On the homology of big mapping class groups

I

Moduli and Friends seminar
IMAR
21 November 2022
28 November 2022

Joint work with Xiaolei Wu (Fudan Univ.)

arXiv preprints : 2211.07470
2212.(.........)

Plan:

talk I

1. Infinite type surfaces
2. Big mapping class groups
3. What was known about H_1 and H^1 (and H_2)
4. Our results: ① $H_i(\mathrm{Map}(\mathcal{B}(\Sigma))) = 0$ $(i \geq 1)$

   Coro: $H_i(\mathrm{Map}(\mathbb{R}^2 \setminus \mathrm{Cantor})) \cong \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$

   ② $H_i(\mathrm{Map}(\mathcal{L}(\Sigma)))$ is uncountable $\forall i \geq 1$ (cond. on Σ)

   E.g.: $H_i(\mathrm{Map}(\mathbb{R}^2 \setminus \mathbb{N}))$ is uncountable $\forall i \geq 1$

talk II

1. Proof of acyclicity
   - ↳ (homologically) dissipated groups
   - ↳ homological stability for big MCGs
2. Proof of uncountability
   - ↳ for Loch Ness monsters
   - ↳ branched double coverings

1. Infinite type surfaces

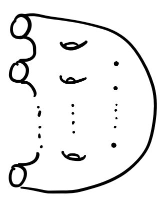
S surface $=$ $\begin{cases} \cdot \text{ Hausdorff} \\ \cdot \text{ second countable} \\ \cdot \text{ locally homeomorphic to } \mathbb{R}^2 \end{cases}$ (countable basis for its topology)

+ connected,
+ orientable

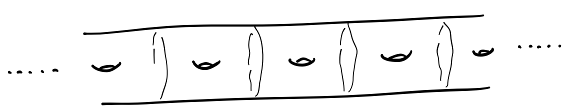
S has finite type $\iff$ $\pi_1(S)$ is finitely generated.

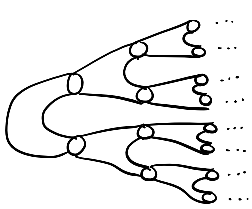
otherwise, S has infinite type (Lemma: then $\pi_1(S) \cong F_\infty$)

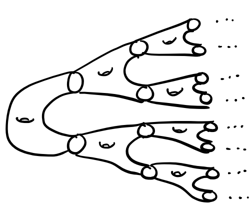
Examples :

$S = \Sigma_{g,b}^{n} =$  (these are all finite-type surfaces)

$S = L = \Omega_1 =$  "Loch Ness monster"

$S = \Omega_2 =$ 

$S =$  $\cong S^2 \setminus \text{Cantor}$ "Cantor tree surface"

$S =$  "Blooming Cantor tree surface"

Motivation / where do these arise "in nature"?

- Dynamics
  - S surface (finite type)
  - Dynamical system on S
    (e.g. Iterated Function System, iterates of a rational map)

$\rightsquigarrow$ attractor or Julia set $K \subseteq S$ compact

Often, $K \cong$ Cantor set, or closed subset of the Cantor set

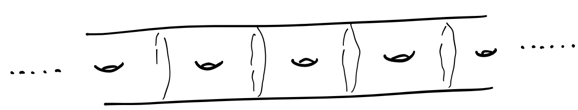
Moduli space of pairs $(S, K) \longleftrightarrow \mathrm{Homeo}(S, K) \cong \mathrm{Homeo}(S \setminus K)$

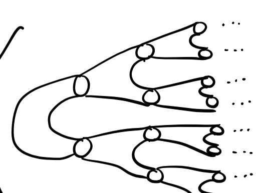
- Leaves of foliations of 3-manifolds

Ends of surfaces

$$\mathrm{Ends}(S) = \varprojlim_{K \text{ cpt} \subseteq S} \left(\pi_0 (S \setminus K) \right)$$

(topologised as an inverse limit of discrete spaces)

E.g. Ends () $= \{L, R\}$

Ends () $\cong$ Cantor set

Lemma

There is a natural topology on $\overline{S} = S \amalg \mathrm{Ends}(S)$ such that

- $\overline{S}$ is a compactification of S ("Freudenthal compactification")
- the subspace topology induced on $\mathrm{Ends}(S)$ agrees with the inverse limit topology described above.

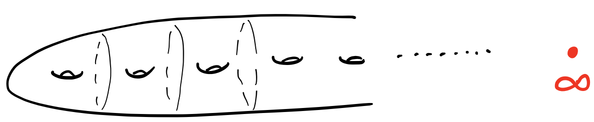
Definition

$e \in \mathrm{Ends}(S)$ is <u>planar</u> $:\Longleftrightarrow$ it has a neighbourhood in $\overline{S}$ that is a surface

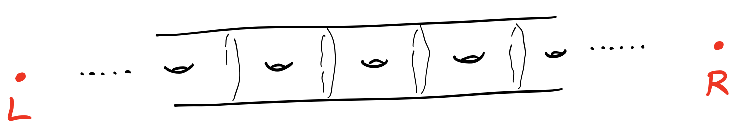
$\Updownarrow$ (Rmk)

it has a neighbourhood in $\overline{S}$ that is a <u>planar</u> surface

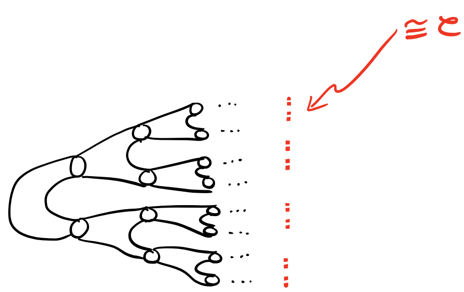
E.g.



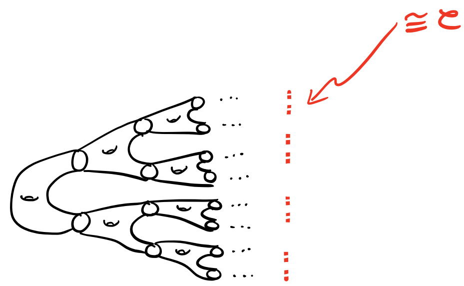
one non-planar end



both ends non-planar



$\cong S^2$ all ends are planar



all ends are non-planar

NB: When considering also non-orientable surfaces, there are 3 types of ends $e \in \mathrm{Ends}(S)$:

<u>planar</u> : $\exists$ nbhd $U \subseteq \overline{S}$ with $U \cap S$ a planar surface

<u>non-planar but orientable</u> : $\exists$ nbhd $U \subseteq \overline{S}$ with $U \cap S$ a non-planar, orientable surface

<u>non-orientable</u> : $\forall$ nbhd $U \subseteq \overline{S}$, the surface $U \cap S$ is non-orientable

Classification

[von Kerékjártó 1923, Richards 1963]

Given $g \in \mathbb{N} \cup \{\infty\}$
a space X that is homeo to a closed $\subseteq$ Cantor
a closed subset $Y \subseteq X$

with $Y \neq \emptyset$ iff $g = \infty$,

$\exists$ unique surface S with genus$(S) = g$
$(\mathrm{Ends}(S), \mathrm{Ends}_{np}(S)) \cong (X, Y)$.

subspace of non-planar ends

Moreover, every surface arises in this way.

In fact, any surface may be constructed from $\mathbb{S}^2$ by:

- removing a closed, totally disconnected subset X,
- taking connected sums with tori along a collection of pairwise disjoint discs in $\mathbb{S}^2 \setminus X$ "converging" to $Y \subseteq X$.

2. Big mapping class groups

Definition: The mapping class group of S is:

$$\mathrm{Map}(S) = \pi_0(\mathrm{Homeo}(S)) = \mathrm{Homeo}(S) / \mathrm{Homeo}_0(S)$$

Remarks:

(1) This is a topological group, as a quotient of $\mathrm{Homeo}(S)$.

compact-open topology

(2) As a space, it is

- Hausdorff
- zero-dimensional (basis of clopen sets)
- Polish —— separable & completely metrisable
- discrete $\Longleftrightarrow$ S has finite type

(3) In fact, $\mathrm{Map}(S)$ is

$$\begin{cases} \text{countable and discrete} & \text{if } S \text{ has finite type} \\ \text{homeomorphic to } \mathbb{N}^{\mathbb{N}} & \text{if } S \text{ has infinite type} \end{cases}$$

("Baire space")

Definition

$E = \mathrm{Ends}(S)$

$E_{np} = \mathrm{Ends}_{np}(S)$

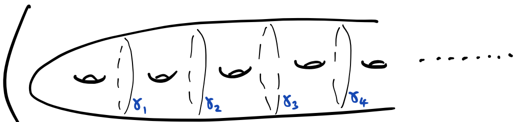
pure mapping class group

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{PMap}(S) \arrow[r, hook] & \mathrm{Map}(S) \arrow[r, two heads] & \mathrm{Homeo}(E, E_{np}) \arrow[r] & 1 \\ & \mathrm{Map}_c(S) \arrow[u, phantom, "\rotatebox{90}{$\subseteq$}"] & & & \end{tikzcd}$$

compactly-supported mapping class group

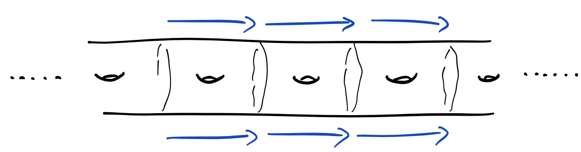
Interesting elements

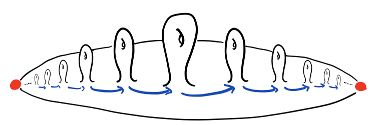
- Dehn twists $\in \mathrm{Map}_c(S)$
- Infinite products of Dehn twists $\in \overline{\mathrm{Map}_c(S)} \setminus \mathrm{Map}_c(S)$

$$\prod_{i=1}^{\infty} T_{\gamma_i} \in \overline{\mathrm{Map}_c} \Bigg($$ 

$$\Bigg)$$

- Handle slides $\in \mathrm{PMap}(S) \setminus \overline{\mathrm{Map}_c(S)}$

e.g. 

more generally: 

Requires at least 2 non-planar ends

Proposition

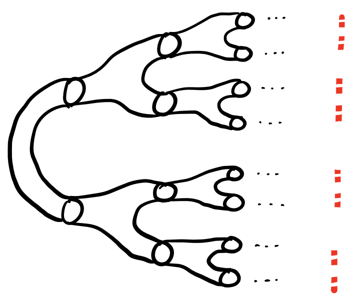
- Dehn twists generate $\mathrm{Map}_c(S)$ [classical]
- Dehn twists and handle slides generate a dense subgroup of $\mathrm{PMap}(S)$ [Patel - Vlamis]
- Hence $\overline{\mathrm{Map}_c(S)} = \mathrm{PMap}(S) \iff S$ has ≤ 1 non-planar end.

What about interesting elements of $\mathrm{Map}(S) \setminus \mathrm{PMap}(S)$?

$\Updownarrow$

interesting elements of $\mathrm{Homeo}(E, E_{np})$

Example :

$S =$ 

$= \mathcal{C}$

$$\mathrm{Map}(S) \twoheadrightarrow \mathrm{Homeo}(\mathcal{C})$$

interesting subgroup : Thompson's group V

This lifts naturally to $\mathrm{Map}(S)$:

$$\begin{tikzcd}
\pi_0(\mathrm{Diff}(\mathbb{S}^2, \mathcal{C})) \arrow[d, hook] \arrow[dr, "(*)"] & \\
\mathrm{Map}(\mathbb{S}^2 \setminus \mathcal{C}) \arrow[r, two heads] & \mathrm{Homeo}(\mathcal{C})
\end{tikzcd}$$

<u>Theorem</u> [Funar – Nerétin '18]

$$\mathrm{Image}((*)) = V^{\pm}$$

the <u>signed</u> <u>Thompson group</u> : $V \subset V^{\pm} \subset \mathrm{Homeo}(\mathcal{C})$

V and $V^{\pm}$ are <u>conjugate</u> in $\mathrm{Homeo}(\mathcal{C})$

3. The homology of mapping class groups

For finite-type surfaces $S = \Sigma_{g,b}^{n}$:

- $H_*(\mathrm{Map}(\Sigma_{0,1}^{n})) = H_*(B_n)$ completely computed [Arnol'd, F. Cohen, Fuchs, 1970s]
- $H_i(\mathrm{Map}(\Sigma_{g,b}^{n}))$ independent of g, b, n when $g \gg i$. [Harer '85]
- stable homology when $g \to \infty$ computed by [Madsen-Weiss '02] ($\mathbb{Q}$ coeff)
  [Galatius '04] ($\mathbb{F}_p$ coeff)

  (reduced to the case $(b,n) = (1,0)$ by [Bödigheimer-Tillmann '01])
- many other calculations known for small i, g, b, n

  (survey : [Bödigheimer-Boes-Kranhold '22])

For infinite-type surfaces

All previously-known results are for degree = 1 or 2 :

Pure mapping class groups

classes represented by separating 1-cycles

- genus ≥ 1 : $H^1(\mathrm{PMap}(S)) \cong H_1^{sep}(\overline{S}^{plan})$ [Aramayona-Patel-Vlamis $g \geq 2$] [Domat-Plummer $g = 1$]

fill in all planar ends of S

$$\left(\cong \mathbb{Z}\{\text{clopen subsets of } E = \mathrm{Ends}_{np}(S)\} \Big/ \begin{matrix} \emptyset = 0 \\ A = E \setminus A \\ A \cup B = A + B \\ \text{if } A \cap B = \emptyset \end{matrix} \right)$$

In particular,

- if S has ≤ 1 non-planar end $\longrightarrow H^1(\mathrm{PMap}(S)) = 0$
- $H^1($PMap( $)) \cong \mathbb{Z}$

- genus $= 0$: $H^1(\mathrm{PMap}(S))$ is uncountable. [Domat-Plummer]

- for any genus : $H_1(\overline{\mathrm{Map}_c(S)})$ contains $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ [Domat]

cardinality of the continuum $\cong 2^{\aleph_0}$

$= H_1(\mathrm{PMap}(S))$ if S has ≤ 1 non-planar end

Full mapping class groups

Cantor set embedded in S

- S finite type :

$$H_1(\mathrm{Map}(S \setminus \mathcal{C})) \cong H_1(\mathrm{Map}(S))$$ [Calegari-Chen]

In particular,

$$H_1(\mathrm{Map}(\mathbb{S}^2 \setminus \mathcal{C})) = 0$$
$$H_1(\mathrm{Map}(\mathbb{R}^2 \setminus \mathcal{C})) = 0$$ [Vlamis]
$$H_1(\mathrm{Map}(\mathbb{D}^2 \setminus \mathcal{C})) = 0$$

- $H_2(\mathrm{Map}(\mathbb{S}^2 \setminus \mathcal{C})) = \mathbb{Z}/2$ [Calegari-Chen]

- $H_1(\mathrm{Map}(\mathbb{R}^2 \setminus \mathbb{N}))$ contains $\bigoplus_{\mathfrak{c}} \mathbb{Q}$

  $H^1(\mathrm{Map}(\mathbb{R}^2 \setminus \mathbb{N})) = 0$ [Malestein-Tao]

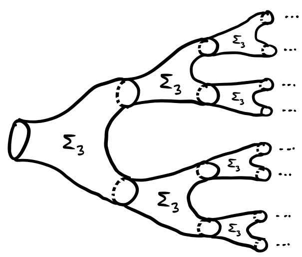
- Under certain coarse geometry-type conditions, [Field-Patel-Rasmussen]

  $H_1(\mathrm{Map}(S))$ is finitely generated

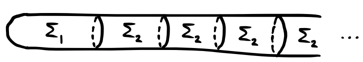
4. Our results (higher degrees)

Σ any surface (without boundary)

$\Sigma_n := \Sigma$ minus interiors of n pairwise-disjoint discs

$B(\Sigma) =$ 

"Binary tree surface"

$R(\Sigma) =$ 

"Ray surface"

__Examples__:

$B(\mathbb{S}^2) \cong \mathbb{D}^2 \setminus \mathcal{C}$ (Cantor tree surface minus a disc)

$R(\mathbb{S}^2) \cong \mathbb{R}^2$

$R(T^2) =$ Loch Ness monster surface

$R(\mathbb{R}^2) \cong \mathbb{R}^2 \setminus \mathbb{N}$ (Flute surface)

Exercise: $B(\mathbb{S}^2 \setminus \mathcal{C}) \cong \mathbb{D}^2 \setminus \mathcal{C}$

$R(\mathbb{S}^2 \setminus \mathcal{C}) \cong \mathbb{S}^2 \setminus \mathcal{C}$

__Theorem__ ① [P.-Wu, arXiv:2211:07470]

For any Σ, $\boxed{\widetilde{H}_*(\mathrm{Map}(B(\Sigma))) = 0}$

In particular, $\widetilde{H}_*(\mathrm{Map}(\mathbb{D}^2 \setminus \mathcal{C})) = 0.$

Corollary

$$H_i(\mathrm{Map}(\mathbb{R}^2 \setminus \mathcal{C})) \cong \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

proof : Apply the Lyndon-Hochschild-Serre spectral sequence to the central extension

$$1 \to \underset{\substack{\uparrow \\ \text{Dehn twist around} \\ \text{the boundary}}}{\mathbb{Z}} \to \mathrm{Map}(\mathbb{D}^2 \setminus \mathcal{C}) \to \mathrm{Map}(\mathbb{R}^2 \setminus \mathcal{C}) \to 1$$

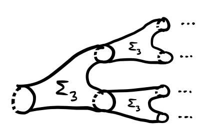
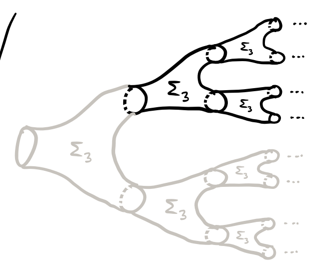
$$E^2 = \begin{array}{c|l} & \quad 0 \\ \hline 1 & H_*(\mathrm{Map}(\mathbb{R}^2 \setminus \mathcal{C})) \\ \hline 0 & H_*(\mathrm{Map}(\mathbb{R}^2 \setminus \mathcal{C})) \\ \hline & 0 \quad 1 \quad 2 \quad \cdots \end{array} \Rightarrow \begin{cases} \mathbb{Z} & \text{in degree } 0 \\ 0 & \text{in degrees} \geq 1 \end{cases}$$

$\square$

Idea of proof of Theorem 1

(more details in talk II)

- Key property :

Map() $\xrightarrow[\text{the identity}]{\text{extend by}}$ Map()

This is a H_*-isomorphism (proof by homological stability techniques)

- Using this, we adapt an argument of [Mather, '71] to prove acyclicity.

<u>Definition</u> $x \in X$ is <u>topologically distinguished</u> if it is not locally homeomorphic to any other point of X.

(If $x \in X$ has open nbhd U
$y \in X$ has open nbhd V
with $(U, x) \cong (V, y)$ then $y = x$.)

<u>Theorem</u> ② [P.-Wu, arXiv: 2212: ....]

If genus$(\Sigma) = 0$
Ends(Σ) has a topologically distinguished point

then

$H_i(\mathrm{Map}(R(\Sigma)))$ is <u>uncountable</u> for all $i \geq 1$

in fact:

$$\boxed{\Lambda^*\Big(\bigoplus_{\underset{2^{\aleph_0}}{c}} \mathbb{Z}\Big) \hookrightarrow H_*(\mathrm{Map}(R(\Sigma)))}$$

In particular, when $\Sigma = \mathbb{R}^2$: $\Lambda^*\Big(\bigoplus_c \mathbb{Z}\Big) \hookrightarrow H_*(\mathrm{Map}(\mathbb{R}^2 \setminus \mathbb{N}))$

<u>Remarks</u>

(1) The proof is (almost) constructive.

(2) Each generator of $\bigoplus_c \mathbb{Z} = \Lambda^1\Big(\bigoplus_c \mathbb{Z}\Big)$ is sent to an infinite product of powers of Dehn twists.

(3) Hence the embedding factors through

$$\Lambda^*\Big(\bigoplus_c \mathbb{Z}\Big) \hookrightarrow H_*(\mathrm{PMap}(R(\Sigma)))$$