Main Idea:

Construct new examples of highly symmetric line arrangement using existing old-fashioned examples as building blocks.

Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of d lines

Consider $\mathcal{D} : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ the dual operator

the set of lines $\mapsto$ the set of points

$$\ell : ax+by+cz=0 \mapsto (a:b:c) \in \mathbb{P}^2_{\mathbb{C}}.$$

For a fixed subset $\overline{m} \subset \mathbb{Z}_{\geq 0}$, we define $\mathcal{D}_{\overline{m}}(\mathcal{L})$ that sends the line arrangement $\mathcal{L}$ to the line arrangement in the dual plane which is the union of the lines containing exactly m points of $\mathcal{D}(\mathcal{L})$ for each $m \in \overline{m}$.

Define $\Lambda_{\overline{n},\overline{m}} = \mathcal{D}_{\overline{n}} \circ \mathcal{D}_{\overline{m}}$

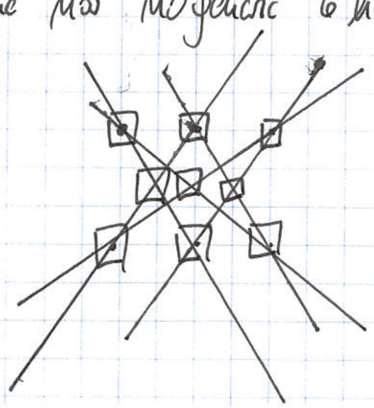
Explain: How does it work?

Consider $\Lambda_{\{2\},\{k\}}$: it returns the union of lines containing exactly k double points of $\mathcal{L}$.
However, for some $\overline{n}, \overline{m} \subset \mathbb{N}$, the result of $\Lambda_{\overline{n},\overline{m}}$ might be empty.

If we take 6 generic lines, we have only 15 double intersections, and

$\Lambda_{\{2\},\{3\}} = \emptyset$ → genericity assumption means no 3 points on a line.
However, we have 5-rich lines & ordinary lines!

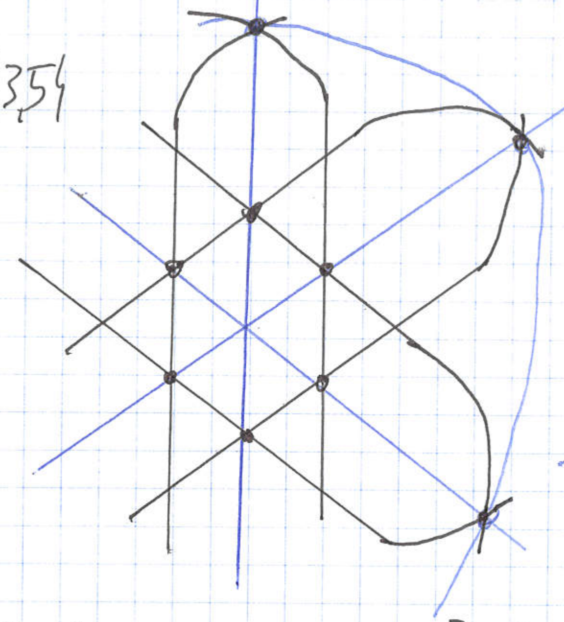
Take now non-generic 6 lines



$\Lambda_{\{2\},\{3\}} \neq \emptyset$!

Take a regular hexagon (that can be difficult to draw by hand).

$\Lambda_{\{2\},\{3,5\}}$



$A_1(10) = \Lambda_{\{2\},\{3,5\}}(H_6)$

$t_4 = 3$

$t_3 = 7$

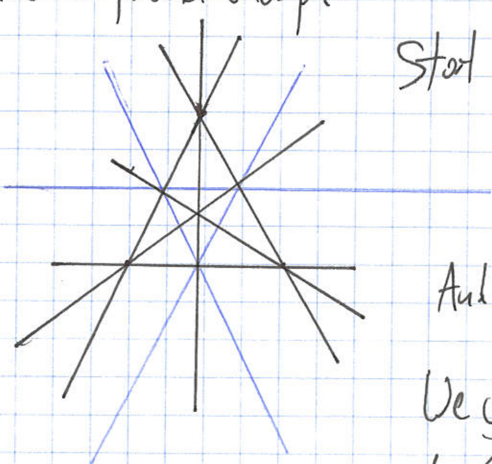
$t_2 = 6$

This arrangement is simplicial!

Def/Theorem: A line arrangement $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{R}}$ of $d \geq 3$ lines with $t_d = 0$ is simplicial $\iff$ $t_2 = 3 + \sum_{r \geq 4} (r-3) t_r$.

this condition means that $M(\mathcal{L}) := \mathbb{P}^2_{\mathbb{R}} \setminus \bigcup_{l_i \in \mathcal{L}} l_i$ has only open simplices as connected components!

Slightly more complicated example:



Start with $A_1(6)$

$t_3 = 4$

$t_2 = 3$

And we apply $\Lambda_{\{2,3\},\{2,3\}}(A_1(6))$

We get a line arrangement with $d = 9$ lines & $t_2 = 6$, $t_3 = 4$, $t_4 = 3$.

We can check that $t_2 = 6 = 3 + 3 \cdot 1 = 6$, so we get a simplicial line arrangement.

However, such a trick is rather rare, sometimes we get something being far from simplicial.

Joint work with X. Roulleau.

- construct new examples of free line arrangements with $d \gg$ lines.
$\to$ to be done, construct new examples of line arrangements with non-trivial monodromy.

We denote by $\Lambda_{n,m} := \Lambda_{\{n \geq 0\}, \{m \geq 0\}}$ = the operator that returns the line arrangement which is the union of the lines containing at least m-points of multiplicity at least n!

We denote by $t_n :=$ the number of n-fold intersections.

Let $S = \mathbb{C}[x,y,z]$,

$C: f=0$ reduced plane curve, $J_f = \langle \partial_x f, \partial_y f, \partial_z f \rangle$.

Definition: Let $C: f=0$ be a reduced plane curve of degree d, $f \in S$.

$M(f) := S / J_f$ the associated Milnor algebra.

C is called m-syzygy if $M(f)$ has the following mgfr (a MGFR):

$$0 \to \bigoplus_{i=1}^{m-2} S(-e_i) \to \bigoplus_{i=1}^{m} S(1-d-d_i) \to S^3(1-d) \to S \to M(f) \to 0$$

with $e_1 \leq e_2 \leq \ldots \leq e_{m-2}$ & $1 \leq d_1 \leq d_2 \leq \ldots \leq d_m$.

The m-tuple $(d_1, \ldots d_m)$ is called the set of the exponents of C.

Def: We say that C is free iff C is 2-syzygy and $d_1 + d_2 = d-1$.
degree $C = d$

First result:

There exists $\mathcal{H}_{57} \subset \mathbb{P}^2_{\mathbb{C}}$ with $d = 57$ lines & $t_2 = 252$, $t_3 = 120$, $t_4 = 72$, $t_8 = 21$.
$\mathcal{H}_{57}$ is free with $(d_1, d_2) = (25, 31)$.

Moreover, it turns out that the moduli space of $\mathcal{H}_{57}$ is 0-dimensional (so it is rigid).

Construction $\mathcal{H}_{57}$ uses the Hesse arrangement of $d = 12$ & $\cancel{t_4} = 9$, $t_2 = 12$.
We apply $\Delta_{2,2}$ getting $\mathcal{H}_{57}$.

<u>Moduli question:</u>

$M = (\{1, \dots n\}, \underset{\uparrow \text{bases}}{B})$ loopless simple matroid.

A realization of M over a given field $\mathbb{F}$ is a matrix $\mathbb{F}^{3 \times n}$ such that for all subsets $P \subset \{1, \dots, n\}$ of size 3 we have

$$(*) \det X_P \neq 0 \iff P \in B,$$

where X_P is the 3×3 submatrix consisting of the columns indexed by P.

∇ The kernels of the linear forms given by columns define an arrangement $\mathcal{L}$ of n lines.
The condition $(*)$ defines an ideal I' in the ring $\mathcal{R} = R[\alpha]$

where $R = \mathbb{Z}[x_{ij},\ i \in \{1,2,3\},\ j \in \{1, \dots n\}]$.

$$I' = \langle \det(X_N),\ \underset{\text{basis}}{N \subset E \text{ not}}\ |N| = 3 \rangle + \langle 1 - \alpha \prod_{B \in \mathcal{B}} \det X_B \rangle < R[\alpha]$$

where $X = (x_{ij}) \in \mathbb{F}^{3 \times n}$ is a $3 \times n$ matrix having the variables x_{ij} as the entries.

Then $R(M; \mathbb{F}) := V(I') \subset \mathbb{A}^{3n+1}_{\mathbb{F}} = \operatorname{Spec} R[\alpha] \to \operatorname{Spec} \mathbb{Z}$

this is a scheme parametrizing equivalence classes of point configurations in $\mathbb{P}^2_{\mathbb{F}}$.

The moduli space $\mathcal{H}_{57}$ is 0-dimensional $\Rightarrow$ $\mathcal{H}_{57}$ is rigid (two points that are Galois conjugate).

<u>Theorem B.</u> There exists a rigid arrangement $\mathcal{O}_{33}$ in $\mathbb{P}^2_{\mathbb{R}}$ with $d = 33$
such that $\cancel{t_2} = 128$, $\cancel{t_3} = 40$, $t_5 = \cancel{16}$, $t_8 = 5$.
$\mathcal{O}_{33}$ is free with $(d_1, d_2) = (15, 17)$ & is defined over $\mathbb{Q}(\sqrt{2})$.

Apply $\Delta_{\{2\},3}$ to a regular octagon $\mathcal{E}_8$. $\Rightarrow$ we get the union of at least 3-rich lines.

We can do more:

$$\mathcal{D}_{61} = \Lambda_{324,3}(\mathcal{C}_{10}), \quad \mathcal{D}_{49} = \Lambda_{423,4}(\mathcal{C}_{12})$$

$t_2 = 335$, $t_3 = 140$

$t_5 = 70$, $t_{10} = 1$

$t_{15} = 5$

defined over $\mathbb{Q}(\sqrt{5})$

free, rigid

$t_2 = 204$, $t_4 = 6$ $t_6 = 6$ $t_{12} = 1$

$t_3 = 96$ $t_5 = 24$ $t_7 = 12$

defined over $\mathbb{Q}(\sqrt{3})$, rigid.
free