal categories).

Nikshych 'os

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Twisted Drinfeld double (Virelizier, '03)

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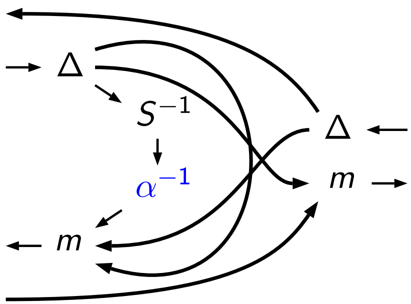

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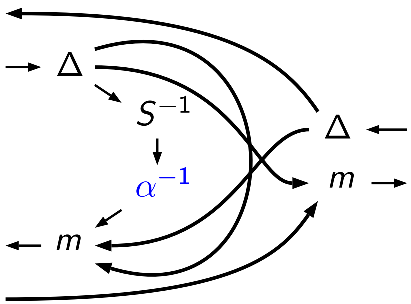

This is NOT a Hopf algebra (if $\alpha \neq \text{id}_H$) but there is a "coproduct"

$$\Delta_{\alpha,\beta} : D_{\alpha\beta} \to D_\alpha \otimes D_\beta$$

and antipode $S_\alpha : D_\alpha \to D_{\alpha^{-1}}$, satisfying graded versions of Hopf axioms. Thus, it is group-graded in the sense of Turaev.

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$K'^{2p} = q^{2p\alpha}$

$$[E', F] = \frac{K' - k'}{q - q^{-1}}$$

and $(K')^{2p} = t^p$. So the twisted Drinfeld double is essentially the semi-restricted $U_q(\mathfrak{sl}_2)$ (after quotient $Kk = 1$ and $t = q^{2\alpha}$)!

G-braiding

The twisted Drinfeld double $\{D_\alpha = H^* \otimes H\}_{\alpha \in \mathrm{Aut}(H)}$ is Aut(H)-braided in the sense that:

There are isomorphisms $\varphi_\alpha : D_\beta \to D_{\alpha\beta\alpha^{-1}}$ and R-matrices $R_{\alpha,\beta} \in D_\alpha \otimes D_\beta$:

$$\varphi_\alpha(h^* \otimes h) = h^* \circ \alpha^{-1} \otimes \alpha(h),$$

$$R_{\alpha,\beta} = \sum \alpha(h_i) \otimes h^i \in D_\alpha \otimes D_\beta$$

where h_i is a basis of H and $h^i \in H^*$ is the dual basis + relations.

Invariants of G-tangles

Consider a $(1,1)$-tangle T equipped with $\rho : \pi_1(X_T) \to G = \mathrm{Aut}(H)$.

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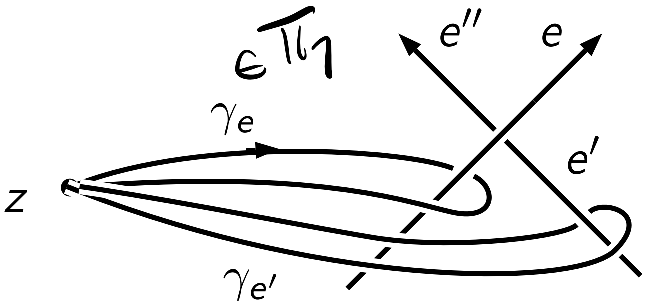

$$\alpha = \rho(\gamma_e)$$
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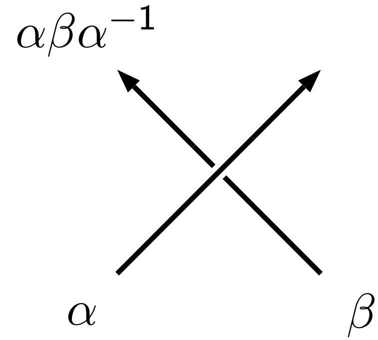

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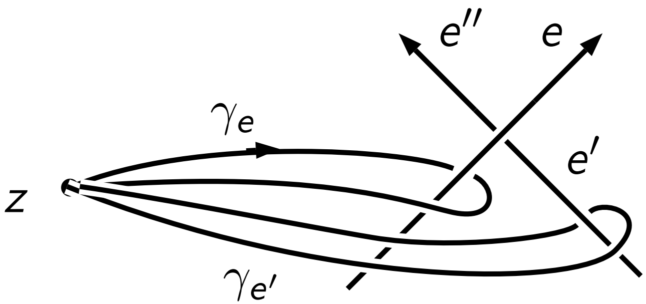

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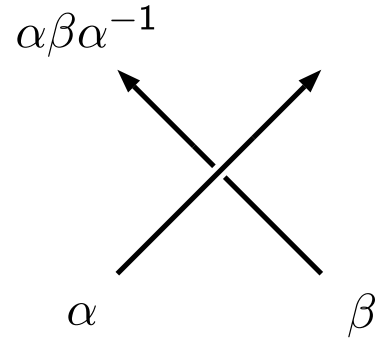
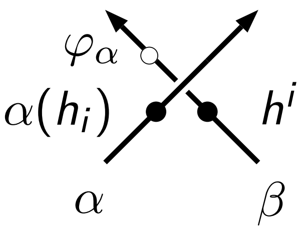
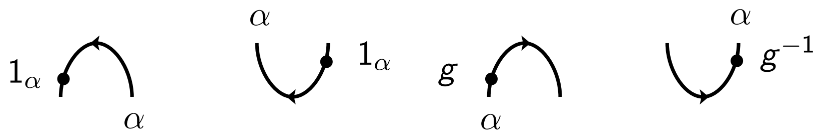


Step 2. Put one copy of $R_{\alpha,\beta} \in D_\alpha \otimes D_\beta$ and φ_α at each crossing (with labels α, β):



Step 3: Put the group-like g such that $S^2_{D(H)} = gxg^{-1}$ on right caps/cups:



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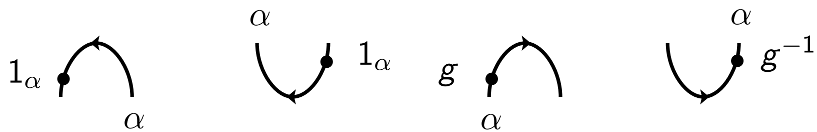

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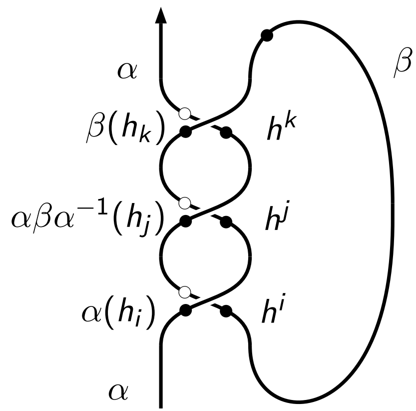

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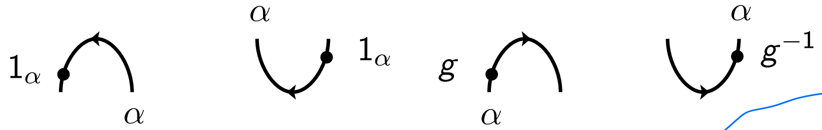

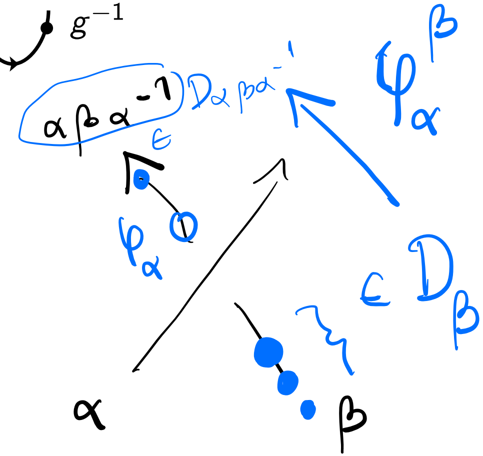

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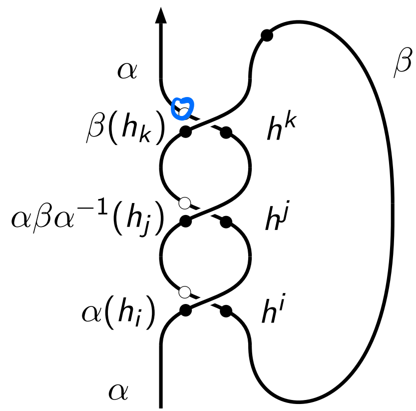

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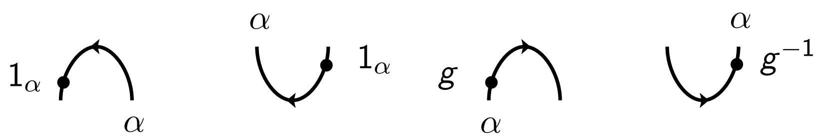

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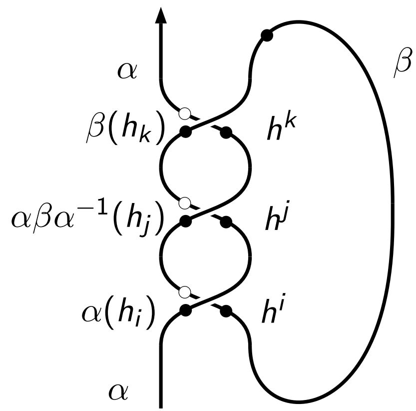

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Here $Z^\rho_H(T) = \varphi_\beta(h_k\alpha\beta\alpha^{-1}(h_j)\varphi_\alpha(h^i g\beta(h_k)\varphi_{\alpha\beta\alpha^{-1}}(h^j\alpha(h_i))))$.

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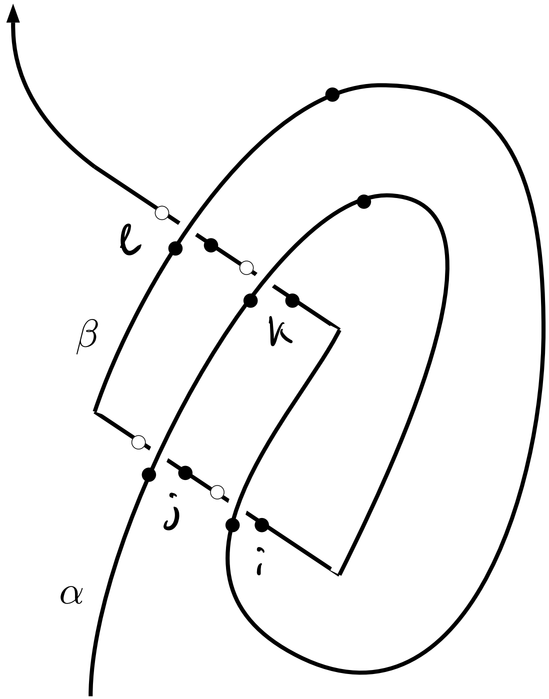
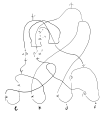
To get an invariant of (K, ρ): need to apply a $\mathrm{Aut}(H)$-invariant functional. For instance

$$P_H^\rho(K) := \epsilon_{D(H)}(Z_H^\rho(T)) \quad \in \mathbb{C}$$

where $\epsilon_{D(H)}(p \otimes h) = p(1)\epsilon(h)$.

Trefoil example





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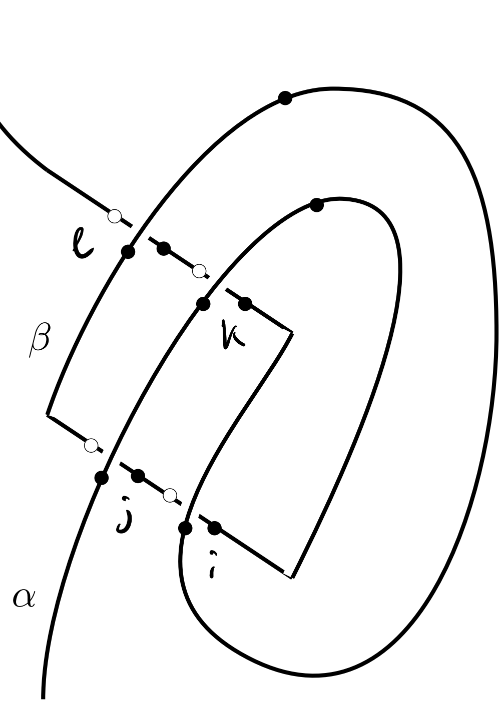

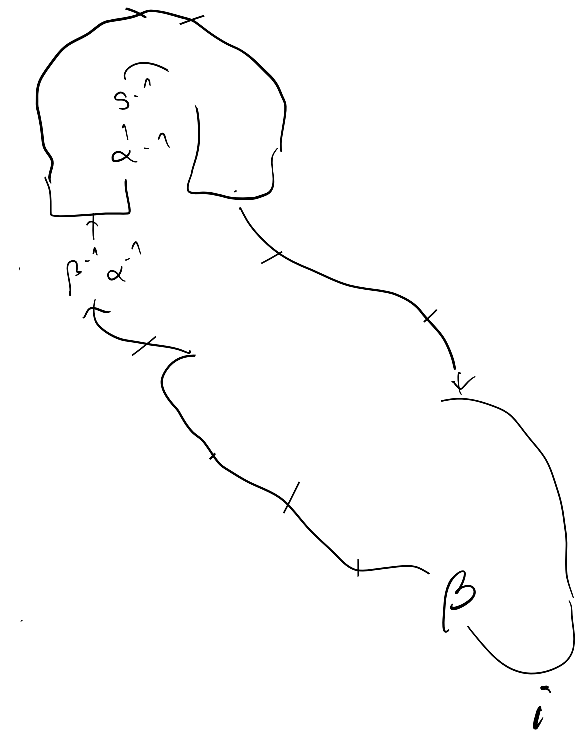

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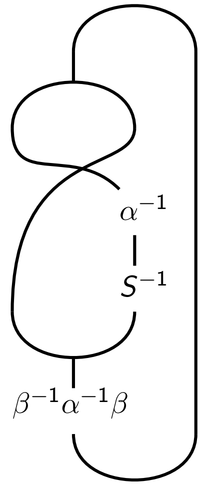

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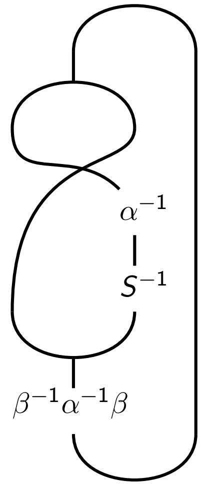


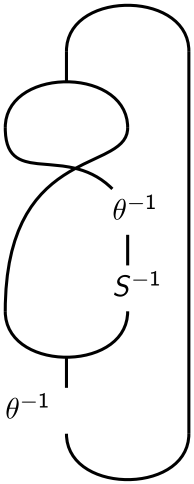
Ex: If $H = \Lambda(\mathbb{C}^n)$, $\mathrm{Aut}(H) = GL(n, \mathbb{C})$ and the above tensor is

$$\mathrm{tr}\left(\Lambda(\beta^{-1}\alpha^{-1}\beta - \alpha^{-1}\beta^{-1}\alpha^{-1}\beta)\right) = \det(\beta^{-1}\alpha^{-1}\beta - \alpha^{-1}\beta^{-1}\alpha^{-1}\beta - I_n)$$

$$= \det\left(\sigma\left(\frac{\partial\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}}{\partial\beta}\right)\right)$$

$$= \tau^{\rho}(M_K, \mu).$$

Trefoil example



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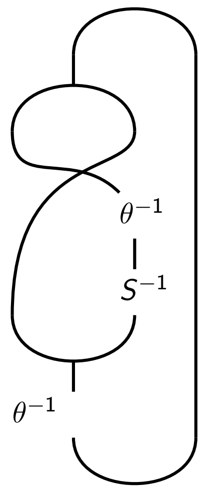

Ex: If $H = B_q/(E^p = 0, K^{2p} = 1)$ at $p = 4$ and $\rho : H_1 M_K \to \mathrm{Aut}(H), [\mu] \to \theta$ where $\theta(E) = tE, \theta(K) = K$ then the above tensor is

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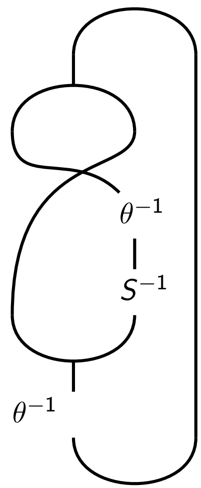

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After $t = q^{-2}x$:$ix^3 + ix^2 + \frac{i}{x^2} + \frac{i}{x^3} + (1+i)x + \frac{1+i}{x} + (1+2i)$

Special cases of $P_H^\rho(K)$:

- (LN '19-22) If $H = \Lambda(\mathbb{C}^n)$, then $\mathrm{Aut}(H) = GL(n, \mathbb{C})$ and we get **twisted Reidemeister torsion**:

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$$P_H^\rho(K) = \text{“ADO polynomial”}$$

  (Akutsu-Deguchi-Ohtsuki, '91) also called a “**colored**” Alexander invariant.

Lifting to polynomials

Suppose H is $\mathbb{Z}$-**graded** and let $H' = H \otimes \mathbb{C}[t^{\pm 1}]$. Then any $\rho : \pi_1(X_K) \to \mathrm{Aut}(H)$ extends to $\rho \otimes h : \pi_1(X_K) \to \mathrm{Aut}(H')$ by

$$\rho \otimes h(\delta)(x) = t^{n|x|}\rho(x)$$

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$\rho \equiv 1$

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$h : \pi_1 \to H_1 = \mathbb{Z}$

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Thus, we define

$$P_H^{\rho}(K, t) := P_H^{\rho \otimes h}(K) \in \mathbb{C}[t^{\pm 1}].$$

$= P_H^h(K)^{\epsilon}$

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Thus, we define

$$P_H^{\rho}(K, t) := P_H^{\rho \otimes h}(K) \in \mathbb{C}[t^{\pm 1}].$$

This is the "twisted" knot polynomial of our main thm:

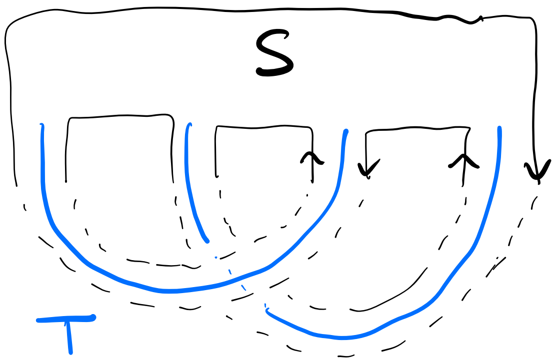
Theorem (LN-van der Veen, '22)

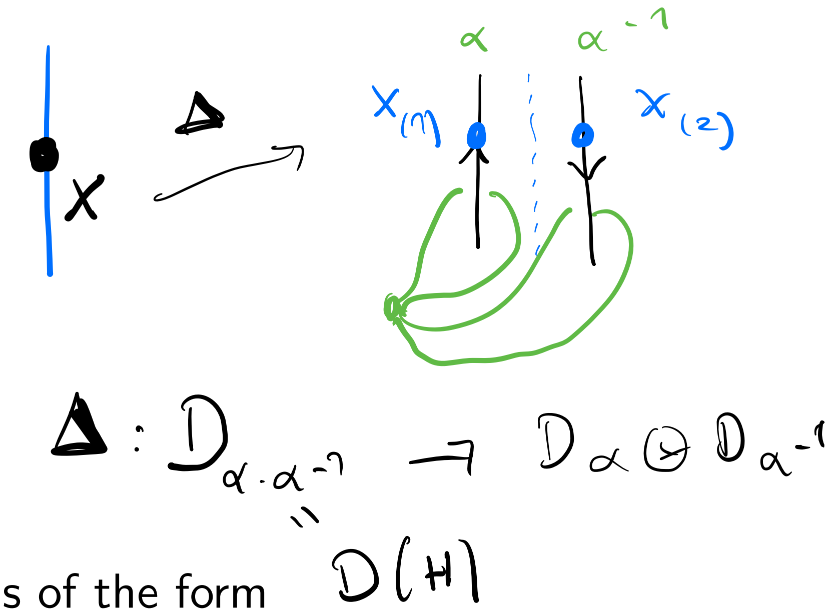
For any f.d. $\mathbb{Z}$-graded Hopf algebra H there is a knot polynomial $P_H^{\rho}(K, t)$, where $\rho : \pi_1(M_K) \to Aut(H)$, such that $\mathfrak{sl}_2$ ADO

$$deg\, P_H(K, t) \leq 2g(K)|H|. = p - 1$$

Sketch of proof ($g(K) = 1$ case)

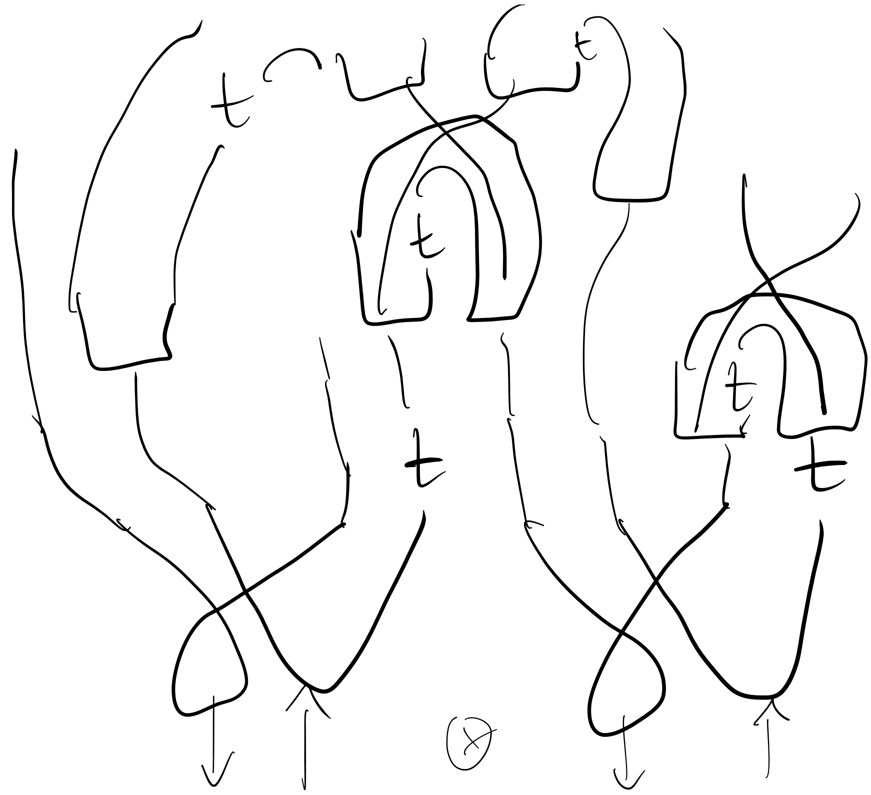
Let S be a genus 1 Seifert surface of K. Isotope S to $\mathbb{D}^2 \cup N(T)$ where T is a $2g$-tangle:





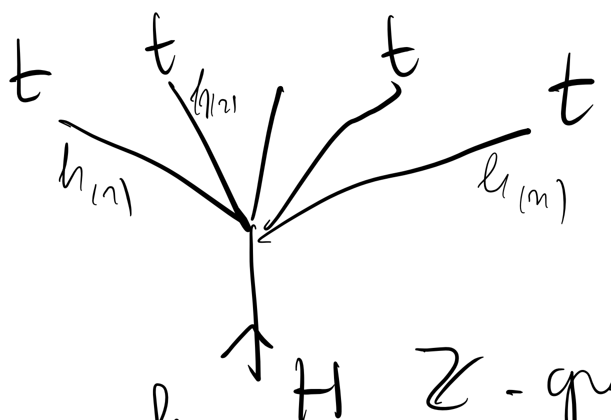
Then the invariant of each band is of the form $\Delta_{\theta,\theta^{-1}}(x), x \in D_{\mathrm{id}_H} = D(H)$! Thus, the θ-twisted invariant of K is computed as

$$Z_H^h(K) = m_t^{(4g)} P(\mathrm{id} \otimes S_{t^{-1}} \circ \Delta_{t,t^{-1}})^{\otimes 2g}(Z_T), \;\; Z_T \in D_{\mathrm{id}_H}.$$



$$\parallel$$

$$t^{|h_{(1)}| + |h_{(2)}| + \dots + |h_{(m)}|} \leq t^{N}$$



H $\mathbb{Z}$-graded

N = max deg of H

Thanks!

Some computations

Let's see what happens with the first two knots with $\Delta_K = 1$, the Kinoshita-Terasaka (KT) and the Conway knot. These are "mutants" so all $\mathfrak{sl}_2$ invariants coincide!

If $H =$ quantum Borel of $\mathfrak{sl}_3$ at $q = i$, and $h(E_1) = xE_1, h(E_2) = yE_2$
$P^h(KT, x, y)$ is:

$$12x^6y^6 - 34x^6y^4 + 34x^6y^2 - 34x^4y^6 + 148x^4y^4 - 228x^4y^2 - \frac{34x^4}{y^2} + 34x^2y^6 - 228x^2y^4 + 496x^2y^2 + \frac{228x^2}{y^2} - \frac{34x^2}{y^4} + \frac{228y^2}{x^2} - \frac{34y^4}{x^2} - \frac{34y^2}{x^4} + \frac{496}{x^2y^2} - \frac{228}{x^4y^2} + \frac{34}{x^6y^2} - \frac{228}{x^2y^4} + \frac{148}{x^4y^4} - \frac{34}{x^6y^4} + \frac{34}{x^2y^6} - \frac{34}{x^4y^6} + \frac{12}{x^6y^6} - 12x^6 + 148x^4 - 496x^2 - \frac{496}{x^2} + \frac{148}{x^4} - \frac{12}{x^6} - 12y^6 + 148y^4 - 496y^2 - \frac{496}{y^2} + \frac{148}{y^4} - \frac{12}{y^6} + 721$$

$E_1 E_{12} E_2$ $|\ | = 4$

Ugly... but degree is 12 (in x^2, y^2) and $|H| = 4$ so our bound is $\deg P \leq 8g(K)$. Thus, it is $g > 1$! There is a SS of $g = 2$ so $g(KT) = 2$.

If H = quantum Borel of $\mathfrak{sl}_3$ at $q = i$, and $h(E_1) = xE_1, h(E_2) = yE_2$
$P^h(CONWAY, x, y)$ is:

$$2y^8x^{12} - 4y^6x^{12} + 2y^4x^{12} - 4y^{10}x^{10} + 4y^8x^{10} + 8y^6x^{10} - 8y^4x^{10} - 4y^2x^{10} + 4x^{10} + 2y^{12}x^8 + 4y^{10}x^8 - 20y^8x^8 + 8y^6x^8 + 12y^4x^8 + 8y^2x^8 + \frac{4x^8}{y^2} + \frac{2x^8}{y^4} - 20x^8 - 4y^{12}x^6 + 8y^{10}x^6 + 8y^8x^6 - 4y^6x^6 - 46y^4x^6 + 46y^2x^6 - \frac{8x^6}{y^2} - \frac{8x^6}{y^4} + \frac{4x^6}{y^6} + 4x^6 + 2y^{12}x^4 - 8y^{10}x^4 + 12y^8x^4 - 46y^6x^4 + 164y^4x^4 - 248y^2x^4 - \frac{46x^4}{y^2} + \frac{12x^4}{y^4} - \frac{8x^4}{y^6} + \frac{2x^4}{y^8} + 164x^4 - 4y^{10}x^2 + 8y^8x^2 + 46y^6x^2 - 248y^4x^2 + 476y^2x^2 + \frac{248x^2}{y^2} - \frac{46x^2}{y^4} - \frac{8x^2}{y^6} + \frac{4x^2}{y^8} - 476x^2 + 4y^{10} + \frac{4y^8}{x^2} + \frac{2y^8}{x^4} - 20y^8 + \frac{4y^6}{x^6} + 4y^6 + \frac{12y^4}{x^4} + \frac{2y^4}{x^8} + 164y^4 + \frac{248y^2}{x^2} + \frac{4y^2}{x^8} - 476y^2 - \frac{8y^6}{x^2} - \frac{46y^4}{x^2} - \frac{476}{x^2} - \frac{8y^6}{x^4} - \frac{46y^2}{x^4} + \frac{164}{x^4} - \frac{8y^4}{x^6} - \frac{8y^2}{x^6} + \frac{4}{x^6} - \frac{20}{x^8} + \frac{4}{x^{10}} - \frac{476}{y^2} + \frac{476}{x^2y^2} - \frac{248}{x^4y^2} + \frac{46}{x^6y^2} + \frac{8}{x^8y^2} - \frac{4}{x^{10}y^2} + \frac{164}{y^4} - \frac{248}{x^2y^4} + \frac{164}{x^4y^4} - \frac{46}{x^6y^4} + \frac{12}{x^8y^4} - \frac{8}{x^{10}y^4} + \frac{2}{x^{12}y^4} + \frac{4}{y^6} + \frac{46}{x^2y^6} - \frac{46}{x^4y^6} - \frac{4}{x^6y^6} + \frac{8}{x^8y^6} + \frac{8}{x^{10}y^6} - \frac{4}{x^{12}y^6} - \frac{20}{y^8} + \frac{8}{x^2y^8} + \frac{12}{x^4y^8} + \frac{8}{x^6y^8} - \frac{20}{x^8y^8} + \frac{4}{x^{10}y^8} + \frac{2}{x^{12}y^8} + \frac{4}{y^{10}} - \frac{4}{x^2y^{10}} - \frac{8}{x^4y^{10}} + \frac{8}{x^6y^{10}} + \frac{4}{x^8y^{10}} - \frac{4}{x^{10}y^{10}} + \frac{2}{x^4y^{12}} - \frac{4}{x^6y^{12}} + \frac{2}{x^8y^{12}} + 649$$

Uglier... but degree is 20 (in x^2, y^2) and $\deg P \leq 8g(K)$ so $g > 2$! There is a SS of $g = 3$ so $g(CONWAY) = 3$.