# Stable cohomology of the IA-automorphism group

Erik Lindell

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**Definition**

- $F_n := \langle x_1, x_2, \dots, x_n \rangle$
- $H_{\mathbb{Z}}(n) := H_1(F_n, \mathbb{Z}) \cong F_n^{\mathrm{ab}} \cong \mathbb{Z}^n$. Let us write $e_i := [x_i]$.
- $H(n) := H_1(F_n, \mathbb{Q}) \cong \mathbb{Q}^n$

The action $\mathrm{Aut}(F_n) \curvearrowright H_{\mathbb{Z}}(n)$ defines a homomorphism

$$\varphi : \mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$$

which is surjective.

**Definition**

The IA-automorphism group is defined by

$$\mathrm{IA}_n := \ker(\varphi).$$

**Remark**

"IA" is an abbreviation of "identity on abelianization".

Some facts:

- For each $n \geqslant 1$, $\mathrm{IA}_n$ is finitely generated by the *Magnus generators* $A_{ij}$ and $B_{ijk}$ defined by

$$A_{ij}(x_l) = \begin{cases} x_j^{-1} x_i x_j, & \text{if } l = i, \\ x_l & \text{otherwise,} \end{cases}$$

$$B_{ijk}(x_l) = \begin{cases} x_i [x_j, x_k] & \text{if } l = i, \\ x_l & \text{otherwise,} \end{cases}$$

  where $[x, y] = xyx^{-1}y^{-1}$ is the commutator.
- For $n \geqslant 2$, it is unknown whether $\mathrm{IA}_n$ is finitely presented.

**Goal:** Understand $H^*(\mathrm{IA}_n, \mathbb{Q})$.

**Why is this interesting?**

- It is related to finiteness properties of the group.
- *Characteristic classes:* We have

$$\mathrm{Aut}(F_n) \cong \pi_0 \,\mathrm{hAut}_*\left(\vee^n S^1\right)$$

and $\mathrm{hAut}_*(\bigvee^n S^1)$ has contractible path components, so

$$B\,\mathrm{Aut}(F_n) \simeq B\,\mathrm{hAut}_*\left(\vee^n S^1\right),$$

which classifies $\vee^n S^1$-fibrations. Similarly, $B\,\mathrm{IA}_n$ classifies such fibrations with trivial monodromy.

**Proposition**

$H^*(\mathrm{IA}_n, \mathbb{Q})$ *is a* $\mathrm{GL}_n(\mathbb{Z})$*-representation.*

**Proof.**

By definition, we have a short exact sequence

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \xrightarrow{\varphi} \mathrm{GL}_n(\mathbb{Z}) \to 1.$$

If $A \in \mathrm{GL}_n(\mathbb{Z})$, let $\tilde{A} \in \varphi^{-1}(A)$ and $C_{\tilde{A}} \in \mathrm{Aut}(\mathrm{IA}_n)$ be conjugation by $\tilde{A}$.

If $\tilde{A}' \in \varphi^{-1}(A)$, then $\tilde{A}^{-1}\tilde{A}' \in \mathrm{IA}_n$, so $C_{\tilde{A}}^{-1} C_{\tilde{A}'}$ is in $\mathrm{Inn}(\mathrm{IA}_n)$. Inner automorphisms of a group act trivially on its cohomology, so this gives us a well-defined action. □

## The Johnson homomorphism and $\mathrm{IA}_n^{ab}$

Let $\mathcal{L}_n(2) := [F_n, F_n]/[F_n, [F_n, F_n]]$ be the second quotient of the lower central series of $F_n$. We then have a homomorphism

$$\tau : \mathrm{IA}_n \to \mathrm{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}(n), \mathcal{L}_n(2))$$
$$\phi \mapsto ([x] \mapsto [x^{-1}\phi(x)])$$

called the *Johnson homomorphism*. We have $\mathcal{L}_n(2) \cong \Lambda^2 H_{\mathbb{Z}}(n)$ and

$$\tau(A_{ij}) = e_i^\vee \otimes (e_i \wedge e_j)$$
$$\tau(B_{ijk}) = e_i^\vee \otimes (e_j \wedge e_k)$$

**Theorem (Kawazumi '05)**

*The Johnson homomorphism induces a* $\mathrm{GL}_n(\mathbb{Z})$*-equivariant isomorphism*

$$\mathrm{IA}_n^{ab} \xrightarrow{\cong} H_{\mathbb{Z}}(n)^\vee \otimes \Lambda^2 H_{\mathbb{Z}}(n).$$

## Stability

Using that $F_{n+1} \cong F_n * \mathbb{Z}$, we have a homomorphism

$$s_n : \mathrm{Aut}(F_n) \to \mathrm{Aut}(F_{n+1})$$
$$\phi \mapsto \phi * \mathrm{id}_{\mathbb{Z}}$$

**Theorem (Hatcher-Vogtmann '04)**

*For $n \geqslant 2* + 2$, $s_n$ induces an isomorphism in homology in degree $*$.*

**Theorem (Galatius '11)**

*We have $\operatorname{colim}_n H_*(\mathrm{Aut}(F_n), \mathbb{Q}) \cong \mathbb{Q}[0]$.*

Note that $s_n$ restricts to a homomorphism $\mathrm{IA}_n \to \mathrm{IA}_{n+1}$. However we have

$$\mathrm{rank}(H_{\mathbb{Z}}(n)^\vee \otimes \Lambda^2 H_{\mathbb{Z}}(n)) = n \cdot \binom{n}{2} \xrightarrow{n\to\infty} \infty,$$

so $H_1(\mathrm{IA}_n, \mathbb{Z})$ *does not* stabilize.

## Interlude - Representation theory of $\mathrm{GL}_n$

The $\mathrm{Aut}(F_n)$-representation $H(n)$ factors through $\mathrm{GL}_n(\mathbb{Z})$.

**Definition**

A $\mathrm{GL}_n(\mathbb{Z})$-representation $V$ is *algebraic* if it is the restriction of a representation of $\mathrm{GL}_n(\mathbb{Q})$.

**Example**

The representation $H(n)$ is algebraic.

Representations of $\mathrm{GL}_n(\mathbb{Q})$ are semi-simple and the irreducible representations are indexed by *bipartitions*, i.e. pairs $(\lambda, \mu)$, of partitions $(\lambda_1 \geqslant \lambda_2 \geqslant \cdots \lambda_l \geqslant 0)$. We write $V_{(\lambda,\mu)}$ for the corresponding representation.

**Example**

- $H(n) \cong V_{(1,0)}$,
- $H(n)^\vee \cong V_{(0,1)}$,
- $H(n)^\vee \otimes \Lambda^2 H(n) \cong V_{(1^2,1)} \oplus V_{(1,0)}$.

## Back to stability

**Corollary**

$H_1(\mathrm{IA}_n, \mathbb{Q}) \cong V_{(1^2,1)} \oplus V_{(1,0)}$.

This is independent of $n$ *as a* $\mathrm{GL}_n(\mathbb{Z})$*-representation*.

**Conjecture (Church-Farb '12)**

*The rational (co)homology groups of* $\mathrm{IA}_n$ *are stably algebraic* $\mathrm{GL}_n(\mathbb{Z})$*-representations and satisfy multiplicity stability, i.e. for* $n \gg *$*, its decomposition*

$$H_*(\mathrm{IA}_n, \mathbb{Q}) \cong \bigoplus_i V_{(\lambda,\mu)_i}$$

*is independent of* $n$.

However, in degrees $* > 1$, we don't even know if there is a range $n \gg *$ such that the (co)homology is finite dimensional!

# Albanese cohomology

## Definition

For $G$ a group, we define

$$H^*_{\mathrm{Alb}}(G,\mathbb{Q}) := \mathrm{im}\left(\Lambda^* H^1(G,\mathbb{Q}) \xrightarrow{\cup} H^*(G,\mathbb{Q})\right)$$

## Corollary

*$H^*_{\mathrm{Alb}}(\mathrm{IA}_n,\mathbb{Q})$ is a quotient of $\Lambda\left(\Lambda^2 H(n)^\vee \otimes H(n)\right)$. In particular, it is an algebraic $\mathrm{GL}_n(\mathbb{Z})$-representation.*

Albanese cohomology may be effectively studied using *abelian (co)cycles*:

## Definition

For $G$ a group and $(g_1,\ldots,g_k) \in G^{\times k}$ pairwise commuting, we define

$$AC(g_1,\ldots,g_k) := \{f_1 \cup \cdots \cup f_k\} \subseteq H^*_{\mathrm{Alb}}(G,\mathbb{Q})$$

for $f_i \in H^1(G,\mathbb{Q})$ such that $f_i([g_j]) = \delta_{ij}$.

A $k$-tuple $(g_1,\ldots,g_k) \in G^{\times k}$, as in the definition, defines a homomorphism

$$\begin{aligned} \phi : \mathbb{Z}^k &\to G \\ e_i &\mapsto g_i \end{aligned}$$

which induces a commutative diagram

$$\begin{array}{ccc} \Lambda^k H^1(G,\mathbb{Q}) & \xrightarrow{\cup} & H^k(G,\mathbb{Q}) \\ \downarrow{\scriptstyle \Lambda^k\phi_*} & & \downarrow{\scriptstyle \phi_*} \\ \Lambda^k H^1(\mathbb{Z}^k,\mathbb{Q}) & \xrightarrow{\cong} & H^k(\mathbb{Z}^k,\mathbb{Q}) \end{array}$$

By definition, any $f_1 \cup \cdots \cup f_k \in AC(g_1,\ldots,g_k)$ is mapped to the generator of $H^k(\mathbb{Z}^k,\mathbb{Q})$.

For $G = \mathrm{IA}_n$:

- Abelian (co)cycles are easy to construct using the generators $A_{ij}$ and $B_{ijk}$.
- We can also use the representation structure.

# Albanese cohomology

This approach was used to obtain the following:

**Theorem (Pettet '05)**

*For $n \geqslant 6$, we have*

$$H^2_{\mathrm{Alb}}(\mathrm{IA}_n, \mathbb{Q}) \cong \Lambda^2(\Lambda^2 H(n)^\vee \otimes H(n))/R_2,$$

*where*

$$R_2 := \left\langle \sum_{i=1}^{n} \begin{pmatrix} ((f_1 \wedge f_2) \otimes e_i) \wedge ((e_i^\vee \wedge f_3) \otimes a) \\ -((f_3 \wedge f_1) \otimes e_i) \wedge ((e_i^\vee \otimes f_2) \otimes a) \end{pmatrix} \right\rangle$$

**Theorem (Katada '22)**

*For $n \geqslant 9$, we have*

$$H^3_{\mathrm{Alb}}(\mathrm{IA}_n, \mathbb{Q}) \cong \Lambda^3(\Lambda^2 H(n)^\vee \otimes H(n))/R_3,$$

*where*

$$R_3 := \langle \mathrm{im}(R_2 \otimes H^1(\mathrm{IA}_n) \xrightarrow{\cup} H^3(\mathrm{IA}_n, \mathbb{Q})) \rangle$$

**Conjecture (Katada '22)**

*For $n \geqslant 3*$, we have*

$$H^*_{\mathrm{Alb}}(\mathrm{IA}_n, \mathbb{Q}) \cong \Lambda^*(\Lambda^2 H(n)^\vee \otimes H(n))/R,$$

*where $R$ is the ideal generated by $R_2$.*

Katada defined a graded representation $W^*(n)$ *(which we will characterize later)* and proved, using abelian cycles, that for $n \geqslant 3*$,

$$W^*(n) \hookrightarrow H^*_{\mathrm{Alb}}(\mathrm{IA}_n, \mathbb{Q}).$$

**Conjecture (Katada '22)**

*For $n \geqslant 3*$, we have*

$$W^*(n) \cong H^*_{\mathrm{Alb}}(\mathrm{IA}_n, \mathbb{Q}) \cong \Lambda^*(\Lambda^2 H(n)^\vee \otimes H(n))/R.$$

**Q.** How can we try to understand $H^*(\mathrm{IA}_n, \mathbb{Q})$ more generally?

Let us limit ourselves to the *algebraic part*:

**Definition**

For $V$ a $\mathrm{GL}_n(\mathbb{Z})$-representation, let

$$V^{\mathrm{alg}} := \operatorname*{colim}_{\substack{W \subset V \\ W \text{ algebraic}}} W \subseteq V.$$

To decompose a $\mathrm{GL}_n(\mathbb{Q})$-representation $V$ into irreducibles, we need to calculate

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Q})}(V_{(\lambda,\mu)}, V) \cong [V_{(\mu,\lambda)} \otimes V]^{\mathrm{GL}_n(\mathbb{Q})},$$

for every bipartition $(\lambda, \mu)$.

Let us write $H^\vee(n) := \mathrm{Hom}_{\mathbb{Q}}(H(n), \mathbb{Q})$ and for $p, q \geqslant 0$

$$K^{p,q}(n) := H(n)^{\otimes p} \otimes H^\vee(n)^{\otimes q}.$$

The duality pairing $H(n) \otimes H^\vee(n) \to \mathbb{Q}$ induces, for $p, q \geqslant 1$ and $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$, equivariant maps

$$\lambda_{i,j} : K^{p,q}(n) \to K^{p-1,q-1}(n).$$

**Definition**

We define

- $T^{p,q}(n) := \bigcap_{i,j} \ker(\lambda_{i,j}) \subseteq K^{p,q}(n)$
- $V_{(\lambda,\mu)} := (S^\lambda \otimes S^\mu) \otimes_{\Sigma_p \times \Sigma_q} T^{p,q}(n)$

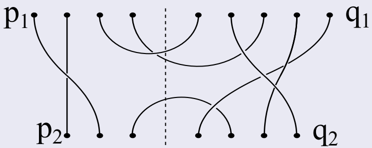
**Slogan:** Decomposing $\mathrm{GL}_n(\mathbb{Q})$-representation $V$ requires

- computing $[V \otimes K^{p,q}(n)]^{\mathrm{GL}_n(\mathbb{Q})}$ as a $\Sigma_p \times \Sigma_q$-representation, for all $p, q \geqslant 0$,
- understanding the maps induced by the $\lambda_{i,j}$.

**Definition**

The *walled Brauer category* $\mathrm{wBr}_n$ is the $\mathbb{Q}$-linear category with

- **Objects:** $(p, q)$ with $p, q \geqslant 0$
- **Morphisms:** spanned by diagrams




**Slogan':** To decompose $V$, we need to understand the functor $[V \otimes K^{-,-}(n)]^{\mathrm{GL}_n(\mathbb{Z})} : \mathrm{wBr}_n \to \mathbb{Q}\text{-mod}$.

**Definition**

The *downward* walled Brauer category $\mathrm{dwBr} \subset \mathrm{wBr}_n$ has the same objects, but the spaces of morphisms are spanned by diagrams with only downward horizontal strands.

For $M \in \mathrm{Rep}(\mathrm{GL}_n(\mathbb{Z}))^{\mathrm{dwBr}^{\mathrm{op}}}$ and $N \in \mathrm{Rep}(\mathrm{GL}_n(\mathbb{Z}))^{\mathrm{dwBr}}$, we will write

$$M \overset{\mathrm{dwBr}}{\otimes} N := \int^{(p,q)\in\mathrm{dwBr}} M(p,q) \otimes N(p,q) \in \mathrm{Rep}(\mathrm{GL}_n(\mathbb{Z})).$$

**Proposition (Kupers–Randal-Williams '19, L. '24)**

*If $A \in (\mathbb{Q}\text{-mod})^{\mathrm{dwBr}}$ has finite length and $B$ is a $\mathrm{GL}_n(\mathbb{Z})$-representation such that*

$$i_*(A) \cong [K^{-,-}(n) \otimes B]^{\mathrm{GL}_n(\mathbb{Z})},$$

*where $i_*$ denotes left Kan extension along $i : \mathrm{dwBr} \hookrightarrow \mathrm{wBr}_n$, then*

$$i^*(K^{-,-}(n)^\vee) \overset{\mathrm{dwBr}}{\otimes} A \cong B^{\mathrm{alg}}.$$

For $p, q \geqslant 0$ the short exact sequence

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z}) \to 1$$

gives us a Hochschild-Serre spectral sequence

$$E_2^{i,j} = H^i(\mathrm{GL}_n(\mathbb{Z}), H^j(\mathrm{IA}_n, \mathbb{Q}) \otimes K^{p,q}(n)) \Rightarrow H^{i+j}(\mathrm{Aut}(F_n), K^{p,q}(n)).$$

Note that $E_2^{0,j} = [H^j(\mathrm{IA}_n, \mathbb{Q}) \otimes K^{p,q}(n)]^{\mathrm{GL}_n(\mathbb{Z})}$.

**Theorem (L. '22)**

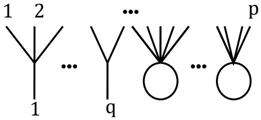
*For $n \geqslant 2* + p + q + 3$, we have*

$$H^*(\mathrm{Aut}(F_n), K^{p,q}(n)) \cong \mathcal{P}(p, q) \otimes \mathrm{sgn}_p \otimes \mathrm{sgn}_q,$$

*where*

$$\mathcal{P}(p,q) := \left\{ \begin{array}{c} \text{Partitions of } \{1, \dots, p\} \text{ with at least } q \\ \text{parts, } q \text{ of which are labeled } 1, \dots, q \\ \text{and remaining unlabeled} \end{array} \right\}$$

We can also think of $\mathcal{P}(p, q)$ as being spanned by graphs like the following:




**Definition**

Let $\mathcal{P}'(p, q) \subseteq \mathcal{P}(p, q)$ be the subspace generated by partitions with no labeled parts of size 1.

**Theorem (Katada '22)**

*For $n \gg *$, we have*

$$W^*(n) \cong \bigoplus_{p-q=*} T^{q,p}(n) \underset{\Sigma_p \times \Sigma_q}{\otimes} (\mathcal{P}'(p,q) \otimes \mathrm{sgn}_p \otimes \mathrm{sgn}_q).$$

$$\left( \cong K^{-,-}(n)^\vee \overset{\mathrm{dwBr}}{\otimes} (\mathcal{P}' \otimes \mathrm{sgn}) \right).$$

# A spectral sequence approach

By analyzing the spectral sequence (under certain added assumptions that we will specify), one obtains that in a stable range of degrees

$$\begin{aligned}&[K^{p,q}(n) \otimes H^*(\mathrm{IA}_n, \mathbb{Q})]^{\mathrm{GL}_n \mathbb{Z}} \\ \cong&(\mathcal{P}(p,q) \otimes \mathrm{sgn}_p \otimes \mathrm{sgn}_q) \otimes \mathbb{Q}[y_4, y_8, \ldots,],\end{aligned}$$

where $|y_{4k}| = 4k$. This is a natural isomorphism and we stably have that $\mathcal{P} \cong i_* \mathcal{P}'$. Habiro and Katada used a similar approach to obtain the following:

**Theorem (Habiro-Katada '22)**

*Suppose that there is a $Q \geqslant 0$ such that for $n \gg *$ and $* \leqslant Q$, $H^*_{\mathrm{Alb}}(\mathrm{IA}_n, \mathbb{Q}) \cong W^*(n)$ and $H^*(\mathrm{IA}_n, \mathbb{Q})$ is algebraic. Then there is a range $n \gg *$ such that*

$$\begin{aligned}H^*(\mathrm{IA}_n, \mathbb{Q}) &\cong H^*_{\mathrm{Alb}}(\mathrm{IA}_n, \mathbb{Q}) \otimes H^*(\mathrm{IA}_n, \mathbb{Q})^{\mathrm{GL}_n(\mathbb{Z})} \\ &\cong W^*(n) \otimes \mathbb{Q}[y_4, y_8, \ldots],\end{aligned}$$

*for $* \leqslant Q$.*

**Q.** What is the source of the invariant classes $y_{4k}$?

Understanding the invariants corresponds to analysing the spectral sequence for $p = q = 0$ :

$$E_2^{i,j} = H^i(\mathrm{GL}_n(\mathbb{Z}), H^j(\mathrm{IA}_n, \mathbb{Q})) \Rightarrow H^{i+j}(\mathrm{Aut}(F_n), \mathbb{Q}).$$

Note that

- For $n \gg i + j > 0$, the target is 0,
- $E_2^{0,j} = H^j(\mathrm{IA}_n, \mathbb{Q})^{\mathrm{GL}_n(\mathbb{Z})}$,
- $E_2^{i,0} = H^i(\mathrm{GL}_n(\mathbb{Z}), \mathbb{Q})$.

**Theorem (Borel '74)**

*For* $n \gg *$, $H^*(\mathrm{GL}_n(\mathbb{Z}), \mathbb{Q}) \cong \Lambda\{x_5, x_9, \ldots\}$, *where* $|x_{4k+1}| = 4k + 1$.

Since the target is zero, the differential

$$d_k : E_k^{0,k} \to E_k^{k+1,0}$$

must be an isomorphism, so we can define the *anti-transgression* map

$$\varphi_k : E_2^{k+1,0} \twoheadrightarrow E_k^{k+1,0} \xrightarrow{d_k^{-1}} E_k^{0,k} \hookrightarrow E_2^{0,k}.$$

**Definition**

We set $y_{4k} := \varphi_{4k}(x_{4k+1}) \in H^{4k}(\mathrm{IA}_n, \mathbb{Q})^{\mathrm{GL}_n(\mathbb{Z})}$.

## Borel Vanishing

This method is only suitable for understanding $H^*(\mathrm{IA}_n, \mathbb{Q})^{\mathrm{alg}}$, but the assumption of algebraicity is also used in the analysis of the spectral sequence. More specifically:

**Theorem (Borel '74)**

*Suppose $V$ is an algebraic $\mathrm{GL}_n(\mathbb{Z})$-representation. If $n \gg *$, then the cup product map*

$$H^0(\mathrm{GL}_n(\mathbb{Z}), V) \otimes H^*(\mathrm{GL}_n(\mathbb{Z}), \mathbb{Q}) \to H^*(\mathrm{GL}_n(\mathbb{Z}), V)$$

*is an isomorphism.*

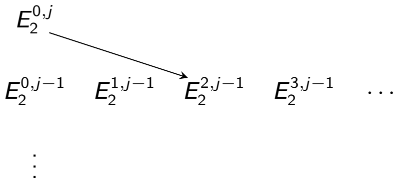
This implies that if for $n \gg *$, we have that $H^*(\mathrm{IA}_n, \mathbb{Q})$ is algebraic, then

$$\begin{aligned} E_2^{i,j} &= H^i(\mathrm{GL}_n, H^j(\mathrm{IA}_n, \mathbb{Q})) \\ &\cong H^i(\mathrm{GL}_n(\mathbb{Z}), \mathbb{Q}) \otimes H^j(\mathrm{IA}_n, \mathbb{Q})^{\mathrm{GL}_n(\mathbb{Z})} \cong E_2^{i,0} \otimes E_2^{0,j}. \end{aligned}$$

This makes the spectral sequence manageable.

## Some slight improvements

Note that for each $k \geqslant 2$, the term $E_k^{0,j}$ only has differentials to strictly lower rows.




$\Rightarrow$ It suffices to assume that Borel vanishing holds up to degree $(j-1)$ to understand $E_2^{0,j}$.

**Proposition**

*Suppose $V$ is a finite dimensional $\mathrm{GL}_n(\mathbb{Z})$-representation. If $n \gg *$, then*

$$H^0(\mathrm{GL}_n(\mathbb{Z}), V) \otimes H^*(\mathrm{GL}_n(\mathbb{Z}), \mathbb{Q}) \to H^*(\mathrm{GL}_n(\mathbb{Z}), V)$$

*is an isomorphism.*

## Some slight improvements

### Theorem (L. '24)

*Suppose that there is a* $Q \geqslant 0$ *such that for* $n \gg *$, $H^*(\mathrm{IA}_n, \mathbb{Q})$ *is finite dimensional for all* $* \leqslant Q$. *Then there is a range* $n \gg *$ *such that*

$$H^*(\mathrm{IA}_n, \mathbb{Q})^{\mathrm{alg}} \cong W^*(n) \otimes \mathbb{Q}[y_4, y_8, \ldots],$$

*for* $* \leqslant Q + 1$.

### Corollary

$$H^2(\mathrm{IA}_n, \mathbb{Q})^{\mathrm{alg}} \cong H^2_{\mathrm{Alb}}(\mathrm{IA}_n, \mathbb{Q}) \cong \Lambda^2(\Lambda^2 H(n)^\vee \otimes H(n))/R_2$$

## Computing the Albanese cohomology

Recall that $W^*(n) \cong K^{-,-}(n)^\vee \overset{\mathrm{dwBr}}{\otimes} (\mathcal{P}' \otimes \mathrm{sgn})$. By the universal properties of coends, we obtain a map

$$\Lambda^*\left(K^{1,2}(n) \otimes (\mathcal{P}'(2,1) \otimes \mathrm{sgn}_2)\right) \to W^*(n)$$

By identifications in the coend, we get that this map factors through $\Lambda^*(H(n) \otimes \Lambda^2 H(n)^\vee)$.

### Theorem (L. '24)

*The map*

$$\Lambda^*(H(n) \otimes \Lambda^2 H(n)^\vee) \to W^*(n)$$

*is surjective and stably, its kernel is* $R$.

### Corollary (Katada '24)

*Stably, we have*

$$H^*_{\mathrm{Alb}}(\mathrm{IA}_n, \mathbb{Q}) \cong \Lambda^*(\Lambda^2 H(n)^\vee \otimes H(n))/R$$

We can summarize the results as follows:

**Theorem (Habiro-Katada '22, L. '22, Katada '24)**

1. *For $n \gg *$, we have*

$$H^*_{\mathrm{Alb}}(\mathrm{IA}_n, \mathbb{Q}) \cong \Lambda^*(\Lambda^2 H(n)^\vee \otimes H(n))/R.$$

2. *Suppose that there is a $Q \geqslant 0$ such that for $n \gg *$, $H^*(\mathrm{IA}_n, \mathbb{Q})$ is finite dimensional for all $* \leqslant Q$. Then there is a range $n \gg *$ such that*

$$H^*(\mathrm{IA}_n, \mathbb{Q})^{\mathrm{alg}} \cong H^*_{\mathrm{Alb}}(\mathrm{IA}_n, \mathbb{Q}) \otimes \mathbb{Q}[y_4, y_8, \ldots],$$

*for $* \leqslant Q + 1$.*

3. *In particular*

$$H^2(\mathrm{IA}_n, \mathbb{Q})^{\mathrm{alg}} \cong H^2_{\mathrm{Alb}}(\mathrm{IA}_n, \mathbb{Q}).$$