Embeddings in Euclidean space and Galois actions

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Moduli and friends
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M smooth, compact w/o boundary

$$\text{emb}(M, \mathbb{R}^n) = \text{space of smooth embeddings } M \hookrightarrow \mathbb{R}^n$$

We know something about π_0 in certain cases. E.g.

- non-empty for $n \geq 2m$ (Whitney) and trivial for $n \geq 2m + 2$ (Hirsch).
- for spheres and in codim ≥ 3, it is trivial in the "metastable range" $2n \geq 3m + 4$ and can more generally be described in homotopy theoretic terms (Haefliger).
- in low dimensions, e.g. 3-manifolds in $\mathbb{R}^6$, 4-manifolds in $\mathbb{R}^7$ (Crowley-Skopenkov, ...).
- Many ...

Homotopy or homology, rationally or at a prime? How do these depend on M and its smooth structure?

Immersions

$$\mathrm{imm}(M, \mathbb{R}^n) = \text{space of smooth immersions } M \to \mathbb{R}^n$$

Smale-Hirsch: the derivative map

$$\mathrm{imm}(M, \mathbb{R}^n) \to \mathrm{Hom}^{inj}(TM, T\mathbb{R}^n)$$

sending f to df, is a weak equivalence provided $m < n$.

$\rightsquigarrow$ an immersion exists if and only if $\exists$ $(n-m)$-dim bundle μ on M with $TM \oplus \mu$ trivial. This typically depends on the smooth structure of M, although for $2n > 3m$ it doesn't (Haefliger-Hirsch).

$\rightsquigarrow$ the rational homotopy type of each component depends fairly little on M, e.g. if the codimension is odd and M is 1-connected, it only depends on the rational homotopy type of M (Abdoulkader).

Look at

$$\overline{\mathsf{emb}}(M, \mathbb{R}^n) := \mathsf{hofiber}(\mathsf{emb}(M, \mathbb{R}^n) \to \mathsf{imm}(M, \mathbb{R}^n))$$

over a fixed immersion $f : M \to \mathbb{R}^n$.

An important and easy special case: $\overline{\mathsf{emb}}(\mathbb{R}^m, \mathbb{R}^n) \simeq *$ and for a finite set S,

$$\overline{\mathsf{emb}}(S \times \mathbb{R}^m, \mathbb{R}^n) \xrightarrow{\simeq} \mathsf{emb}(S, \mathbb{R}^n)$$

Note that $\overline{\mathsf{emb}}(M, \mathbb{R}^n)$ may be empty! We won't be able to say much about $\pi_0 \overline{\mathsf{emb}}(M, \mathbb{R}^n)$ in what follows. But we'll try say something about the homotopy type of each component.

Theorem (Arone-Turchin,Fresse-Turchin-Willwacher)

Let M be a closed m-manifold and $f : M \to \mathbb{R}^n$ an immersion. Under certain assumptions, there is a map

$$\overline{\mathrm{emb}}(M, \mathbb{R}^n) \to \mathbb{R}\mathrm{map}_{\mathrm{Fin}_{\leq k}}(\mathrm{map}(-, M), P_n^{\mathbb{Q}})$$

which, restricted to each path component, is rationally ℓ-connected for $\ell = 3 - n + (k+1)(n - m - 2)$ and $2 \leq k \leq \infty$.

Here:

- $\mathbb{R}\mathrm{map}_{\mathrm{Fin}_{\leq k}}$ the space of derived maps between functors ${\mathrm{Fin}_{\leq k}}^{\mathrm{op}} \to \mathrm{Spaces}$,
- $\mathrm{map}_{\mathrm{Fin}_{\leq k}}(-, M)$ is the functor $S \mapsto \mathrm{map}(S, M)$,
- $P_n^{\mathbb{Q}}$ is the Sullivan realization of $S \mapsto H^*(\mathrm{emb}(S, \mathbb{R}^n), \mathbb{Q})$ (left adjoint to polynomial forms $\Omega^*_{poly} : \mathrm{Spaces}^{\mathrm{op}} \to \mathrm{cdga}$).

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which, restricted to each path component, is rationally ℓ-connected for $\ell = 3 - n + (k+1)(n - m - 2)$ and $2 \leq k \leq \infty$.

- AT require codimension roughly $\geq m$. Then all spaces are path-connected. FTW require $n - m \geq 3$ and that the immersion $f : M \to \mathbb{R}^n$ factors through $\mathbb{R}^{n-2}$.

Goal: explain a different proof, with a slightly weaker assumption, which also gives some partial results on torsion.

By adjunction,

$$\mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathrm{map}(-,M), P_n^{\mathbb{Q}}) \simeq \mathbb{R}\mathrm{map}_{\mathsf{Fin}}(H^*(E_n,\mathbb{Q}), \Omega^*(\mathrm{map}(-,M)))$$

and FTW express the homotopy groups of the right hand side as the homology of a certain graph complex. The graphs are like those describing $H^*(E_n, \mathbb{Q})$, but with extra hairs that are labelled by elements in a chosen rational model for M.

Homotopy invariance

Corollary (Arone-Lambrechts-Volic)

Let $M \to M'$ be a rational homotopy equivalence of m-dimensional submanifolds of $\mathbb{R}^k$, $k \leq n-2$. Then

$$\overline{\text{emb}}(M, \mathbb{R}^n)_{incl.} \simeq_{\mathbb{Q}} \overline{\text{emb}}(M', \mathbb{R}^n)_{incl.}$$

Corollary

Let $i : M \to \mathbb{R}^n$ and $i' : M' \to \mathbb{R}^n$ be embeddings. Suppose i and i' can be isotoped to homotopic embeddings in some submanifold $N \subset \mathbb{R}^n$ of codim ≥ 2 (i.e. there is $j : M \to M'$ such that $i \sim i'j$ as maps $M \to N$). If $j : M \xrightarrow{\simeq_{\mathbb{Q}}} M'$ then

$$\overline{\text{emb}}(M, \mathbb{R}^n)_i \simeq_{\mathbb{Q}} \overline{\text{emb}}(M', \mathbb{R}^n)_{i'}$$

Corollary

Two homotopic compressible embeddings $i : M \hookrightarrow N \subset \mathbb{R}^n$ have rat. homotopy equivalent $\overline{\mathrm{emb}}(M, \mathbb{R}^n)_i$.

$\rightsquigarrow$ c.f. Skopenkov: if M is a 3-manifold with $H_1(M)$ having no 2-torsion, any two embeddings in $\mathbb{R}^6$ which are compressible in S^4 or $S^2 \times S^2$ are isotopic.

$\rightsquigarrow$ No longer true if the word compressible is dropped. For example, the component of the Hopf link $S^a \amalg S^b \to \mathbb{R}^{a+b+1}$ does not have the same (rat.) homotopy type as that of the unlink.

Detecting configurations using barycenters

Suppose M is triangulated, i.e. $M \cong |X|$ for some (non-singular) simplicial set X.

Task: describe $\overline{\text{emb}}(M, \mathbb{R}^n)$ in terms of X, as much as possible.

For a parallel, let us look at the space of continuous maps $\text{map}(M, N)$

$$\begin{aligned} \text{map}(M, N) &\cong \text{map}(|X|, N) \\ &\simeq \text{map}(\text{hocolim}_{s \in \Delta} X_s, N) \\ &\simeq \text{holim}_{s \in \Delta} \text{map}(X_s, N) \end{aligned}$$

Of course, this can't work for embeddings. Face maps $X_s \to X_{s-1}$ not injective, so $[s] \mapsto \text{emb}(X_s, N)$ isn't a functor.

Get around: thickening

Definition

Let $\mathcal{O}$ be the poset of open subsets $T \subset [0,1]$ such that $\pi_0 T$ is finite, $\partial[0,1] \subset T$ and $T \neq [0,1]$.

There is a functor

$$\mathcal{O} \to \Delta^{\mathrm{op}}$$

sending T to $\pi_0([0,1] \backslash T)$. (Example of face and deg. maps) This is an ∞-localization. That is,

$$\{\text{simplicial objects on } \mathcal{C}\} \leftrightarrows \{\text{functors } \mathcal{O} \to \mathcal{C} \text{ which send isotopy equivalences to weak equivalences.}\}$$

Definition

Given $T \in \mathcal{O}$, let $\Delta^n_T = \{(t_1 \leq \cdots \leq t_n) : t_i \in T\} \subset \Delta^n$.

(examples)

Assume (for simplicity) that X is such that every face of a non-deg simplex is non-deg (e.g. a simplicial complex). Define

$$U(T) = \bigcup_{\sigma \in X} \Delta_T^{|\sigma|} \subset \bigcup_{\sigma \in X} \Delta^{|\sigma|} = |X|$$

If $T \subset T'$ then $U(T) \subset U(T')$, i.e. U defines a functor from $\mathcal{O}$ to the poset of open subsets of M.

Definition

Let

$$\Gamma : \mathcal{O}^{\mathrm{op}} \to \mathrm{Spaces}$$

be the functor that sends T to

$$\overline{\mathrm{emb}}(U(T), N) = \mathrm{hofiber}[\mathrm{emb}(U(T), N) \to \mathrm{imm}(U(T), N)] \; .$$

- For $T \in \mathcal{O}$ with s interior components, $U(T) \simeq X_s$.
- And $\overline{\mathrm{emb}}(U(T), N) \simeq \overline{\mathrm{emb}}(X_s \times \mathbb{R}^m, N) \simeq \mathrm{emb}(X_s, N)$, at least if $n - m \geq 3$.

Formally, Γ extends to a cosimplicial space $\Delta \to$ Spaces,

- $\Gamma^s \simeq \mathrm{emb}(X_s, N)$
- for each $\theta : [s] \to [t]$ in Δ there is a homotopy commutative diagram

$$\begin{array}{ccc} \Gamma^s & \overset{\simeq}{\longleftrightarrow} & \overline{\mathrm{emb}}(X_s \times \mathbb{R}^m, \mathbb{R}^n) \\ {\scriptstyle \theta_*}\big\downarrow & & \big\downarrow{\scriptstyle f_\theta^*} \\ \Gamma^t & \overset{\simeq}{\longleftrightarrow} & \overline{\mathrm{emb}}(X_t \times \mathbb{R}^m, \mathbb{R}^n) \end{array}$$

  where $f_\theta : X_t \times \mathbb{R}^m \hookrightarrow X_s \times \mathbb{R}^m$ is any embedding such that $\pi_0 f_\theta = \theta^* X$. $\rightsquigarrow$ θ^* does not depend on the smooth structure of M up to homotopy.

Theorem (B.-Lambrechts-Songhafuou-Pryor)

$\overline{\mathrm{emb}}(M, N) \simeq \operatorname{holim}_{\Delta} \Gamma$ *provided* $n - m \geq 3$.

(Generalises Sinha's cosimplicial model for 1-dimensional M.)

Step of proof.

Embedding calculus: $\overline{\mathrm{emb}}(-, N)$ is a homotopy sheaf with respect to covers $\{U_i \to M\}$ where each finite subset $S \subset M$ is contained in some U_i. The open subsets $\{U(T)\}$ form one such cover: assume $S \subset \Delta^m$, some top. dim. simplex. Pick $T \in \mathcal{O}$ containing all the coordinates of S. Then $S \subset U(T)$. □

More generally,

$$\overline{\mathrm{emb}}(M, N) \to \operatorname{holim}_{[s]\in\Delta} \operatorname{holim}_{S\subset X_s, |S|\leq k} \text{"}\mathrm{emb}(S, N)\text{"}$$

is highly-connected.

Lemma

The map

$$\mathrm{holim}_{\Delta_{\leq s}}\Gamma \to \mathrm{holim}_{\Delta_{\leq s-1}}\Gamma$$

is ℓ-connected with $\ell = \frac{1}{m}[s(n-m-2)-n+2)]$.

Corollary

The associated spectral sequence converges for $n-m \geq 3$.

Using results of Farjoun:

Corollary

The natural map

$$(\mathrm{holim}_{\Delta}\Gamma)_{\mathbb{Q}} \to \mathrm{holim}_{\Delta}\Gamma_{\mathbb{Q}}$$

is a weak equivalence when restricted to each component.

Corollary

$\mathrm{emb}(M,\mathbb{R}^n)$ *and* $\overline{emb}(M,\mathbb{R}^n)$ *are nilpotent spaces.*

Formality and a proof of the AT-FTW theorem

Proposition (B.-Horel)

Under assumptions (...), the cosimplicial space $\Gamma_{\mathbb{Q}}$ is formal, i.e. there is a zigzag of weak equivalences between $\Gamma_{\mathbb{Q}}$ and its homology $H \circ \Gamma_{\mathbb{Q}}$.

$\Rightarrow \Gamma_{\mathbb{Q}} : \Delta \to \mathsf{Spaces}_{\mathbb{Q}}$ factors as

$$\Delta \xrightarrow{X} \mathsf{Fin} \xrightarrow{P_n^{\mathbb{Q}}} \mathsf{Spaces}_{\mathbb{Q}}$$

since both are formal and have the same homology. And so

$$\operatorname*{holim}_{\Delta} \Gamma_{\mathbb{Q}} \simeq \operatorname*{holim}_{s \in \Delta} P_n^{\mathbb{Q}}(X_s) \simeq \mathbb{R}\mathrm{map}^h_{\mathsf{Fin}}(\mathrm{map}(-, M), P_n^{\mathbb{Q}})$$

Putting it all together,

$$\overline{\mathrm{emb}}(M, \mathbb{R}^n) \xrightarrow{\sim} \mathrm{holim}\Gamma \to \mathrm{holim}(\Gamma_{\mathbb{Q}}) \xrightarrow{\sim} \mathbb{R}\mathrm{map}^h_{\mathsf{Fin}}(\mathrm{map}(-, M), P_n^{\mathbb{Q}})$$

The proof of proposition uses crucially

Theorem (Cirici-Horel)

Let $F : \mathcal{C} \to \mathrm{Spaces}_{\mathbb{Q}}$ *be a functor such that* $F(c)$ *has finite-dim. homology for every* $c \in \mathcal{C}$ *and* $\exists\ u \in \mathbb{Q}^{\times}$ *of infinite order,* $\alpha \in \mathbb{Q}$ *and an endomorphism* $\sigma : F \Rightarrow F$ *such that*

$$H_k(\sigma_c) : H_k(F(c)) \to H_k(F(c))$$

is multiplication by $u^{\alpha k}$*. Then* F *is formal.*

Constructing the action

Definition

The configuration category of a manifold U is the topological category whose objects are ordered configurations (of any cardinality) and morphisms are paths of such, where points are allowed to fuse (but collisions cannot be undone).

An embedding $U \to V$ determines $\mathrm{con}(U) \to \mathrm{con}(V)$ over Fin.

(B-Weiss) For $n - m \geq 3$, the following square is ho. cartesian

$$\begin{array}{ccc} \mathrm{emb}(M, \mathbb{R}^n) & \longrightarrow & \mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathrm{con}(M), \mathrm{con}(\mathbb{R}^n)) \\ \downarrow & & \downarrow \\ \underset{V \in \mathcal{O}_1(M)}{\mathrm{holim}} \ \mathrm{emb}(U, \mathbb{R}^n) & \longrightarrow & \underset{V \in \mathcal{O}_1(M)}{\mathrm{holim}} \ \mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathrm{con}(V), \mathrm{con}(\mathbb{R}^n)) \end{array}$$

where $\mathcal{O}_1(M)$ is the poset of open subsets of M diffeo to $\mathbb{R}^m$.

Definition

Let Γ' be the functor $\mathcal{O} \to \text{Spaces}$ which to $T \in \mathcal{O}$ assigns the homotopy fiber of

$$\mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathrm{con}(U(T)), \mathrm{con}(\mathbb{R}^n)) \to \operatorname*{holim}_{V \in \mathcal{O}_1(U(T))} \mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathrm{con}(V), \mathrm{con}(\mathbb{R}^n))$$

over the image of the fixed immersion $f : M \to \mathbb{R}^n$.

By the cartesianness, there is a weak equivalence of "cosimplicial" spaces $\Gamma \to \Gamma'$.

Definition

Replace all instances of $\mathrm{con}(\mathbb{R}^n)$ by $\mathrm{con}(\mathbb{R}^n)_{\mathbb{Q}}$ to define $\Gamma'_{\mathbb{Q}}$.

Lemma

The canonical map $\Gamma \to \Gamma'_{\mathbb{Q}}$ is a rationalization.

Galois symmetries on $\mathrm{con}(\mathbb{R}^n)$

Theorem (Fresse, Horel)

The restriction $GT_{\mathbb{Q}} \simeq \mathbb{R}\mathrm{Aut}_{\mathsf{Fin}}(\mathrm{con}(\mathbb{R}^2)_{\mathbb{Q}}) \to \mathbb{Q}^\times$ is surjective.

(B-Horel) Can lift this to an action $GT_{\mathbb{Q}} \to \mathbb{R}\mathrm{Aut}_{\mathsf{Fin}}(\mathrm{con}(\mathbb{R}^n)_{\mathbb{Q}})$ whose restriction to $\mathbb{Q}^\times$ is surjective, and for every $u \in \mathbb{Q}^\times$, a lifted automorphism $u^\sharp$ of $\mathrm{con}(\mathbb{R}^n)_{\mathbb{Q}}$ is pure, i.e. it acts on the homology of object spaces $H_k(\mathrm{emb}(S, \mathbb{R}^n), \mathbb{Q})$ as multiplication by $u^{k/(n-1)}$. This follows from an additivity thm:

$$\mathrm{con}(\mathbb{R}^2)_{\mathbb{Q}} \boxtimes \mathrm{con}(\mathbb{R}^{n-2})_{\mathbb{Q}} \simeq \mathrm{con}(\mathbb{R}^n)_{\mathbb{Q}}$$

$\rightsquigarrow$ get an action on the map

$$\mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathrm{con}(M), \mathrm{con}(\mathbb{R}^n)_{\mathbb{Q}}) \to \operatorname*{holim}_{V \in \mathcal{O}_1(M)} \mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathrm{con}(V), \mathrm{con}(\mathbb{R}^n)_{\mathbb{Q}})$$

$\rightsquigarrow$ Lifts to an action on $\Gamma'_{\mathbb{Q}}$ if the image of the basepoint immersion f factors through

$$\operatorname*{holim}_{V \in \mathcal{O}_1(M)} \mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathrm{con}(V), \mathrm{con}(\mathbb{R}^{n-2})_{\mathbb{Q}})$$

For that it is enough that, for each x, $df_x : T_x M \to \mathbb{R}^n$ factors through $\mathbb{R}^{n-2}$, *rationally* (more below).

$\rightsquigarrow$ under this assumption on f, $u^\sharp$ lifts to an endomorphism of $\Gamma'_{\mathbb{Q}}$ which is pure. Hence $\Gamma_{\mathbb{Q}}$ is formal by Cirici-Horel.

Take the classifying map for the tangent bundle of M

$$\tau : M \to BO(m)$$

By Smale-Hirsch, a choice of an immersion is a lift of τ to $Gr_{n,m}$:

$$M \to BO(m) \times BO(n-m)$$

whose composition to $BO(n)$ is null-homotopic. Now we can ask whether this extends to $Gr_{n-2,m}$, *rationally*. There are two cases, depending on the parity of $n-m$.

- if $n-m = 2k-1$, the dual Pontryagin class p^*_{k-1} is zero, and
- if $n-m = 2k$, p^*_{k-1} is a square

Torsion

Given a prime p, we can

- define a cosimplicial space Γ_p whose homotopy limit tries to approximate the p-completion of $\overline{\mathrm{emb}}(M, \mathbb{R}^n)$
- Galois action on the p-completion of $\mathrm{con}(\mathbb{R}^n)$ induces an action on Γ_p
- have purity in a range, so get partial collapse at E_2 in the (Bousfield-Kan) spectral sequence for Γ_p.
- extends results for 1-dim source, e.g.

$$\pi_i \mathrm{emb}(S^1, \mathbb{R}^n) \otimes \mathbb{Z}_{(p)} \cong \oplus_{t-s=i} E^2_{-s,t} \otimes \mathbb{Z}_{(p)}$$

for $i < 2(p+n) - 4$.

Thanks for listening!