On the homology of big mapping class groups II

Moduli and Friends seminar
IMAR
~~21 November 2022~~
28 November 2022

Joint work with Xiaolei Wu (Fudan Univ.)

arXiv preprints : 2211.07470
2212.(..........)

Plan:

Last week

1. Infinite type surfaces
2. Big mapping class groups
3. What was known about H_1 and H^1 (and H_2)
4. Our results: ① $H_i(\mathrm{Map}(B(\Sigma))) = 0 \quad (i \geq 1)$

   Coro: $H_i(\mathrm{Map}(\mathbb{R}^2 \setminus \mathrm{Cantor})) \cong \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$

   ② $H_i(\mathrm{Map}(R(\Sigma)))$ is uncountable $\forall i \geq 1$ (cond. on Σ)

   E.g.: $H_i(\mathrm{Map}(\mathbb{R}^2 \setminus \mathbb{N}))$ is uncountable $\forall i \geq 1$

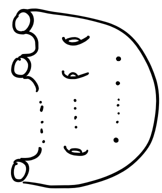
Today

0. Recap of
1. Proof of acyclicity
   - (homologically) dissipated groups
   - homological stability for big MCGs
2. Proof of uncountability
   - for Loch Ness monsters
   - branched double coverings

0. Recap.

Finite type surfaces S (connected, orientable) are classified:

$$S = \Sigma_{g,b}^{n} =$$



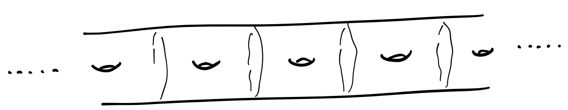
Infinite type surfaces (where $\pi_1(S)$ is allowed to be non-finitely generated but we still assume that ∂S is compact) may be much wilder:

Examples:

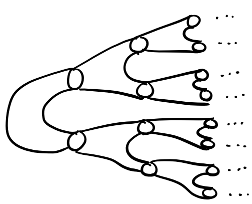
$$S = L = \Omega_1 =$$

"Loch Ness monster"

$$S = \Omega_2 =$$

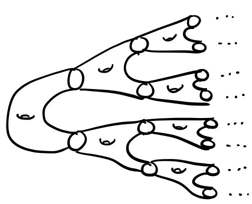


$$S =$$



$\cong S^2 \setminus \text{Cantor}$ "Cantor tree surface"

$$S =$$



"Blooming Cantor tree surface"

etc...

Motivation:

- Dynamics (complement of an attractor for a dynamical system on a finite-type surface S)
- Leaves of foliations of 3-manifolds.

Thm Surfaces are completely classified by: [von Kerékjártó, 1923 Richards, 1963]

- $b = \#$ of ∂-components $\in \mathbb{N}$
- $g =$ genus $\in \mathbb{N} \cup \{\infty\}$
- $\mathrm{Ends}(S) =$ space of ends
  $\cup\!\!|$
- $\mathrm{Ends}_{np}(S) =$ subspace of non-planar ends

$\in \left\{ \begin{array}{l} \text{pairs of spaces } Y \subseteq X \text{ where:} \\ \cdot\, Y \text{ is closed in } X \\ \cdot\, X \cong \text{closed subset of the Cantor set} \end{array} \right\} \Big/ \cong$

Moreover, every combination of $(b, g, \mathcal{E}, \mathcal{E}_{np})$ is realised by some surface S, as long as $g = \infty$ iff $\mathcal{E}_{np} \neq \emptyset$.

Definition: The mapping class group of S is:

$$\mathrm{Map}(S) = \pi_0(\mathrm{Homeo}_\partial(S)) = \mathrm{Homeo}_\partial(S) / \mathrm{Homeo}_0(S)$$

Homeo_∂: id on ∂S; Homeo_0: isotopic to id_S

It is called "big" if S has infinite type.

Interesting subgroups:

pure mapping class group

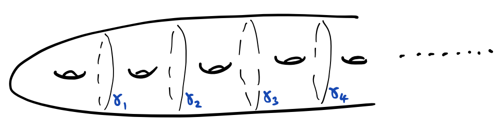
$$1 \to \mathrm{PMap}(S) \hookrightarrow \mathrm{Map}(S) \twoheadrightarrow \underbrace{\mathrm{Homeo}(\mathcal{E}, \mathcal{E}_{np})}_{\substack{\text{group of homeomorphisms} \\ \text{of } \mathcal{E} \text{ sending } \mathcal{E}_{np} \subseteq \mathcal{E} \\ \text{onto itself.}}} \to 1$$

$$\mathrm{PMap}(S) \supseteq \mathrm{Map}_c(S)$$

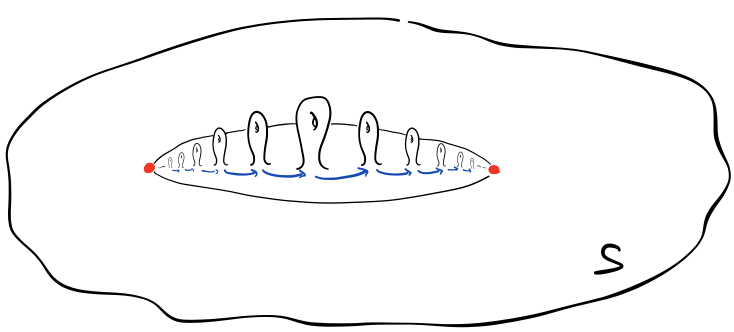
compactly-supported mapping class group

Interesting elements:

- Infinite products of Dehn twists.

e.g. $\prod_{i=1}^{\infty} T_{\gamma_i}$ for $S =$ 

- "Handle slides":



Requires at least 2 non-planar ends

Previous results about H_*(big MCGs):

Pure subgroup:

classes represented by separating 1-cycles

- genus ≥ 1 : $H^1(PMap(S)) \cong \underbrace{H_1^{sep}(\overline{S}^{plan})}_{\text{countable}}$ [Aramayona-Patel-Vlamis $g \geq 2$] [Domat-Plummer $g = 1$]

  fill in all planar ends of S

- genus $= 0$: $H^1(PMap(S))$ is uncountable. [Domat-Plummer]
- $|Ends_{np}(S)| \leq 1$: $H_1(PMap(S)) \supseteq \bigoplus_{c} \mathbb{Q}$ [Domat]

  cardinality of the continuum $\cong 2^{\aleph_0}$

Full mapping class group:

- S finite type :

  Cantor set embedded in S

  $$H_1(Map(S \setminus \mathcal{C})) \cong H_1(Map(S))$$

- $H_2(Map(S^2 \setminus \mathcal{C})) = \mathbb{Z}/2$

[Calegari-Chen]

- $H_1(Map(\mathbb{R}^2 \setminus \mathbb{N})) \supseteq \bigoplus_{c} \mathbb{Q}$
- $H^1(Map(\mathbb{R}^2 \setminus \mathbb{N})) = 0$

[Malestein-Tao]

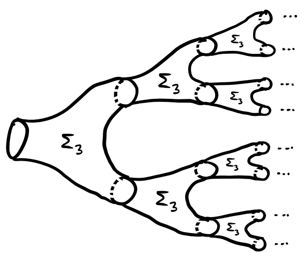
1. Acyclicity of binary tree surfaces

Definition:

Σ any surface (without boundary)

$\Sigma_n := \Sigma$ minus interiors of n pairwise-disjoint discs

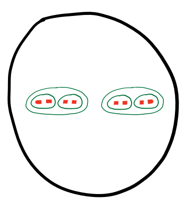
$B(\Sigma) :=$



"Binary tree surface"

E.g. $B(S^2) =$ Cantor tree surface $\setminus \mathring{D}^2$

$\cong D^2 \setminus \mathcal{C}$



Theorem: [P.-Wu, arXiv:2211.07470]

$$\widetilde{H}_*(\mathrm{Map}(B(\Sigma))) = 0$$

Corollary: $$H_*(\mathrm{Map}(\mathbb{R}^2 \setminus \mathcal{C})) \cong \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & * \text{ odd} \end{cases}$$

proof: Central extension

$$1 \to \mathbb{Z} \to \mathrm{Map}(\mathbb{D}^2 \setminus \mathcal{C}) \to \mathrm{Map}(\mathbb{R}^2 \setminus \mathcal{C}) \to 1$$

Dehn twist around the boundary

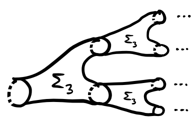
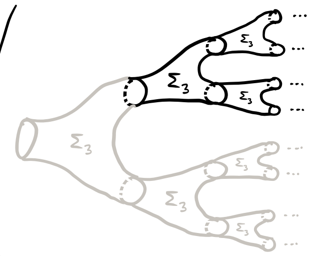
Lyndon-Hochschild-Serre spectral sequence

$$E^2 = \begin{array}{c|cccc} & & 0 & & \\ 1 & & H_*(\mathrm{Map}(\mathbb{R}^2 \setminus \mathcal{C})) & & \\ 0 & & H_*(\mathrm{Map}(\mathbb{R}^2 \setminus \mathcal{C})) & & \\ \hline & 0 & 1 & 2 & \cdots \end{array} \Rightarrow H_*(\mathrm{Map}(\mathbb{D}^2 \setminus \mathcal{C})) \cong \underbrace{\begin{cases} \mathbb{Z} & * = 0 \\ 0 & * \geq 1 \end{cases}}_{\text{Thm}}$$

$$E^2 = \begin{tikzcd}[row sep=small] & & & 0 & & & & \\ 1 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 & \mathbb{Z} \arrow[ull, "d_2"'] & 0 & \mathbb{Z} \arrow[ull, "d_2"'] & 0 & \mathbb{Z} \arrow[ull, "d_2"'] \\ & 0 & 1 & 2 & 3 & 4 & 5 & 6 \; \cdots \end{tikzcd}$$

//

Idea of proof of Theorem

e : Map () $\xrightarrow[\text{the identity}]{\text{extend by}}$ Map ()

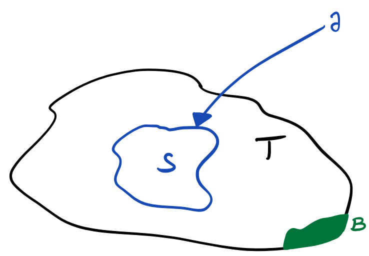
Two steps

(1) $e_* : H_*(\mathrm{Map}(B(\Sigma))) \longrightarrow H_*(\mathrm{Map}(B(\Sigma)))$ is an isomorphism

(2) $\widetilde{H}_*(\mathrm{Map}(B(\Sigma))) = 0$

Proof of (2) assuming (1):

T space
$S, B \subseteq T$ disjoint closed subspaces
$\partial := S \cap (\overline{T \setminus S})$



(*) $\quad \pi_0\left(\mathrm{Homeo}_\partial(S)\right) \xrightarrow[\text{on } T\setminus S]{\text{extend by id}} \pi_0\left(\mathrm{Homeo}_B(T)\right)$

Thm [P.-Wu]

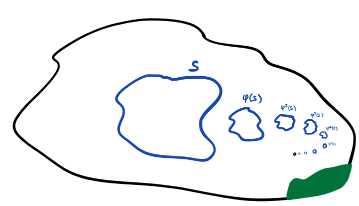
Suppose that

(1) (*) induces $\cong$ on H_*

(2) $\exists\, \varphi \in \mathrm{Homeo}_B(T)$ such that

(a) $\varphi^k(S) \cap S = \emptyset \qquad$ for all $k \geq 1$

(b) $\bigcup_{i=k}^{\infty} \varphi^i(S)$ is closed in $T \qquad$ for all $k \geq 0$

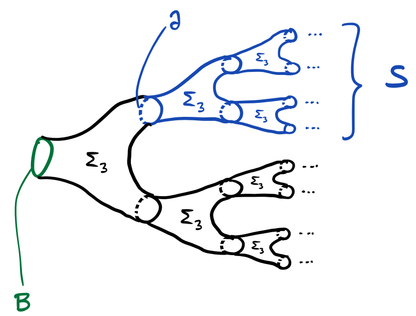


Then $\widetilde{H}_*\left(\pi_0\left(\mathrm{Homeo}_B(T)\right)\right) = 0$.

Rmks

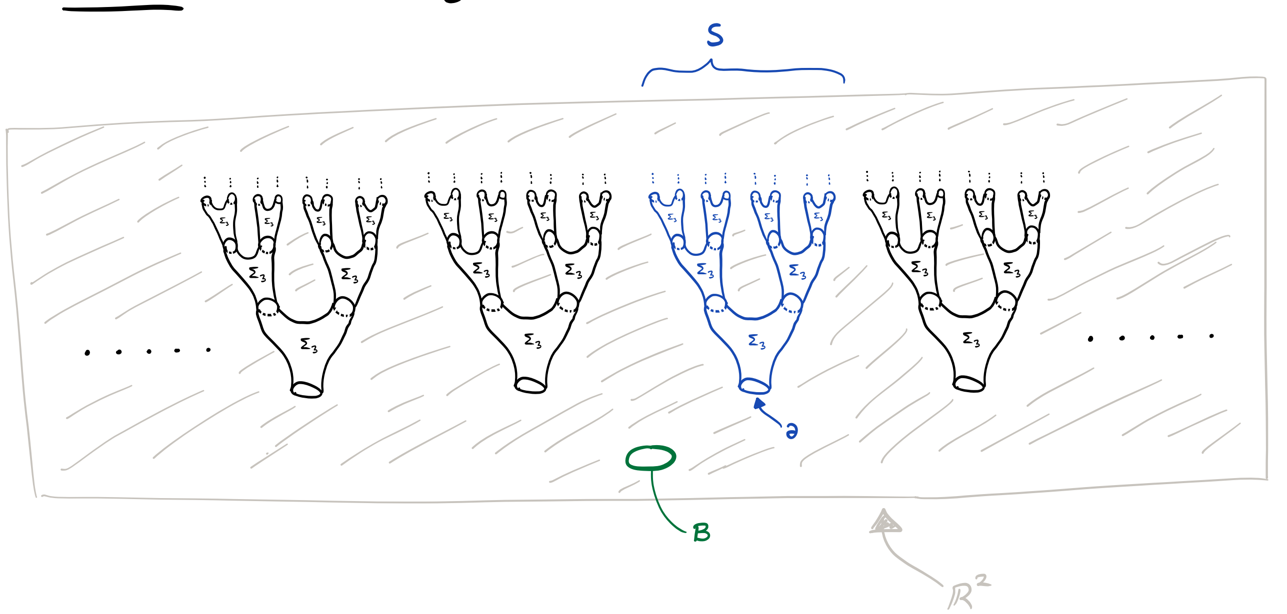
- Inspired by Mather's <u>infinite iteration argument</u>, which he used in 1971 to show that $\widetilde{H}_*(\mathrm{Homeo}_c(\mathbb{R}^d)) = 0$.
- Related to the notion of <u>dissipated groups</u> [Berrick], which are all acyclic.
- We call groups satisfying the hypotheses above <u>homologically dissipated</u>.

We will apply this to the setting:

$T =$ 

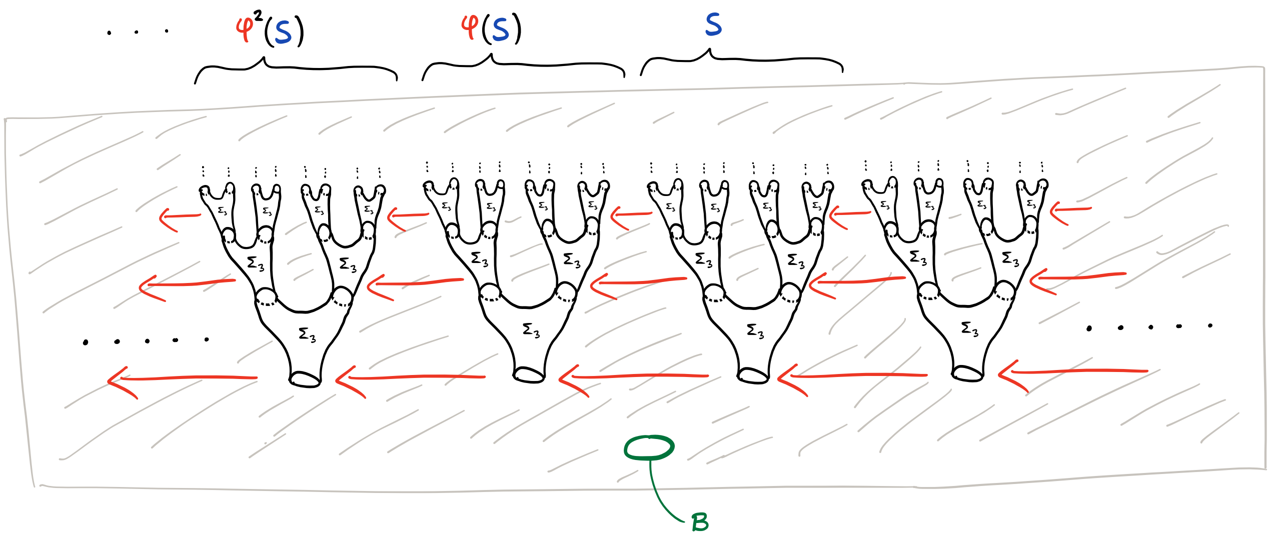
So assuming step (1), it is enough to construct a homeomorphism φ of $B(\Sigma)$, fixing its boundary, satisfying (a) and (b) above.

<u>Lemma</u>: The following is a homeomorphic picture of $T = B(\Sigma)$:



(The proof uses the classification of surfaces and the fact that $\left(\coprod_{i=1}^{\infty} \mathbb{C}\right)^+ \cong \mathbb{C}$.)

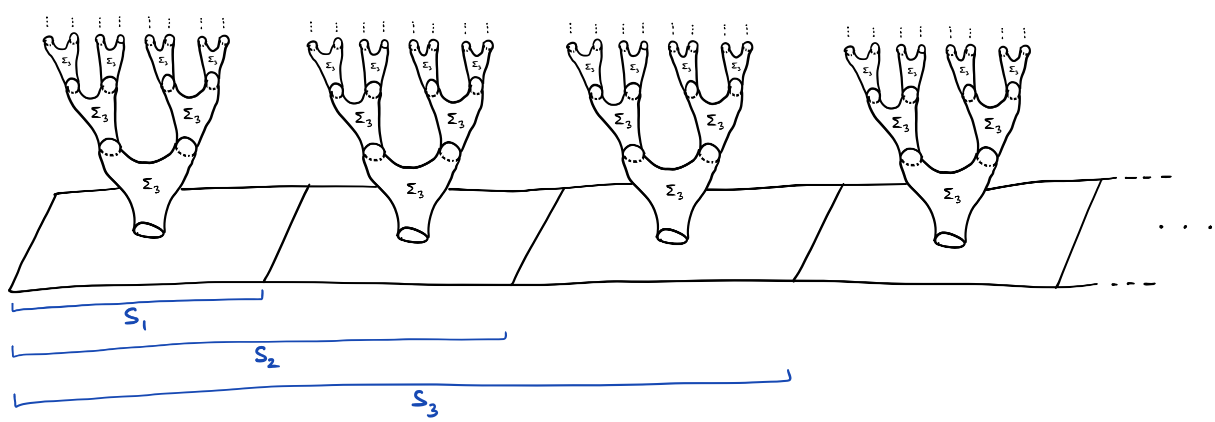
In this picture it is obvious how to define φ:



Hence ① ⇒ ②.

<u>Proof of ①</u>.

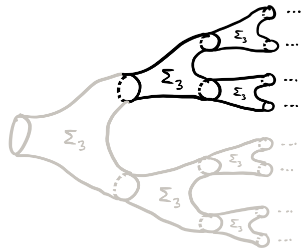
Consider the sequence of subsurfaces



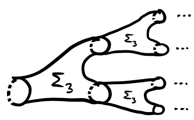
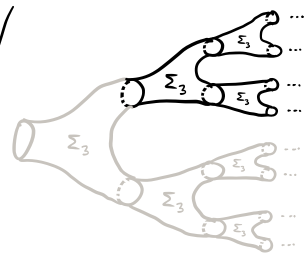
and the homomorphisms (induced by extending homeomorphisms by the identity)

$$\mathrm{Map}(S_1) \longrightarrow \mathrm{Map}(S_2) \longrightarrow \mathrm{Map}(S_3) \longrightarrow \cdots$$

Obs: Each pair of spaces (S_n, S_{n-1}) is isotopy equivalent to the pair



so each homomorphism $\mathrm{Map}(S_{n-1}) \longrightarrow \mathrm{Map}(S_n)$ is isomorphic to

Map () $\xrightarrow{e}$ Map ()

It therefore suffices to prove:

(∗∗) $H_i(\mathrm{Map}(S_{n-1})) \longrightarrow H_i(\mathrm{Map}(S_n))$ is an isomorphism for all $i \leqslant \frac{n-2}{2}$

i.e. <u>homological stability</u> for the sequence $\cdots \to \mathrm{Map}(S_n) \longrightarrow \mathrm{Map}(S_{n+1}) \to \cdots$

Steps:

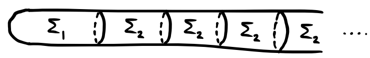
(A) The homological stability machine of [Randal-Williams-Wahl '17] reduces this to proving that <u>X_n is $(n-2)$-connected</u>
(X_n = a certain simplicial complex on which $\mathrm{Map}(S_n)$ acts)

(B) Gradually simplify X_n and then prove that it is $(n-2)$-connected.
[Many technical substeps!]

Rmk:
- Step ① is similar to Step 1 of the proof of [Szymik-Wahl '19] that $\tilde{H}_*(V) = 0$. (V = Thompson's group)
- But our Step ② (infinite iteration argument & homologically dissipated groups) is different to their Step 2 (algebraic K-theory).

2. Uncountability of the homology of linear surfaces

Definition:

$R(\Sigma) :=$ 

"Ray surface"

Examples: $R(T^2) =$ Loch Ness Monster surface

$R(\mathbb{R}^2) = R(S^2 \setminus pt) \cong \mathbb{R}^2 \setminus \mathbb{N}$ ("Flute surface")

Definition: $x \in X$ is *topologically distinguished* if it is not locally homeomorphic to any other point of X.

(If $x \in X$ has open nbhd U
$y \in X$ has open nbhd V
with $(U, x) \cong (V, y)$ then $y = x$.)

E.g.:

- Spaces that have topologically distinguished points:
  - $\{*\}$
  - $[0, \omega] \cong \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$
  - $[0, \omega^\alpha]$ (closed interval of ordinals in the order topology)
  - $\mathcal{C} \sqcup$ (any of the above)
- Spaces that do not have topologically distinguished points:
  - manifolds (except for the 1-point space)
  - the Cantor set $\mathcal{C}$

Theorem [P. - Wu, arXiv: 2212: ....]

If genus$(\Sigma) = 0$

Ends(Σ) has a topologically distinguished point

then

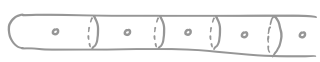
$H_i(\mathrm{Map}(R(\Sigma)))$ is <u>uncountable</u> for all $i \geq 1$

in fact :

$$\boxed{\Lambda^*\Big(\underbrace{\bigoplus}_{\mathfrak{c}} \mathbb{Z}\Big) \hookrightarrow H_*(\mathrm{Map}(R(\Sigma)))}$$

$\mathfrak{c} = 2^{\aleph_0}$

Idea of proof :

(for $\Sigma = \mathbb{R}^2$,
so $R(\Sigma) =$  .... $\cong \mathbb{R}^2 \setminus \mathbb{N}$)

① Prove that $\Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \hookrightarrow H_*(\mathrm{Map}(L))$

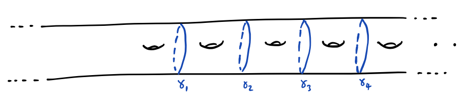
adapting methods of [Domat]

L — Loch Ness Monster surface

② Deduce that $\Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \hookrightarrow H_*(\mathrm{Map}(\mathbb{R}^2 \setminus \mathbb{N}))$

adapting methods of [Malestein - Tao]

Proof of Step ①

$L' = L \setminus pt \cong$ 

For an infinite subset $A \subseteq \mathbb{N}$, consider

$$\prod_{a \in A} \left(T_{\gamma_a}\right)^{n_a} \in \mathrm{Map}(L')$$

Thm [Domat] If $\{n_a \mid a \in A\}$ is unbounded then this element is $\neq 0$ in $\mathrm{Map}(L')^{ab}$.

Consider

$$f_A = \prod_{a\in A} (T_{\gamma_a})^{a!} \in \mathrm{Map}(L')^{ab} \qquad \text{for infinite } A \subseteq \mathbb{N}$$

Obs (1) These pairwise commute (already in $\mathrm{Map}(L')$).

(2) $f_A \neq 0$

(3) if $A \cap B$ is finite then $f_A f_B \neq 0$

] by [Domat]

(4) f_A is divisible:

- $f_{A\cap \mathbb{N}_{\geq n}}$ is divisible by n
- $f_{A\cap \mathbb{N}_{\geq n}} \cdot f_A^{-1}$ = finite product of Dehn twists, supported on a subsurface $\cong \Sigma_{g,2}$ for some finite $g \geq 3$.

[Birman, Powell] $\Rightarrow$ product of commutators.

Choose an uncountable (continuum) family of infinite $\subseteq \mathbb{N}$ whose pairwise intersections are finite. (E.g. $\mathbb{N} \cong \mathbb{Q}$ and choose a sequence in $\mathbb{Q}$ converging to each $a \in \mathbb{R}$.)

$$\begin{tikzcd}
\bigoplus_c \mathbb{Z} \arrow[r, "\text{Obs (1)}"] \arrow[d, hook] & \mathrm{Map}(L') \arrow[d, two heads] \\
\bigoplus_c \mathbb{Q} \arrow[r, "{\text{Obs (4)}}"', "{\text{injective by Obs (2) + (3) + } \varepsilon}"'] & \mathrm{Map}(L')^{ab}
\end{tikzcd}$$

Coro [Domat] $H_1(\mathrm{Map}(L'))$ is uncountable.

But one can deduce more :

- Every injection $A \hookrightarrow B$ of abelian groups splits if A is divisible.
- Hence the inclusion $\bigoplus_c \mathbb{Z} \hookrightarrow \bigoplus_c \mathbb{Q}$ factors through $\mathrm{Map}(L')$.
- The induced map $H_*\left(\bigoplus_c \mathbb{Z}\right) \longrightarrow H_*\left(\bigoplus_c \mathbb{Q}\right)$ is injective.

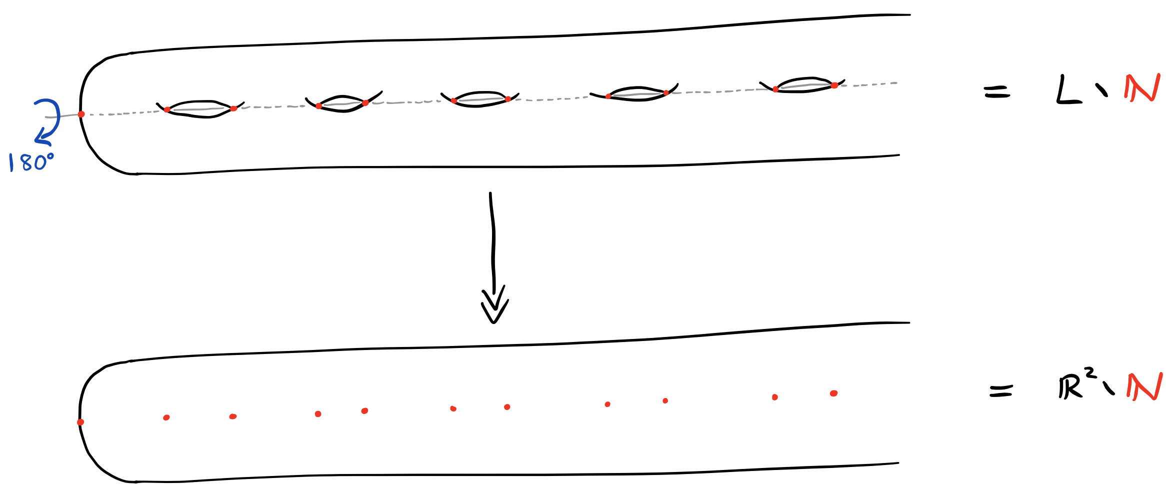
$$\begin{tikzcd}
H_*\left(\bigoplus_c \mathbb{Z}\right) \arrow[rr] \arrow[d, phantom, "\cong" rotate=90] & & H_*\left(\bigoplus_c \mathbb{Q}\right) \arrow[d, phantom, "\cong" rotate=90] \\
\Lambda^*\left(\bigoplus_c \mathbb{Z}\right) \arrow[rr] \arrow[dr] & & \Lambda^*\left(\bigoplus_c \mathbb{Q}\right) \\
& H_*(\mathrm{Map}(L')) \arrow[ur] &
\end{tikzcd}$$

Thus $\Lambda^*\left(\bigoplus_c \mathbb{Z}\right)$ injects into $H_*(\mathrm{Map}(L'))$.

Applying the Birman exact sequence $1 \to \pi_1(L) \to \mathrm{Map}(L') \to \mathrm{Map}(L) \to 1$ during the argument $\Rightarrow$ $\Lambda^*\left(\bigoplus_c \mathbb{Z}\right)$ injects into $H_*(\mathrm{Map}(L))$.

<u>Proof of Step ②</u>

Double covering : [Malestein - Tao]

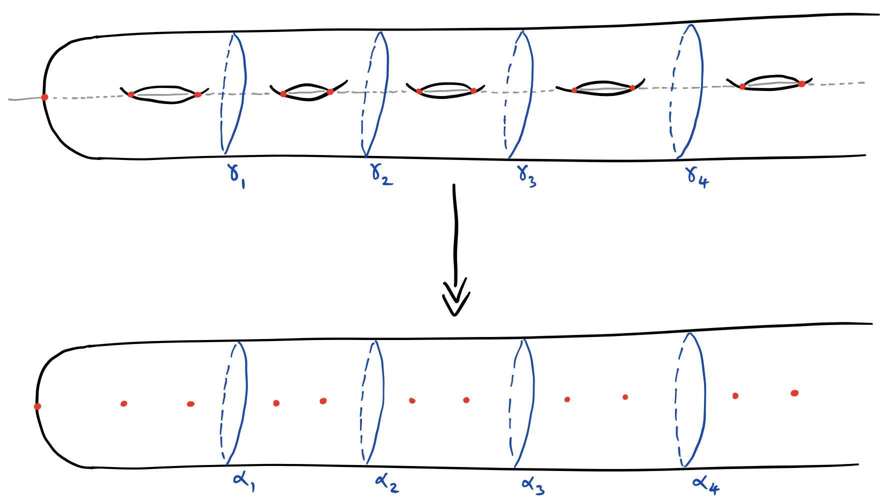


<u>Lemma</u> : The action $\mathrm{Map}(\mathbb{R}^2 \setminus N) \curvearrowright \pi_1(\mathbb{R}^2 \setminus N)$ preserves the index-2 subgroup corresponding to this double covering.

Thus we have homomorphisms

$$\mathrm{Map}(\mathbb{R}^2 \setminus \mathbb{N}) \xrightarrow[\text{unique lifting}]{} \mathrm{Map}(L \setminus \mathbb{N}) \xrightarrow[\substack{\text{unique extension}\\ \text{after filling in all}\\ \text{planar ends}}]{} \mathrm{Map}(L)$$

Note that $(T_{\alpha_i})^2 \longmapsto T_{\gamma_i}$



Hence we have a lift:

The generator corresp. to $A \subseteq \mathbb{N}$ is sent to $\prod_{a \in A} (T_{\alpha_a})^{2a!}$.

The generator corresp. to $A \subseteq \mathbb{N}$ is sent to $\prod_{a \in A} (T_{\gamma_a})^{a!}$.

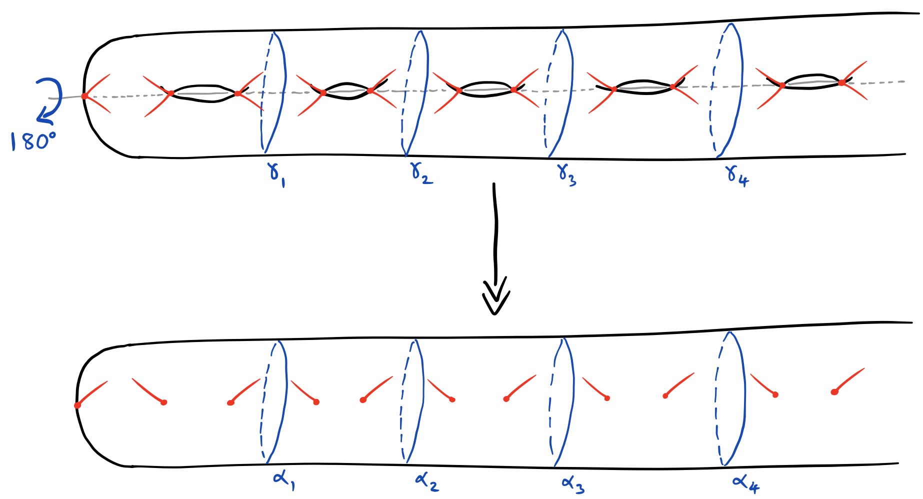
$$\begin{tikzcd}
 & \bigoplus_{c} \mathbb{Z} \arrow[dl, "(2)"'] \arrow[dr, "(1)"] & \\
\mathrm{Map}(\mathbb{R}^2 \setminus \mathbb{N}) \arrow[r] & \mathrm{Map}(L \setminus \mathbb{N}) \arrow[r] & \mathrm{Map}(L)
\end{tikzcd}$$

By the previous step, the map (1) is injective on H_*. Hence so is (2).

$\square$

BONUS: Generalising the proof to any Σ of genus 0 where $E = \mathrm{Ends}(\Sigma)$ has a topologically distinguished point.

We only have to modify Step ②.



$= L \smallsetminus (E \vee E).\omega$

$X.\omega = \coprod_{i=1}^{\infty} X$

$= \mathbb{R}^2 \smallsetminus E.\omega \cong R(\Sigma)$

Lemma: The action $\mathrm{Map}(\mathbb{R}^2 \smallsetminus E.\omega) \curvearrowright \pi_1(\mathbb{R}^2 \smallsetminus E.\omega)$ preserves the index-2 subgroup corresponding to this double covering.

The proof uses in an essential way the fact that E has a topologically distinguished point (and we have arranged that each embedding $E \hookrightarrow \mathbb{R}^2$ sends this point to a branch point of the branched double covering $L \twoheadrightarrow \mathbb{R}^2$).

Then we have:

$$\begin{tikzcd}
& \bigoplus_c \mathbb{Z} \arrow[dl] \arrow[dr] & \\
\mathrm{Map}(\mathbb{R}^2 \smallsetminus E.\omega) \arrow[r] \arrow[d, equal] & \mathrm{Map}(L \smallsetminus (E \vee E).\omega) \arrow[r] & \mathrm{Map}(L) \\
\mathrm{Map}(R(\Sigma)) & &
\end{tikzcd}$$

$\square$

Rmk The theorem becomes false if we remove the hypothesis that $\mathrm{Ends}(\Sigma)$ has a topologically distinguished point.

For example, if Σ = Cantor tree surface

then $\mathcal{R}(\Sigma) \cong$ Cantor tree surface

$H_i(\mathrm{Map}(\mathcal{R}(\Sigma)))$ is not uncountable for $i = 1, 2$.

$$\begin{array}{l} H_i(\mathrm{Map}(\mathcal{R}(\Sigma))) \\ \quad \cong \\ H_i(\mathrm{Map}(\text{Cantor tree surface})) \cong \begin{cases} 0 & i=1 \quad \text{[Calegari]} \\ \mathbb{Z}/2 & i=2 \quad \text{[Calegari-Chen]} \end{cases} \end{array}$$

Rmk The proof may be generalised slightly to prove that:

$$\Lambda^*\left(\bigoplus_{\mathfrak{c}} \mathbb{Z}\right) \hookrightarrow H_*(\mathrm{Map}(\mathcal{R}(\Sigma) \# S)),$$

whenever:

- Σ has genus 0, no boundary and $\mathrm{Ends}(\Sigma)$ has a topologically distinguished point e_0.
- S has finite genus, finitely many ∂-components and $\mathrm{Ends}(S)$ has no point that is locally homeomorphic to e_0.