Homological stability of symplectic groups

Ismael Sierra
Nathalie Wahl

University of Toronto
Copenhagen University

Introduction: what is homological stability?

Homological Stability I: motivation

$G =$ (discrete) group.

$BG =$ classifying space of principal G-bundles.

$H^*(BG) =$ characteristic classes of such bundles.

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Algebraic interpretation: $H_*(BG) = Tor_*^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z})$ and $H^*(BG) = Ext^*_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z})$.

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Problem: this is very hard!

Homological Stability II: a (partial) solution

Key idea (Quillen): Many groups of interest arise in families:

$$G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \cdots .$$

Thus, get a family of classifying spaces

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(i) Do we have *homological stability*? i.e. are the maps $H_d(BG_n) \to H_d(BG_{n+1})$ isomorphisms for $d << n$?
(ii) Can we compute the *stable homology*? i.e. can we compute $\operatorname{colim}_n H_d(BG_n)$?

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If answer to both is yes then we get partial computations!... and improving stability range becomes relevant!

Stable homology: examples

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Examples:

(i) $G_n = GL_n(\mathbb{F}_q)$: stable homology completely described by Quillen '72.

(ii) $G_n = GL_n(R)$: stable homology related to $K(R) =$ algebraic K-theory of R.
Example: $R =$ number field and $\mathbb{Q}$ coefficients then known by Borel '74.

(iii) $G_n = MCG(\Sigma_{n,1})$: stable homology known!
$H^*(BG_\infty, \mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \cdots]$. (Madsen–Weiss '07)

(iv) $G_n = Sp_{2n}(\mathbb{Z})$: $H^*(BSp_\infty(\mathbb{Z}), \mathbb{Q}) = \mathbb{Q}[x_2, x_6, x_{10}, \dots]$ by Borel '74.
Integral computations possible by 9 authors (Calmès–Dotto–Harpaz–Hebestreit–Land–Moi–Nardin–Nikolaus–Steimle).

The“Classical” argument I: set-up

Question: how does one prove homological stability? for which families?

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Set-up:

1. $(\mathsf{C}, \oplus, 0) =$ (braided) monoidal category.
2. $X \in \mathsf{C} =$ *stabilizing object*.
3. $A \in \mathsf{C} =$ choice of object, usually take $A = 0$.
4. $G_n = \mathrm{Aut}_{\mathsf{C}}(A \oplus X^{\oplus n})$.
5. $G_n \hookrightarrow G_{n+1}$ given by $- \oplus \mathrm{id}_X$.

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(ii) $\mathsf{C} = \begin{cases} \text{objects=} & \Sigma_{n,1} \\ \text{morphisms=} & \text{diffeomorphisms rel boundary/isotopy} \end{cases}$,

$\oplus =$ boundary connected sum,

$A = \varnothing$, $X = \Sigma_{1,1}$. Then $G_n = MCG(\Sigma_{n,1})$.

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(iii) $\mathsf{C} =$ category of skew-symmetric bilinear forms over $\mathbb{Z}$, $\oplus =$ orthogonal direct sum, $A = 0$, $X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ hyperbolic form. Then $G_n = Sp_{2n}(\mathbb{Z})$.

The Classical argument III: the method

There are canonical **destabilization complexes** $W_n(A, X)$ associated to the above data, dimension $= n - 1$.

Informally: $p-$simplices $\Longleftrightarrow$ ways of destabilizing $X^{\oplus p+1}$ from $A \oplus X^{\oplus n}$ $\Longleftrightarrow$ embeddings $X^{\oplus p+1} \to A \oplus X^{\oplus n}$ whose complement is $A \oplus X^{\oplus n-p-1}$.

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Precise: If $W_n(A, X)$ is $\frac{n-c}{k}$ connected, $c \in \mathbb{Z}$, $\in \mathbb{Z}_{>0}$ then $H_d(BG_n) \to H_d(BG_{n+1})$ is an iso if $d \leq \frac{n-c+2}{\max\{2,k\}}$.

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Limitation of method: we can never get something better than $\frac{n-\text{const}}{2}$. In fact, this issue is intrinsic: braid groups!

The Classical method: applications

(i) $G_n = GL_n(R)$: get result $\frac{n-c}{2}$, $c =$ constant depending on ring. (Quillen, Maazen, Van der Kallen, ...).

(ii) $G_n = MCG(\Sigma_{n,1})$ (Harer, Ivanov, Boldsen, Randal-Williams, Galatius–Kupers–Randal-Williams, Harr-Vistrup–Wahl). Best bound $\lesssim \frac{2n}{3}$.

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Result of Harr–Vistrup–Wahl manages to do so!

Results

Result I

Theorem (S., Whal)
Let R be a ring with finite unitary stable rank (usr). Let $c = 0$ if R is a PID and $c = 2usr(R) + 2$ otherwise. Then

$$H_d(BSp_{2g}(R)) \to H_d(BSp_{2(g+1)}(R))$$

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As we will see, 2/3 slope is related to MCG 2/3 slope...

Result II

Define the *"odd" symplectic groups* $Sp_{2g+1}(R) := Stab_{Sp_{2g+2}(R)}(e_1)$, where standard basis is $e_1, f_1, \ldots, e_{g+1}, f_{g+1}$.

Have $Sp_0(R) \subset Sp_1(R) \subset Sp_2(R) \subset Sp_3(R) \subset \cdots$.

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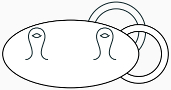
New slope $= 1/3$.

The proof

The geometric idea I

Key idea (Harr–Vistrup–Wahl) $\Sigma_{n+1,1}$ can be obtained by attaching **two** handles to $\Sigma_{n,1}$.



Thus, we can define new family by attaching **one** handle at a time: $\Sigma_{0,1}, \Sigma_{0,2}, \Sigma_{1,1}, \Sigma_{1,2}, \Sigma_{2,1}, \Sigma_{2,2}, \dots$.

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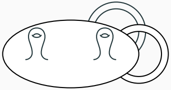

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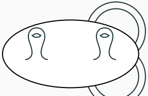
Issues:

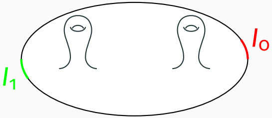
1. One has to be careful attaching handles!
2. How to create a nice categorical set-up?



The geometric idea II

Take C = category of *bidecorated surfaces*.

Objects = (Σ, I_0, I_1).



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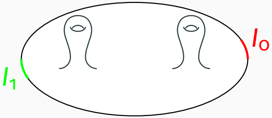

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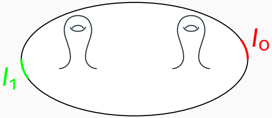

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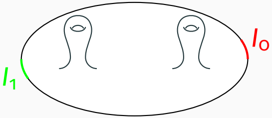

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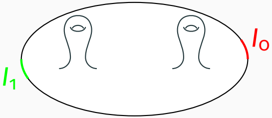

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$W_n(\varnothing, X) =$ *complex of disordered arcs.*

Vertices: non-separating arcs from b_0 to b_1 (up to isotopy).

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This implies stability result of slope $2/3$ for $MCG(\Sigma_{g,1})$.

Formed spaces with boundary I

Key insight: Action on homology (with R coefficients) gives a map $MCG(\Sigma_{g,1}) \to Sp_{2g}(R)$. Want to find $Sp_{2g+1}(R)$ with maps $MCG(\Sigma_{g,2}) \to Sp_{2g+1}(R)$. Then study new family.

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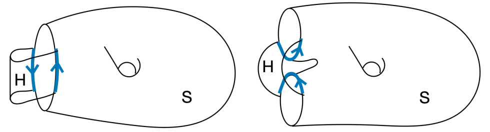
Solution: Category F_∂ of formed spaces with boundary.

- Objects $= (M, \lambda, \partial)$, $M =$ f.g. free R-module, $\lambda =$ skew-symmetric bilinear form, $\partial : M \to R$.
- Morphisms$=$ module maps preserving λ, ∂.
- Monoidal structure $\#$:
  $$(M_1, \lambda_1, \partial_1)\#(M_2, \lambda_2, \partial_2) = \left(M_1 \oplus M_2, \begin{pmatrix} \lambda_1 & \partial_1^T \partial_2 \\ -\partial_2^T \partial_1 & \lambda_2 \end{pmatrix}, \partial_1 + \partial_2\right).$$
- $X = (R, 0, \mathrm{id})$.

Formed spaces with boundary II

Geometric interpretation: Functor from bidecorated surfaces to F_∂ defined by $(\Sigma, I_0, I_1) \mapsto (H_1(\Sigma \cup_{I_0 \sqcup I_1} H), \lambda, \partial)$ where $\partial : H_1(\Sigma \cup_{I_0 \sqcup I_1} H) \cong H_1(\Sigma, I_0 \sqcup I_1) \to \tilde{H}_0(I_0 \sqcup I_1) \cong R$ is boundary map.



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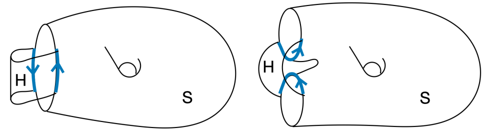

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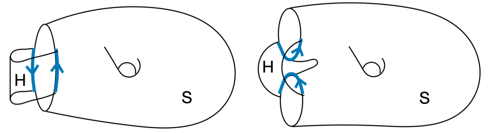

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Functor is monoidal! This gives geometric meaning to $\#$.

Algebraic meaning uses $(M_1 \oplus M_2)^\vee \cong M_1^\vee \oplus M_2^\vee$ and $\Lambda^2(M_1 \oplus M_2)^\vee \cong \Lambda^2 M_1^\vee \oplus M_1^\vee \otimes M_2^\vee \oplus \Lambda^2 M_2^\vee$.

Even and odd symplectic groups

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Fun fact: There is a braiding in full subcategory generated by X so get $B_n \to Aut(X^{\#n}) = Sp_{n-1}(R)$... this is (reduced) Bureau representation!

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Meaning: condition says there is $b \in M$ such that $\partial b = 1$ and $\lambda(a, b) = 1$: "b connects the two sides of a".

Destabilization complex and arcs I

Geometrically $W_n =$ disordered arc complex.

Question: what happens algebraically?

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Jointly non-separating: $\{\lambda(a_0, -), \ldots, \lambda(a_p, -), \partial\}$ unimodular in $M^\vee$.

Destabilization complex and arcs II

What about ordering condition?

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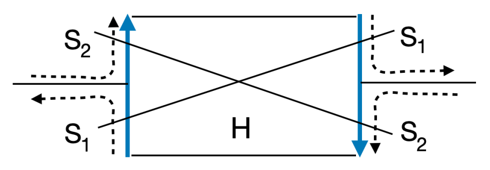

Figure 2: Two disordered arcs crossing once inside the handle

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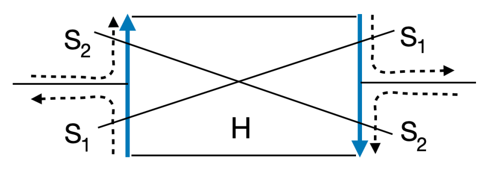

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Vertices= non-separating arcs.

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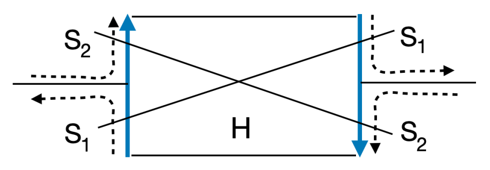

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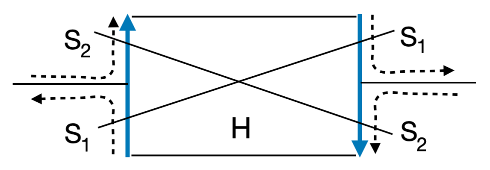

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Connectivity of disordered arc complex

Theorem (S.–Wahl) $D(M, \lambda, \partial)$ is $\frac{n-c}{3}$-connected, where $c = 5$ if R is a PID and $c = 2usr(R) + 6$ in general.

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Using the above one proves stability theorem!

Finishing the proof

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Key: Complex of non-separating arcs related to unimodular vectors complexes… that has connectivity of slope 1 in fact!

Key algebraic ingredient: understanding how X-genus decreases when we cut algebraic arcs… problem is that the X-genus (algebraic version of “number of handles”) generally drops by 2 and not by 1, that causes slope $1/3$ and not $1/2$.

Further possible works and applications

1. In the geometric arc complex, slope 1/3 stability is optimal. What about in the algebraic arc complex? (nothing known...)
2. Use this to get a classical proof 2/3-slope stability for diffeomorphism groups of some high-dimensional manifolds.
3. What about quadratic symplectic groups? issue is the non-separating arc complex... all other steps work analogously and stability of slope $> 1/4$ has new geometric implications!
4. Can one use similar methods to improve the slope 1/4 connectivity in the paper "Uniform twisted homological stability" by Miller–Patzt–Petersen–Randal-Williams? Maybe go to slope 1/3? (Ideal conjecture says it is 1/2 and connects to number theory)