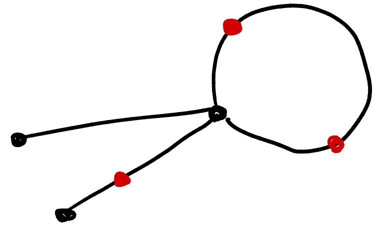
Stable and unstable homology of graph braid groups

(jt w/ B. An + G. Drummond-Cole)

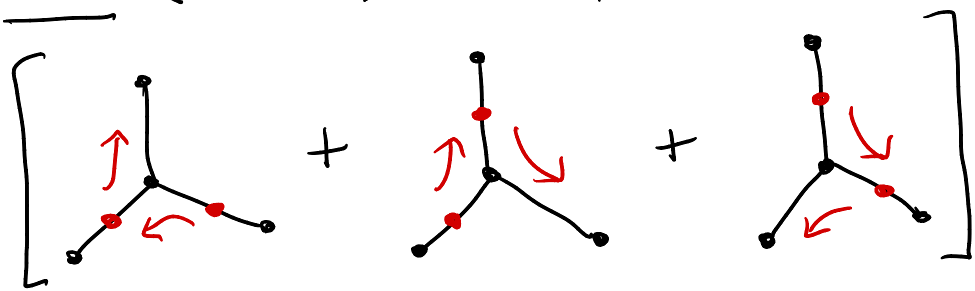
Γ a graph



$\in B_3(\Gamma)$

$$B_k(\Gamma) = \{(x_1, \ldots, x_k) \in \Gamma^k \mid x_i \neq x_j \text{ if } i \neq j\} / \Sigma_k$$

<u>Ex</u> $(k=2,\ \Gamma = \lambda)$ $\qquad S^1 \xhookrightarrow{\sim} B_2(\lambda)$



$\neq 0 \in \pi_1$

Thm (Abrams) $B_k(\Gamma)$ is aspherical. 1

Ex (Ghrist) $B_2(K_5) \simeq \#_7 \mathbb{RP}^2$ "graph braid groups"

Def The i^{th} **Ramos number** of Γ is

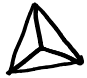
$$\Delta^i_\Gamma = \max_{|W|=i} |\pi_0(\Gamma \setminus W)|,$$

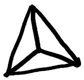
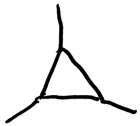
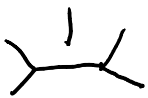
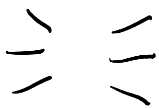
where W runs over sets of essential vertices (valence ≥ 3).

<u>Thm</u> (ADK) Fix a field $\mathbb{F}$ and $i>1$. If Γ is a connected graph with an essential vertex, then

$$\dim H_i(B_k(\Gamma);\mathbb{F}) \sim \left[\sum_{\substack{|W|=i \\ \Delta_0 = \Delta_\Gamma^i}} \frac{1}{(\Delta_\Gamma^i - 1)!} \prod_{w\in W} (d(w)-2) \right] k^{\Delta_\Gamma^i - 1}.$$

In fact, we show the Betti numbers are eventually polynomial in k, so the theorem amounts to calculating the degrees and leading coefficients.

Ex ($\Gamma = K_4 =$ )

i	$K_4 \setminus W$	$\Delta^i_{K_4}$	Asymptotic Betti
0		1	1
1		1	Ko-Park
2		2	$\binom{4}{2}\frac{1}{1!}k = 6k$
3		4	$\binom{4}{3}\frac{1}{3!}k^3 = \frac{2}{3}k^3$
4		6	$\binom{4}{4}\frac{1}{5!}k^5 = \frac{1}{120}k^5$
≥ 5	?	0	0

<u>Question</u> Why eventual polynomial growth?

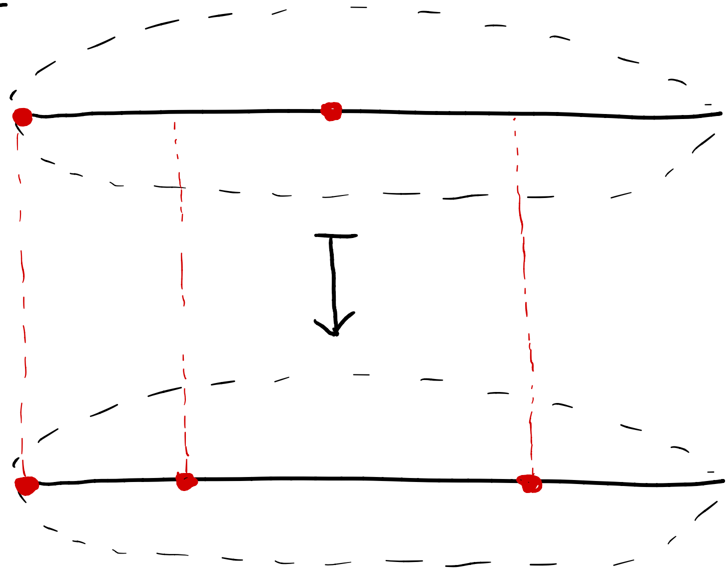
<u>Thm</u> (Hilbert) A finitely generated graded module over $\mathbb{F}[x_1, \ldots, x_n]$ exhibits eventual polynomial growth of degree $\leq n-1$.

$e \in E = E(\Gamma)$

$\sigma_e : B_k(\Gamma) \to B_{k+1}(\Gamma)$

$H_*(B(\Gamma)) \circlearrowleft \mathbb{Z}[E]$

$B(\Gamma) := \coprod_{k \geq 0} B_k(\Gamma)$



Thm (ADK) $H_*(B(\Gamma))$ is finitely generated over $\mathbb{Z}[E]$.

Perspective Homological stability

Space	Stable homology	Generation
$B_k(M)$	constant	$\mathbb{Z}[\sigma]$
$\mathrm{Conf}_k(M)$	constant characterwise	FI
$B_k(\Gamma)$	polynomial	$\mathbb{Z}[E]$
$\mathrm{Conf}_k(\Gamma)$	?	?

Question: Why degree $\Delta_\Gamma^i - 1$?

To bound the degree from below, we identify a submodule with this growth.

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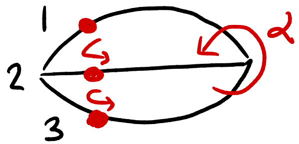
essential vertex $v \in \Gamma$ $\rightsquigarrow$ $\lambda \hookrightarrow \Gamma$ $\rightsquigarrow$ "star class" in $H_1(B_2(\Gamma))$

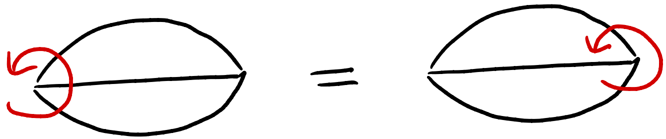
set W of essential vertices $\rightsquigarrow$ $\coprod_W \lambda \hookrightarrow \Gamma$ $\rightsquigarrow$ "W-torus" in $H_{|W|}(B_{2|W|}(\Gamma))$

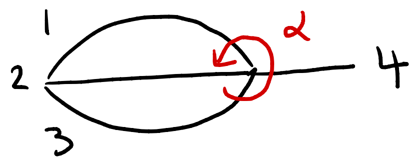
Observation: The action of $\mathbb{Z}[E]$ on a W-torus α factors through $\mathbb{Z}[E] \twoheadrightarrow \mathbb{Z}[\pi_0(\Gamma \setminus W)]$.

Ex 

$e_1\alpha = e_2\alpha = e_3\alpha$

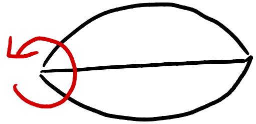
Sometimes the action factors further.

Ex  "θ-relation"

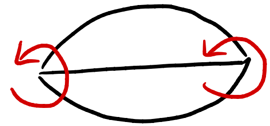
$\Longrightarrow \quad e_1\alpha = e_2\alpha = e_3\alpha = e_4\alpha$ 

We say that a W-torus is <u>rigid</u> if each star factor intersects at least two components of $\Gamma \setminus W$.

Ex



not rigid



rigid

Prop (1) The W-torus α is rigid iff $\mathbb{Z}[\pi_0(\Gamma \setminus W)] \twoheadrightarrow \mathbb{Z}[E] \cdot \alpha$ is an isomorphism.

(2) If W maximizes $|\pi_0(\Gamma \setminus W)|$, then a rigid W-torus exists.

This proposition implies that the degree of growth is at least $\Delta^i_\Gamma - 1$:

$$\operatorname{rk} H_i(B_k(\Gamma)) \geq \operatorname{rk} \mathbb{Z}[\pi_0(\Gamma \setminus W)]_{k-2i}$$

$$= \binom{k-2i+|\pi_0(\Gamma \setminus W)|-1}{|\pi_0(\Gamma \setminus W)|-1}$$

$$= \binom{k-2i+\Delta_\Gamma^i-1}{\Delta_\Gamma^i-1}$$

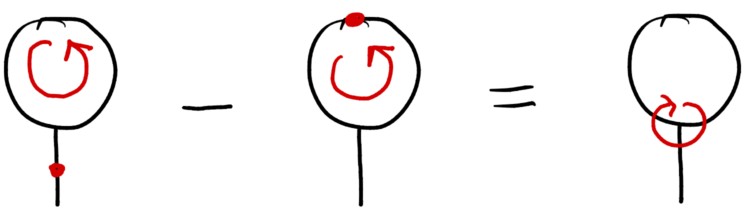
$$= \frac{(k-2i+\Delta_\Gamma^i-1)(k-2i+\Delta_\Gamma^i-2)\cdots(k-2i)}{(\Delta_\Gamma^i-1)!}$$

$$\sim \frac{1}{(\Delta_\Gamma^i-1)!}\, k^{\Delta_\Gamma^i-1}$$

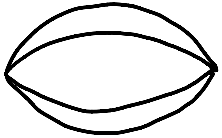
Question what about the upper bound? Why does nothing grow faster than a rigid torus?

If $H_*(B(\Gamma))$ were generated by tori, the Θ-relation would be an answer.

Ex A loop in Γ gives a class in $H_1(B_1(\Gamma))$ not in the span of all tori, but



"Θ-relation"

<u>Ex</u>  Θ_4 $\quad B_3(\Theta_4) \simeq \#_3 T^2$

The fundamental class λ is not in the span of all tori, but one can show that $(e_i - e_j)\lambda$ is.

So loop classes and Θ-classes grow slowly modulo tori. The theorem implies that relations of this form hold for all classes.

Perspective Generators and relations

Problem Give a list of atomic graphs generating $H_i(B(\Gamma))$ for some class of graphs Γ.

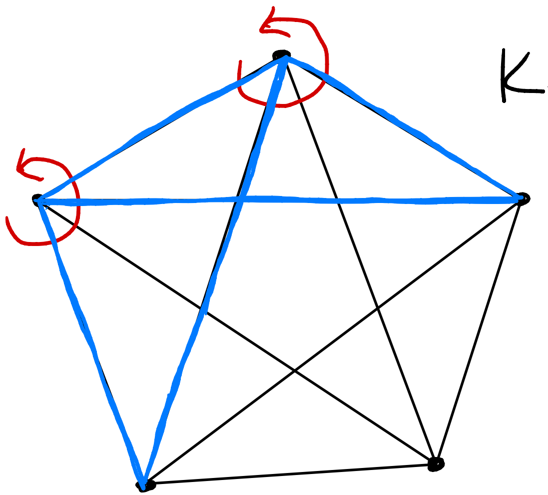
Thm (Ko–Park) For $i=1$ and all graphs, $\{\bigcirc, \curlywedge\}$ is a generating set.

Thm (AK) For $i=2$ and planar graphs, Θ_4 is the only new generator. For non-planar graphs, there are more.

Asymptotically, the only generator is $\curlywedge$.

Perspective Torsion

Thm (Ko–Park) There is (2-)torsion in $H_1(B(\Gamma))$ iff Γ is non-planar.



K_5

Θ-relation $\Longrightarrow \alpha = (-1)^5 \alpha$

No other torsion is known.

<u>Conjecture</u> (?) $H_*(B(\Gamma))$ has no odd torsion and, if Γ is planar, also no even torsion.

Asymptotically, the conjecture holds:

$$\dim H_i(B(\Gamma); \mathbb{F}_p) \sim \dim H_i(B(\Gamma); \mathbb{Q}).$$

Any torsion must arise from exotic classes with slow growth.

Question: Why the leading coefficient?

So far, for each W maximizing $|\pi_0(\Gamma_W)|$, we get $\mathbb{Z}[\pi_0(\Gamma_W)] \subseteq H_{|W|}(B(\Gamma))$ for each rigid W-torus. We claim that

(1) tori for distinct W are (stably) linearly independent, and

(2) there are (stably) $\prod_{w \in W} (d(w)-2)$ linearly independent tori for each W.

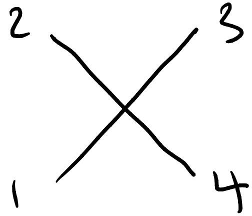
Assuming so, we obtain the bound

$$\operatorname{rk} H_i(B_k(\Gamma)) \geq \bigoplus_{\substack{|W|=i \\ |\pi_0(\Gamma, W)| = \Delta^i_\Gamma}} \mathbb{Z}[\pi_0(\Gamma, W)]_{k-2i}^{\prod_{w\in W}(d(w)-2)}$$

$$\sim \sum_W \frac{1}{(\Delta^i_\Gamma - 1)!} \prod_{w\in W} (d(w)-2)\, k^{\Delta^i_\Gamma - 1}.$$

Point (2) is somewhat surprising, as there are naively $\binom{d(w)}{3}$ star classes (up to sign) at each vertex w.

Ex

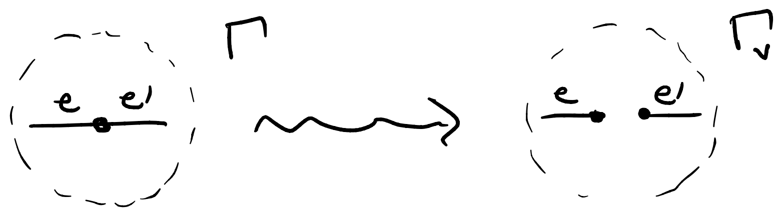


$$\alpha_{123} - \alpha_{124} + \alpha_{134} - \alpha_{234} = 0$$

$$e_4\alpha_{123} - e_3\alpha_{124} + e_2\alpha_{134} - e_1\alpha_{234} = 0$$

"X-relations"

To prove that tori dominate, we argue inductively using a long exact sequence associated to exploding a bivalent vertex v:



This sequence takes the form

$$\cdots \to H_i(B_k(\Gamma_v)) \to H_i(B_k(\Gamma)) \to H_{i-1}(B_{k-1}(\Gamma_v)) \to \cdots .$$

Using this sequence, we show that $H_i(B(\Gamma))/_{\text{tori}}$ lies in a Serre subcategory of "tame" $\mathbb{Z}[E]$-modules. Tameness is a slow growth condition controlled by the geometry of Γ, allowing tameness relative to Γ to be derived from tameness relative to Γ_v.

The map $H_i(B_{k-1}(\Gamma_v)) \longrightarrow H_i(B_k(\Gamma_v))$ is multiplication by $e-e'$, so we obtain the SES

$$0 \to \mathrm{Tor}_0^{e-e'} H_i(B_\bullet(\Gamma_v))/\mathrm{tors} \to H_i(B_\bullet(\Gamma))/\mathrm{tors}$$
$$\downarrow$$
$$\mathrm{Tor}_1^{e-e'} H_{i-1}(B_\bullet(\Gamma_v)) \to 0.$$

The result for i and Γ, using the "geometric" nature of tameness, implies that the kernel is tame.

To show the cokernel is tame, it suffices to show that the two modules

$$\frac{\mathrm{Tor}_1^{e-e'} H_{i-1}(B_\bullet(\Gamma_v))}{\mathrm{tor}_1} + \mathrm{tor}_1 \cap \mathrm{Tor}_1^{e-e'} H_{i-1}(B_\bullet(\Gamma_v))$$

are tame. The first uses the result for $i-1$ and Γ_v, and the second uses a spectral sequence and most of the relations from earlier.

Conjecture / theorem-in progress The dimension of $H_i(\mathrm{Conf}_k(\Gamma); \mathbb{F})$ is

$$\sim \left[\sum_{W} \frac{e^{r_\Gamma^W}}{2^i(\Lambda_\Gamma^i - 1)!} \prod_{w \in W} (d(w)-2) \right] k^{\Lambda_\Gamma^i - 1} k!,$$

where r_Γ^W is the number of components of $\Gamma \setminus W$ containing an essential vertex, $\Lambda_\Gamma^W = |\pi_0(\Gamma \setminus W)| - r_\Gamma^W$, and $\Lambda_\Gamma^i = \max\limits_{|W|=i} \Lambda_\Gamma^W$ is the $i^{\underline{th}}$ "leaf-cutting number" of Γ.