State-sum homotopy invariants of maps from 3-manifolds to 2-types

Moduli and Friends Seminar

Kürşat Sözer (joint with Alexis Virelizier)

McMaster University

Contents

Functorial Topological Quantum Field Theories

n-dimensional cobordism category

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- closed oriented $(n-1)$-dimensional manifolds as objects,
- diffeomorphism classes (relative ∂) of n-dimensional oriented cobordisms as morphisms.
- $\circ$: gluing manifolds along common boundary components

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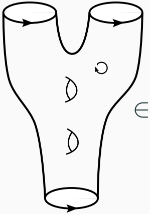

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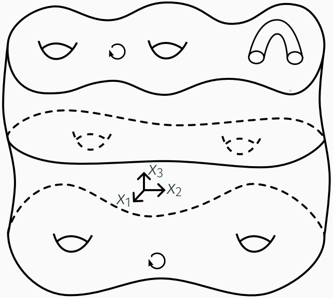

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$\rightarrow (\mathrm{Cob}_n, \coprod)$ is a symmetric monoidal category.

Functorial topological quantum field theories

The category $\mathrm{Vect}_{\mathbb{C}}$ of complex vector spaces has

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- linear transformations as morphisms.

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An n-dimensional topological quantum field theory (TQFT) is a symmetric monoidal functor

$$Z : \mathrm{Cob}_n \to \mathrm{Vect}_{\mathbb{C}}.$$

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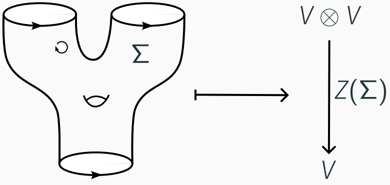

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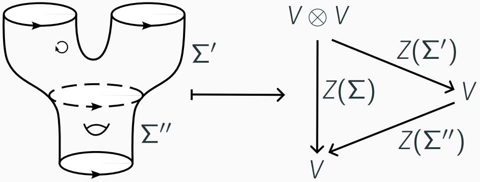

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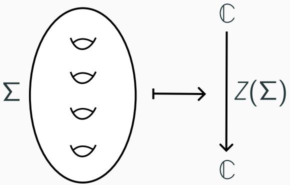

n-dimensional TQFTs produce numerical diffeomorphism invariants of closed n-manifolds which are multiplicative with respect to disjoint union operation and behave well under cut-paste operations.

Examples of TQFTs in Low-dimensions

1-dimensional TQFTs

- $\mathrm{Ob}(\mathrm{Cob}_1)$: closed oriented 0-dimensional manifolds
- $\mathrm{Ob}(\mathrm{Cob}_1)$: finitely many oriented points
- Under the operation $\coprod$, the collection $\mathrm{Ob}(\mathrm{Cob}_1)$ is generated by two objects; namely, $\bullet_+$ and $\bullet_-$.

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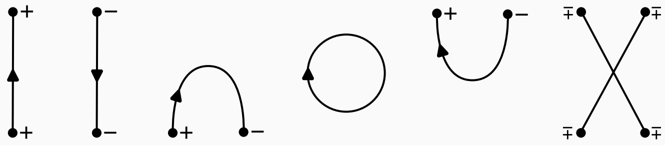

1-dimensional TQFTs

Given a 1-dimensional TQFT $Z\colon \mathrm{Cob}_1 \to \mathrm{Vect}_{\mathbb{C}}$ with the following data

- $Z(\bullet +) = V$
- $Z(\bullet -) = W$
- $Z(\mathrm{ev})\colon V \otimes W \to \mathbb{C}$
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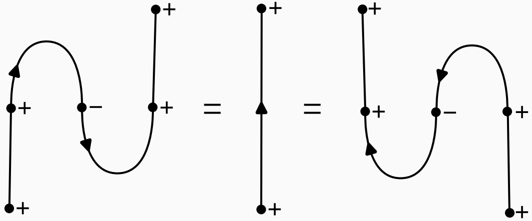
The equivalence of the following cobordisms implies that $Z(\mathrm{ev})$ is a nondegenerate bilinear pairing, so $W \cong V^*$.



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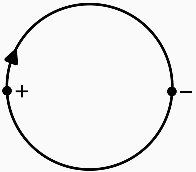
Let us compute the numerical invariant associated with connected closed oriented 1-manifold, namely the circle.



$$\xrightarrow{\;Z\;} \quad \begin{array}{ccc} \mathbb{C} & & 1 \\ \downarrow & & \downarrow \\ V \otimes V^* & \ni & \sum_i e_i \otimes e_i^* \\ \downarrow & & \downarrow \\ \mathbb{C} & & \boldsymbol{dim}(V) \end{array}$$

2-dimensional TQFTs

- $\mathrm{Ob}(\mathrm{Cob}_2)$: closed oriented 1-dimensional manifolds
- $\mathrm{Ob}(\mathrm{Cob}_2)$: finitely many oriented circles

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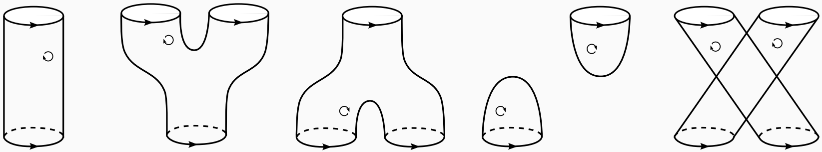

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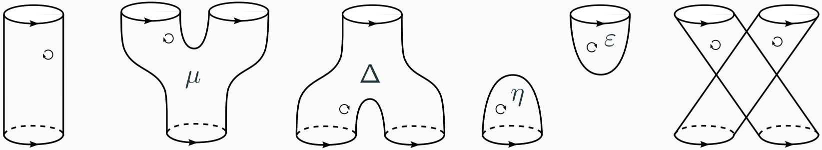
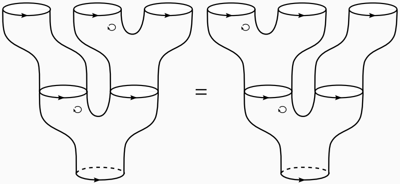
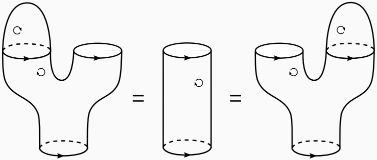


Given a 2-dimensional TQFT $Z\colon \mathrm{Cob}_2 \to \mathrm{Vect}_{\mathbb{C}}$, with the following data

- $Z(S^1) = V$
- $Z(\mu)\colon V \otimes V \to V$
- $Z(\Delta)\colon V \to V \otimes V$
- $Z(\eta)\colon \mathbb{C} \to V$
- $Z(\varepsilon)\colon V \to \mathbb{C}$.





Definition

Frobenius algebra is a finite dimensional associative, unital algebra V equipped with a nondegenerate bilinear form $\sigma \colon V \otimes V \to \mathbb{C}$ satisfying $\sigma(a \cdot b, c) = \sigma(a, b \cdot c)$ for all $a, b, c \in V$.

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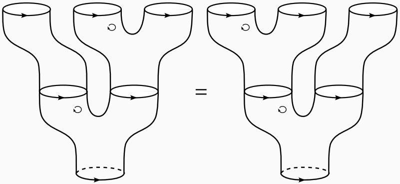

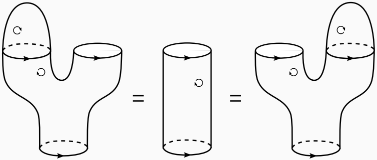

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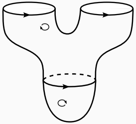
$\sigma :=$ 

Theorem (Abrams, Kock)

2-dimensional TQFTs are classified by commut. Frobenius algebras.

Homotopy Quantum Field Theories

Endowing cobordism category with continuous maps

Fix a connected pointed CW complex (X, x) (called the *target space*). The n-dimensional X-cobordism category XCob_n has

- closed oriented pointed $(n-1)$-dimensional manifolds equipped with continuous pointed maps as objects
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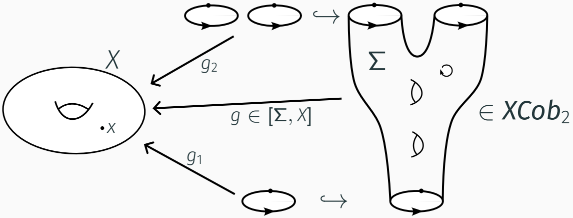

Homotopy quantum field theories

Definition (Turaev)

An n-dimensional homotopy quantum field theory with target X (X-HQFT) is a symmetric monoidal functor

$$Z : \mathrm{XCob}_n \to \mathrm{Vect}_{\mathbb{C}}.$$

- For $X = \{\bullet\}$, we have X-HQFT= TQFT.
- For any target X, we have $\mathrm{Cob}_n \hookrightarrow \mathrm{XCob}_n$ by introducing points on connected components and taking constant maps.
- When $X \simeq K(G, 1)$ for some group G, one can replace continuous pointed maps on objects of XCob_n with pointed homotopy classes of continuous pointed maps.

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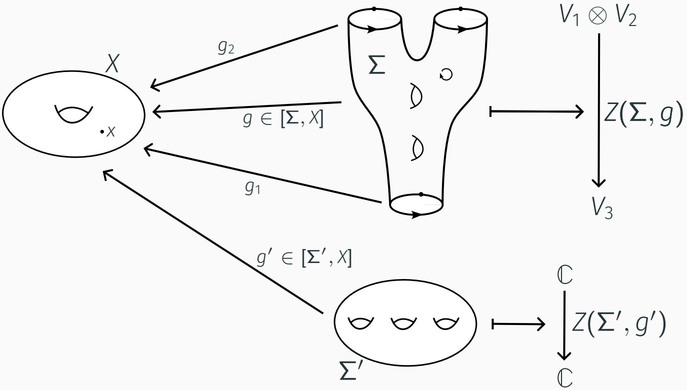

Cohomological HQFTs

Example (Turaev)

For any cohomology class $\theta \in H^n(X, \mathbb{C}^*)$, there exists an n-dimensional X-HQFT, called *cohomological X-HQFT*,

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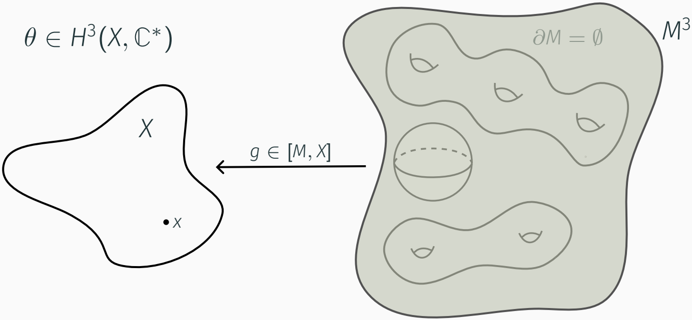

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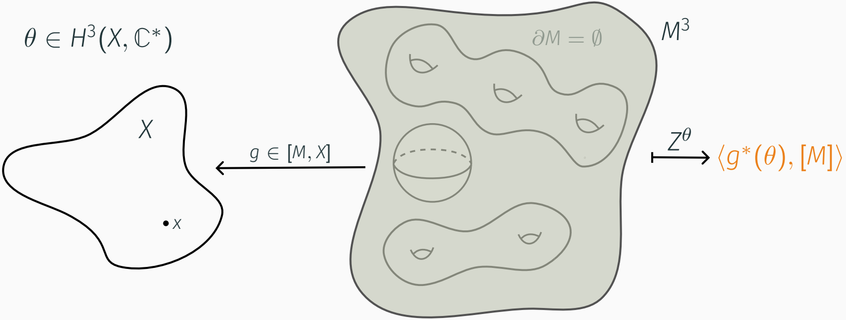

n-dimensional X-HQFTs produce numerical invariants of homotopy classes of maps defined from a closed n-manifold to X.

Results on 3-dimensional TQFTs and HQFTs

3-dimensional TQFTs

There are two main constructions of 3-dimensional TQFTs:

- $\underbrace{\text{Turaev-Viro}}_{\substack{\text{using repres.}\\ \text{of } U_q(sl_2)}}\text{-}\underbrace{\text{Barrett-Westbury}}_{\substack{\text{generalizing to spherical}\\ \text{fusion cats}}}$ state-sum TQFT
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- Turaev-Virelizier: these two TQFTs are related by the center construction. More precisely, the center $Z(\mathcal{C})$ of a spherical fusion category $\mathcal{C}$ is a modular tensor category and for a closed oriented 3-manifold M, we have $\tau_{RT}^{Z(\mathcal{C})}(M) = \tau_{TV}^{\mathcal{C}}(M)$.

In 3d, surgery and state-sum TQFTs are related by the center construction on the corresponding algebraic notions.

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The relationship between 3d TQFTs extends to 3d surgery and state-sum HQFTs with aspherical targets.

Homotopy n-types and Main Theorem

Homotopy n-types

Definition

A *homotopy n-type* is a top. space X with $\pi_i(X, x) = 0$ for all $i > n$.

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Definition

A *crossed module* is a group homomorphism $\chi : E \to H$ with H acts on E (denoted $h \cdot e = {}^h e$ for $h \in H$ and $e \in E$) such that

- χ is H-equivariant (H acts on itself by conjugation) i.e. $\chi({}^h e) = h\chi(e)h^{-1}$ for all $h \in H$ and $e \in E$
- χ satisfies Peiffer identity, i.e. ${}^{\chi(e)}e' = ee'e^{-1}$ for all $e, e' \in E$.

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- **Theorem (MacLane-Whitehead):** Crossed modules model homotopy 2-types. For any homotopy 2-type X, there exists a crossed module $\chi : E \to H$ such that $X \simeq B\chi$ where $\pi_1(B\chi, x) = \text{coker}(\chi)$ and $\pi_2(B\chi, x) = \ker(\chi)$.

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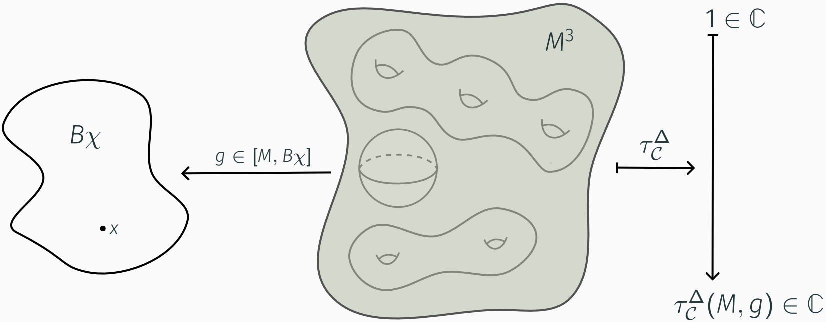
This result generalizes the state-sum TQFT/HQFT results as follows:

- $\chi = \mathrm{id}_H \implies \tau_{\chi}^{\Delta}$ is equivalent to 3d state-sum TQFT τ_{TV}.
- $\chi : E \hookrightarrow H \implies \tau_{\chi}^{\Delta}$ is equivalent to τ_{TV}^{Δ} with $B\chi \simeq K(\mathrm{coker}\chi, 1)$.

What type of invariant such an HQFT yield?

Given a spherical χ-fusion category $\mathcal{C}$ over $\mathbb{C}$. Then for any pair (M, g) where M is a closed oriented 3-manifold and $g \in [M, B\chi]$ is a homotopy class, the $B\chi$-HQFT $\tau_{\mathcal{C}}^{\Delta}$ yields a numerical invariant $\tau_{\mathcal{C}}^{\Delta}(M, g) \in \mathbb{C}$ which is multiplicative with respect to disjoint union operation.



Our main goal is to explain how this number is derived.

Spherical Fusion Categories

Graphical calculus for monoidal categories

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Graphical calculus for monoidal categories

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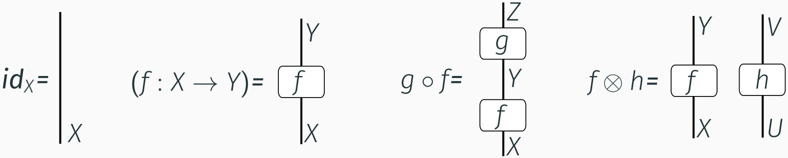

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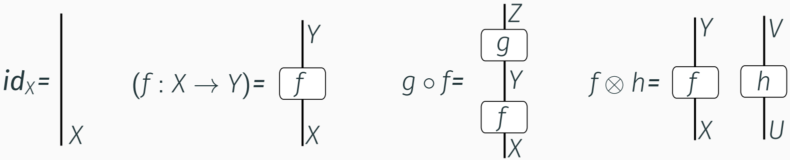

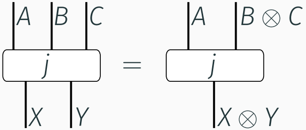

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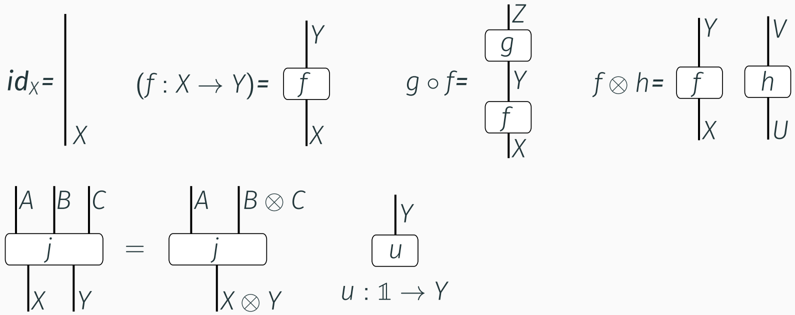

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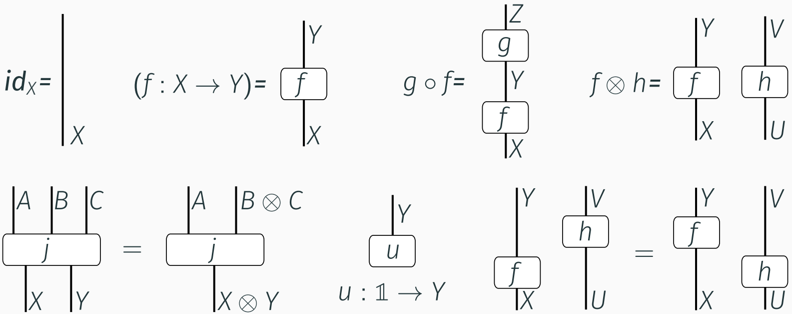

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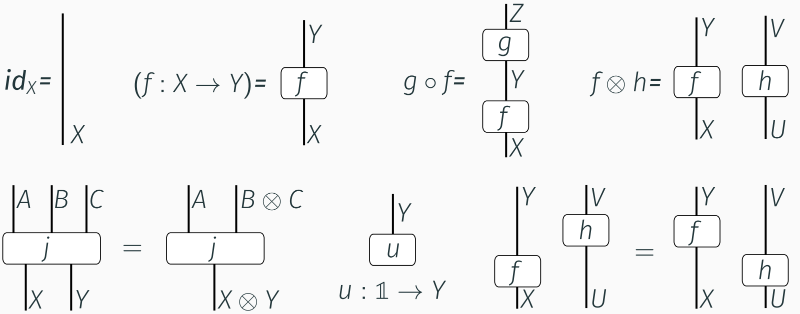

A monoidal category $(\mathcal{C}, \otimes)$ is $\mathbb{C}$-*linear* if for any two objects X, Y of $\mathcal{C}$,

- $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a $\mathbb{C}$-vector space,
- $\circ$ and $\otimes$ are $\mathbb{C}$-bilinear.

Rigid and pivotal categories

A *rigid* category is a monoidal category $(\mathcal{C}, \otimes)$ which admits both a left duality $\{({}^\vee X, \mathrm{ev}_X : {}^\vee X \otimes X \to \mathbb{1})\}_{X \in \mathcal{C}}$ and a right duality $\{(X^\vee, \widetilde{\mathrm{ev}}_X : X \otimes X^\vee \to \mathbb{1})\}_{X \in \mathcal{C}}$.

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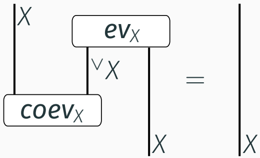

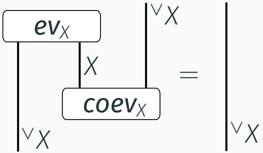

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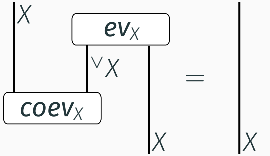

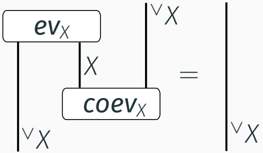

A *pivotal* category is a rigid category with distinguished (pivotal) duality such that the objects of left and right dualities coincide:

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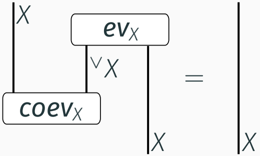

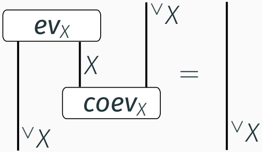

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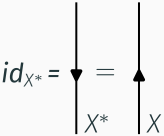

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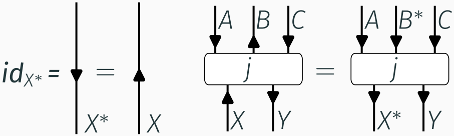

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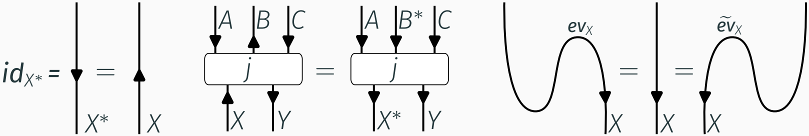

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Definition

A fusion $\mathbb{C}$-category is a monoidal $\mathbb{C}$-linear category $\mathcal{C}$ such that there exists a finite set I of simple objects of $\mathcal{C}$ satisfying the conditions

- $\mathbb{1} \in I$,
- $\mathrm{Hom}_{\mathcal{C}}(i,j) = 0$ for any distinct $i, j \in I$,
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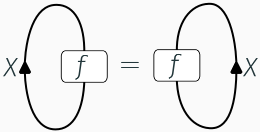
Example Representations of a finite group.

Example Representations of quantum groups.

Example Given a finite group G, we have a category $\mathcal{G}$; $\mathrm{Ob}(\mathcal{G}) = G$ and $\mathrm{Hom}_{\mathcal{G}}(g,h) = \delta_{g,h}\mathbb{C}$ for all $g, h \in G$ where $g \otimes h = gh$ for all $g, h \in G$ and $k \otimes l = kl$ for all $k, l \in \mathbb{C}$.

Spherical categories and graphical calculus on a sphere

A *spherical* category is a pivotal category whose left and right traces coincide. That is, for any endomorphism f, we have $\mathrm{tr}_l(f) = \mathrm{tr}_r(f)$.



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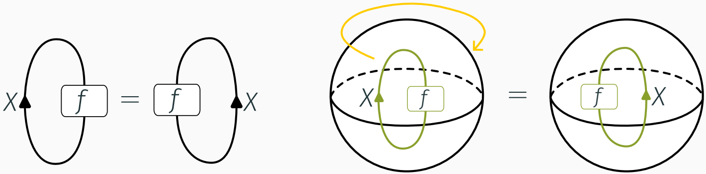

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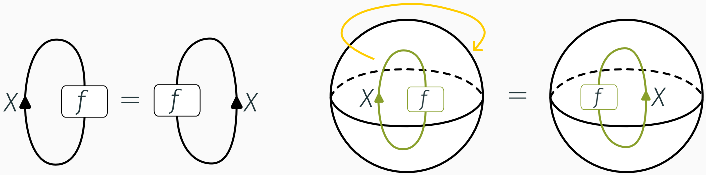

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Let $\mathcal{C}$ be a pivotal fusion cat. and I be a repres. of simple objects;

- left dimension of an object $X \in \mathcal{C}$: $\dim_l(X) = \mathrm{tr}_l(\mathrm{id}_X) \in \mathbb{C}$,
- right dimension of an object $X \in \mathcal{C}$: $\dim_r(X) = \mathrm{tr}_r(\mathrm{id}_X) \in \mathbb{C}$,
- dimension of $\mathcal{C}$: $\dim(\mathcal{C}) = \sum_{i \in I} \dim_l(i)\dim_r(i)$.

Crossed module graded monoidal categories

Let $\chi : E \to H$ be a crossed module. A *χ-graded category* is a $\mathbb{C}$-linear monoidal category $(\mathcal{C}, \otimes)$ which is

- E-Hom graded, i.e. $\mathrm{Hom}_{\mathcal{C}}(X,Y) = \oplus_{e\in E}\mathrm{Hom}^{e}_{\mathcal{C}}(X,Y)$ for all $X, Y \in \mathcal{C}$.
- endowed with a subclass $\mathcal{C}_{\mathrm{hom}}$ and a degree map $|\cdot| : \mathcal{C}_{\mathrm{hom}} \to H$ such that
  - $X = \oplus_{i=1}^{n} X_i$ where $X_i \in \mathcal{C}_{\mathrm{hom}}$,
  - For $X, Y \in \mathcal{C}_{\mathrm{hom}}$, we have $\mathrm{Hom}^{e}_{\mathcal{C}}(X,Y) = 0$ if $|Y| \neq \chi(e)|X|$,
  - For $X, Y \in \mathcal{C}_{\mathrm{hom}}$, we have $X \otimes Y = \oplus_{i=1}^{n} Z_i$ with $|Z_i| = |X||Y|$,
  - $|\mathbb{1}| = 1 \in H$,
  - For any homogeneous morphisms α, β with $s(\alpha) \in \mathcal{C}_{\mathrm{hom}}$, we have $|\alpha \otimes \beta| = |\alpha| \left({}^{|s(\alpha)|}|\beta| \right) \in E$,
  - $|a_{X,Y,Z}| = |l_X| = |r_X| = 1 \in E$ for all $X, Y, Z \in \mathcal{C}$.

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A *pivotal structure* on a χ-graded monoidal category $\mathcal{C}$ is a pivotal duality where all evaluation morphisms ev_X and $\widetilde{\mathrm{ev}}_X$ are homogeneous of degree $1 \in E$.

A pivotal χ-graded monoidal category $\mathcal{C}$ is *spherical* if for any degree 1 endomorphism $f \in \mathrm{Hom}^{1_E}_{\mathcal{C}}(X, X)$, left and right traces coincide.

Spherical χ-fusion categories

Definition (S.-Virelizier)

A spherical χ-fusion category (over $\mathbb{C}$) is a spherical χ-graded category $(\mathcal{C}, \otimes)$ such that

- $\mathcal{C}$ is E-semisimple, i.e. for any $e \in E$ and $X \in \mathcal{C}$, we have $X = \oplus^{e}_{i \in J} X_i$ where each X_i is simple (i.e. $\mathrm{End}^1(X_i) \cong \mathbb{C}$),
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Example: Consider the category $\mathbb{C}\mathcal{G}_\chi$ whose

- $\mathrm{Ob}(\mathbb{C}\mathcal{G}_\chi) = H$
- $\mathrm{Hom}_{\mathbb{C}\mathcal{G}_\chi}(x, y) = \{e \in E \mid y = \chi(e)x\}\mathbb{C}$ for $x, y \in H$.
- monoidal product of objects $x \otimes y = xy$
- monoidal product of morphisms $(x \xrightarrow{e} y) \otimes (z \xrightarrow{f} t) = xy \xrightarrow{e^x f} zt$.

A Hopf χ-coalgebra is a family $\{A_x\}_{x \in H}$ of $\mathbb{C}$-algebras endowed with

- coassociative algebra homoms. $\{\Delta_{x,y} : A_{xy} \to A_x \otimes A_y\}_{x,y \in H}$
- counitary algebra homomorphism $\varepsilon : A_1 \to \mathbb{C}$
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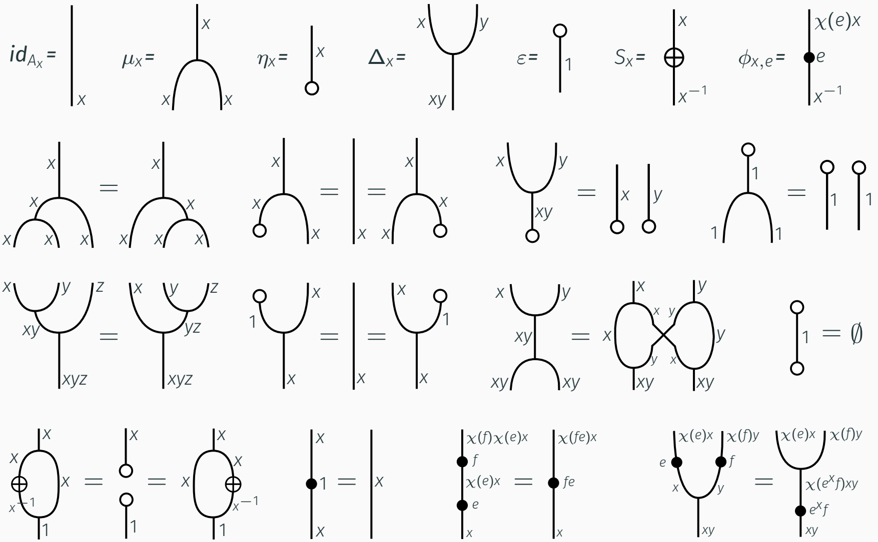
Theorem (S.-Virelizier)

The category mod(A) of representations of a Hopf χ-coalgebra $A = \{A_x\}_{x\in H}$ is χ-fusion if A_1 is semisimple and each A_x is nonzero and finite dimensional.

Hopf χ-coalgebras: Graphical definition



State-sum Homotopy Invariants of Maps

χ-labeling of a triangulation

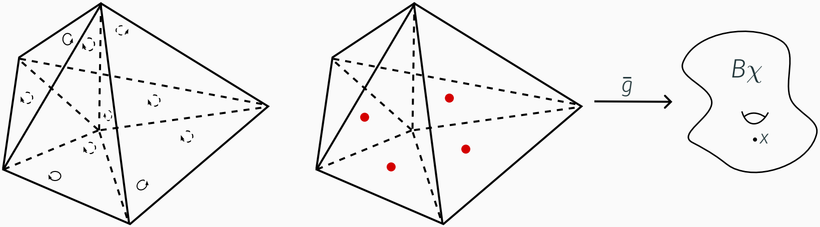
Let M be a closed oriented 3-manifold and $g \in [M, B\chi]$ be a homotopy class of a map.

- Given a triangulation Δ of M with oriented 2-faces $\Delta^{(2)} \subset \Delta$.
- Encode the data of g by specifying a χ-labeling (α, β) where

$$(\alpha : \Delta^{(2)} \to H, \beta : \Delta^{(1)} \to E)$$

Question: **How do we specify a χ-labeling?**

Step 1: Choose a representative $\bar{g}$ of g mapping centers of 3-simplices to the basepoint $x \in B\chi$.



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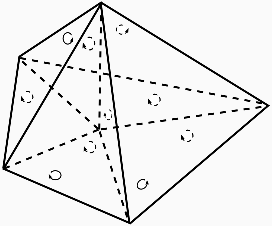
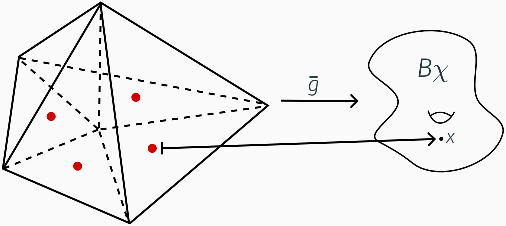


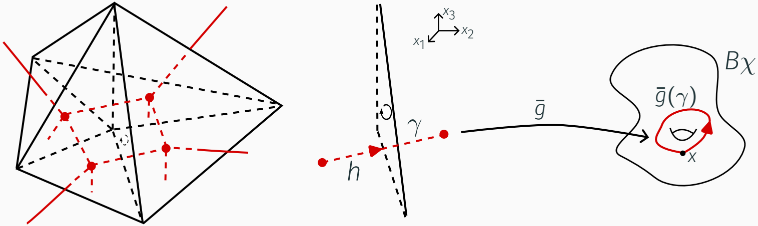


χ-labeling of a triangulation

Step 2: Choose arcs connecting central points and orient each arc using the orientation of the corresponding 2-face and the orientation of M.

Step 3: Label an arc γ by an element of H which corresponds to the homotopy class of a loop $\bar{g}(\gamma) \subset (B\chi)^1 \subset B\chi$.

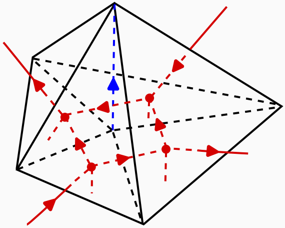
Step 4: $\alpha : \Delta^{(2)} \to H$ maps a 2-face to the H-label of the corresponding arc.

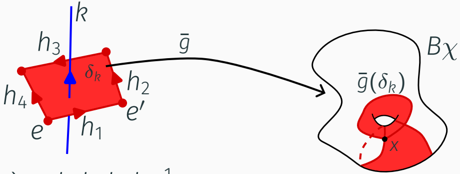


χ-labeling of a triangulation

Step 5: Around an oriented edge k of Δ form a disk δ_k whose boundary is the concatenation of the arcs obtained above.

Step 6: For a central point a adjacent to k, label the pair (k, a) by an element of E which corresponds to the relative homotopy class of $\bar{g}|_{\delta_k} \subset B\chi$ in $\pi_2(B\chi, (B\chi)^1, x) = E$.





$$\chi(e) = h_1 h_2 h_3 h_4^{-1}$$
$$\chi(e') = h_2 h_3 h_4^{-1} h_1$$

- **Lemma:** $\{\chi\text{-labelings of } \Delta^{(2)}\}/\text{Gauge group} \cong [M, B\chi]$.

The state-sum invariant

Given a spherical χ-fusion category $\mathcal{C}$ and a set $I = \sqcup_{h \in H} I_h$ of representatives of simple objects.

A *coloring* is a map $c : \Delta^{(2)} \to I$ such that $c(r) \in I_{\alpha(r)}$ for all $r \in \Delta^{(2)}$.

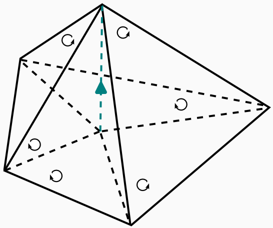
Assigning scalar $|c|$ to a coloring c

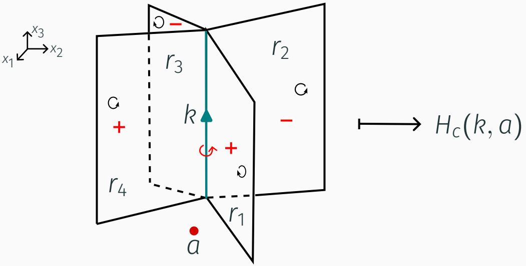
Given a spherical χ-fusion category $\mathcal{C}$ and a set $I = \sqcup_{h\in H} I_h$ of representatives of simple objects.
A *coloring* is a map $c : \Delta^{(2)} \to I$ such that $c(r) \in I_{\alpha(r)}$ for all $r \in \Delta^{(2)}$.

Given a coloring $c : \Delta^{(2)} \to I$, we obtain a scalar $|c| \in \mathbb{C}$ as follows.

- To each pair (k, a) of an oriented edge k and a central point a adjacent to k, we assign a vector space

$$H_c(k,a) = \mathrm{Hom}_{\mathcal{C}}^{\beta(k,a)}(\mathbb{1}, c(r_1)^{\varepsilon_1} \otimes c(r_2)^{\varepsilon_2} \otimes \cdots \otimes c(r_n)^{\varepsilon_n}).$$





Assigning scalar $|c|$ to a coloring c

- Doing this assignment for all oriented edges, we obtain a finite-dimensional $\mathbb{C}$-vector space

$$H_c = \otimes_{\substack{\text{oriented}\\ \text{edges } k}} H_c(k, a_k).$$

- **Lemma:** $H_c(k, a_k)$ and $H_c(-k, a_k)$ are dual to each other. This yields a vector $*_k \in H_c(k, a_k) \otimes H_c(-k, a_k)$.

$$f \in H_c(k, a) = \mathbf{Hom}_{\mathcal{C}}^{\beta(k,a)}(\mathbb{1}, c(r_1) \otimes c(r_2)^* \otimes c(r_3)^* \otimes c(r_4))$$

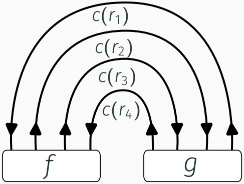
$$g \in H_c(-k, a) = \mathbf{Hom}_{\mathcal{C}}^{-\beta(k,a)}(\mathbb{1}, c(r_4)^* \otimes c(r_3) \otimes c(r_2) \otimes c(r_1)^*)$$

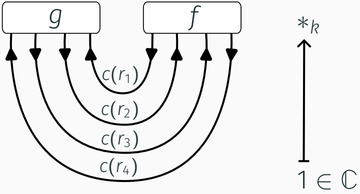
Assigning scalar $|c|$ to a coloring c

- Doing this assignment for all oriented edges, we obtain a finite-dimensional $\mathbb{C}$-vector space

$$H_c = \otimes_{\substack{\text{oriented}\\ \text{edges } k}} H_c(k, a_k).$$

- **Lemma:** $H_c(k, a_k)$ and $H_c(-k, a_k)$ are dual to each other. This yields a vector $*_k \in H_c(k, a_k) \otimes H_c(-k, a_k)$.



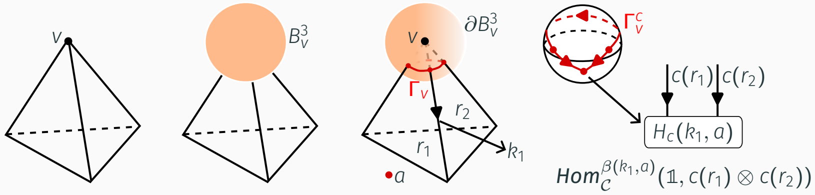


- Each coloring c produces a vector $*_c = \otimes_{\substack{\text{unoriented}\\ \text{edges } \tilde{k}}} *_{\tilde{k}} \in H_c$.

Assigning scalar $|c|$ to a coloring c

The next step involves vertices:

- For any vertex v of Δ, choose a 3-ball neighborhood B_v^3 of v.
- The intersection $\partial B_v^3 \cap \Delta^{(2)}$ yields a graph Γ_v on ∂B_v^2.



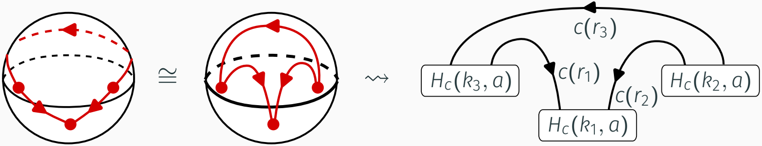
- A coloring c assigns to each vertex of Γ_v a Hom-vector space in $\mathcal{C}$.
- The assigned vector space is precisely $H_c(k_1, a)$ where k_1 is the corresponding edge and oriented away from v.

Assigning scalar $|c|$ to a coloring c

- Then each vertex v of Δ and a coloring c yields a dual vector

$$\mathbb{F}_{\mathcal{C}}(\Gamma_v^c) : \underbrace{H_c(k_1, a) \otimes H_c(k_2, a) \otimes \cdots \otimes H_c(k_n, a)}_{H_c(\Gamma_v^c)} \to \mathbb{C}$$

where k_i's are the edges incident to v and oriented away from v.



- Repeating this process for all vertices, we obtain

$$\otimes_{v \in \Delta} H(\Gamma_v^c)^* \cong \otimes_v \otimes_{k_v} H_c(k_v, a_v)^* \cong \otimes_{\substack{\text{oriented} \\ \text{edges } k}} H_c(k, a_k)^* = H_c^*$$

- Denote the image of $\otimes_{v \in \Delta} \mathbb{F}_{\mathcal{C}}(\Gamma_v^c) \in \otimes_{v \in \Delta} H(\Gamma_v^c)^*$ under these isomorphisms by $V_c \in H_c^*$.
- Lastly, the scalar $|c|$ is obtained by the evaluation $V_c(*_c) \in \mathbb{C}$.

The state-sum invariant

The state-sum invariant of a pair (M, g) is defined as

$$\tau_{\mathcal{C}}^{\Delta}(M,g) = (\dim \mathcal{C}_1^1)^{-(\#\text{ 3-simplices of }\Delta)} \sum_{\substack{\text{colorings}\\ c:\Delta^{(2)}\to I}} \left(\prod_{r\in\Delta^{(2)}} \dim(c(r)) \right) |c| \in \mathbb{C}.$$

where $\mathcal{C}_1^1$ is the fusion subcategory of $\mathcal{C}$ consisting of degree 1 objects and degree 1 morphisms.

Recall that the inputs for $\tau_{\mathcal{C}}^{\Delta}(M,g)$ are

- triangulation Δ of M,
- χ-labeling (α, β) of Δ associated to g.
- spherical χ-fusion category $\mathcal{C}$,
- representative set I of simple objects of $\mathcal{C}$,

Theorem (S.-Virelizier)

$\tau_{\mathcal{C}}^{\Delta}(M,g)$ is independent of the choices of Δ, (α, β), and I.

Thanks for your attention!