Binomial rings and homotopy theory

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Moduli and friends seminar, November 7th 2022

Algebraic invariants of homotopy types

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Answer : One can use cohomology with its cup product structure. We have $H^*(\mathbb{CP}^2) \cong \mathbb{Z}[x]/x^3$ with $|x| = 2$ and $H^*(S^2 \vee S^4) \cong \mathbb{Z}[x]/x^2 \times \mathbb{Z}[y]/y^2$ with $|x| = 2, |y| = 4$.

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Answer : One can use Steenrod operations. In $H^*(\mathbb{CP}^2; \mathbb{F}_2)$, there is a non-trivial Steenrod operation

$$Sq^2(x) = x^2$$

This remains true on $H^*(\Sigma\mathbb{CP}^2; \mathbb{F}_2)$:

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On the other hand, all Steenrod operations are trivial in the cohomology of $S^3 \vee S^5$.

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How to distinguish the homotopy type of $F_2(L_{7,1})$ and $F_2(L_{7,2})$?

Answer : [Salvatore Longoni, 2004]. The universal cover $\tilde{F}_2(L_{7,1})$ and $\tilde{F}_2(L_{7,2})$ are not homotopy equivalent because there is a non-trivial triple Massey product in $H^5(\tilde{F}_2(L_{7,2}); \mathbb{Q})$ whereas all Massey products are trivial in $H^*(\tilde{F}_2(L_{7,1}); \mathbb{Q})$.

Massey products

The Massey products and Steenrod operations come form the fact that there is highly structured multiplication at the chain level. Namely $C^*(X; R)$ is a dg-algebra and an E_∞-algebra.

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Construction

Given three cohomology classes $[x]$, $[y]$ *and* $[z]$ *in a dg-algebra* A *such that* $[x][y]=0$ *and* $[y][z]=0$, *we may form their triple Massey product*

$$\langle [x],[y],[z]\rangle = \{xb+az, db=yz \ \mathit{and}\ da=xy\}$$

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where $W \xrightarrow{\simeq} \mathbb{F}_2$ is a $\mathbb{F}_2[C_2]$ projective resolution.

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This induces a map

$$W^{C_2} \otimes C^* = (W \otimes C^*)^{C_2} \to C^*$$

taking homology, we get

$$Sq : H^*(C_2; \mathbb{F}_2) \otimes H^*(X; \mathbb{F}_2) \to H^*(X; \mathbb{F}_2)$$

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Definition

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In general, there are Σ_n-equivariant maps

$$W \otimes (C^*)^{\otimes n} \to C^*$$

satisfying compatibilities

Highly structured cochains

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$$[n] \mapsto \{f : \Delta^n \to X\}.$$

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Piecewise linear differential forms.

$$\Omega^*_{poly}(\Delta^n) := \mathbb{Q}[x_0, \ldots, x_n, dx_0, \ldots, x_n]/(\sum_{i=0}^{n} x_i = 1, \sum_{i=0}^{n} dx_i = 0)$$

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Theorem

The functor Ω^*_{poly} is quasi-isomorphic to $C^*(-;\mathbb{Q})$. In particular the cohomology of $\Omega^*_{poly}(X)$ is naturally the cohomology of X with its cup-product structure.

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The functor Ω^*_{poly} is a left adjoint functor

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is rationalisation when X is of finite type.

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Definition

1. *finite type : homology is degreewise finitely generated.*
2. *Nilpotent : connected, fundamental group is nilpotent and acts nilpotently on higher homotopy groups.*

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More generally, if X is not nilpotent (but still finite type). We have

$$\pi_1(X_{\mathbb{Q}}) \cong \pi_1(X)^{\wedge}_{\mathbb{Q}}$$

(Malcev completion)

Examples

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where $\mathfrak{pb}_n$ *is the Drinfeld-Konho Lie algebra.*

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Take $X = K(GL_{\infty}(\mathbb{Z}), 1)$*. Then* $X_{\mathbb{Q}}$ *has higher homotopy groups. They are the rational higher K-groups of* $\mathbb{Z}$ *(non trivial, computed by Borel).*

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Theorem (Mandell, 2006)

Let $X \mapsto C^*(X)$ the singular cochain functor.

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Theorem (Toën, 2020)

Same theorem for $X \mapsto \mathbb{Z}^X$.

$$\mathbb{Z}^{(-)} : \mathrm{HoS} \to \mathrm{Ho}(\mathrm{cRing})^{op}$$

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Fullness fails very badly !

Binomial ring

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Proposition

The ring $\mathrm{Num}[x_1, \dots, x_n]$ *is the free binomial ring on n variables.*

$$\mathrm{Hom}_{\mathrm{BRing}}(\mathrm{Num}[x_1, \dots, x_n], R) = R^n$$

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The functor $X \mapsto \mathbb{Z}^X$ from HoS to $\mathrm{Ho}(\mathrm{cBRing})^{op}$ is a left adjoint. The right adjoint is denoted $A \mapsto \langle A \rangle$.

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Remark

The forgetful functor

$$\mathrm{Ho}(\mathrm{cBRing}) \to \mathrm{Ho}(\mathrm{cRing})$$

is not fully faithful.

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Remark

The finite type hypothesis comes from the fact that we work with cochains instead of chains.
If we could define a chain functor with values in simplicial "binomial corings", there would be hope of being able to remove this hypothesis.

Sketch of proof

Any nilpotent space X is the limit of a tower

$$X \to \dots X_n \to X_{n-1} \to \dots \to X_0$$

where the map $X_n \to X_{n-1}$ is a principal fibration with fiber $K(A_n, i_n)$ with A_n a finitely generated abelian group and with $i_n \geq 1$ and the sequence i_n grows to $+\infty$.

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We can further reduce to proving that

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So we have

$$\langle \mathrm{Sym}^{bin}(DK^{-1}\mathbb{Z}[n]) \rangle \simeq \mathbb{R}\mathrm{map}_{\mathrm{cBRing}}(A, \mathbb{Z}) \simeq$$

$$\mathbb{R}\mathrm{map}_{\mathrm{cAb}}(DK^{-1}\mathbb{Z}[n], \mathbb{Z}) \simeq \mathbb{R}\mathrm{map}_{\mathrm{Ch}^*(\mathbb{Z})}(DK^{-1}\mathbb{Z}[n], \mathbb{Z}) \simeq K(\mathbb{Z}, n)$$

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We dualize to get $\mathbb{Z}[1]$

$$\mathbb{Z}[1]^n = \mathbb{Z}^n = Hom(\mathbb{Z}^n, \mathbb{Z})$$

with inner face maps given by the diagonals $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ and outer face maps given by the zero map $0 \to \mathbb{Z}$.

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$$\Delta : \mathrm{Num}[x] \to \mathrm{Num}[x, y]$$

given by $\Delta(f)(x, y) = f(x + y)$.
This can be identified with the cobar construction of $\mathrm{Num}[x]$.

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Vandermonde's identity :

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So the cobar construction of $\mathrm{Num}[x]$ has the same cohomology as $\mathbb{Z}^{S^1}$.