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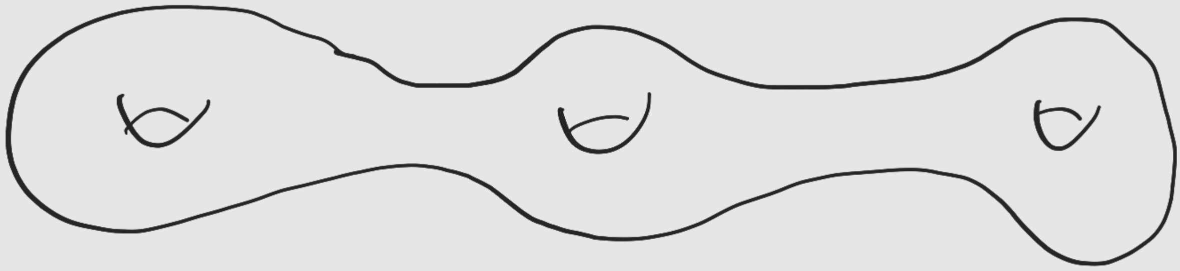
V_g denotes a 3-dimensional handlebody of genus g



$= V_3$

$\partial V_g = \Sigma_g$

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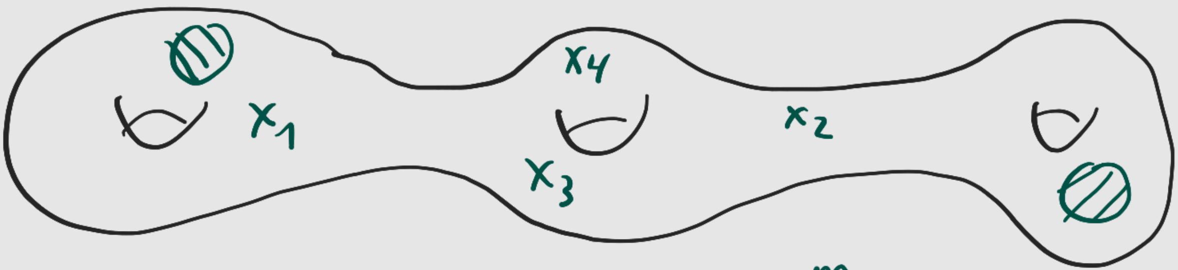
$V_{g,n}^m$ denotes a 3-dimensional handlebody of genus g, with n marked points and m marked disks, both on its boundary.



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$$\partial V_{g,n}^m = \Sigma_{g,n}^m$$

$\mathrm{Diff}(V_{g,n}^m)$ is required to preserve the marked points and marked disks pointwise.

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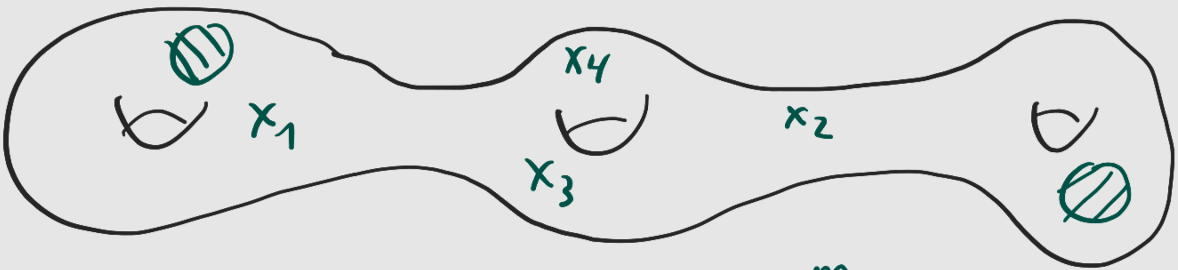

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$\mathrm{Diff}(V_{g,n}^m)$ is required to preserve the marked points and marked disks pointwise.

I will in a first time only discuss the case $n=0$.

- Moduli space of surfaces and some Teichmüller theory

$M = \Sigma_g^m$ a surface of genus g with m boundary curves

$\mathcal{M}_g = B\mathrm{Mod}(\Sigma_g)$ classifies:

- Algebraic structure
- complex structures
- Symplectic structures
- conformal structures
- Riemannian metrics

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For this talk I will use the point of view of <u>Riemannian metrics</u>.

Working assumption: $2g-2+n+m>0$

Let S be a reference surface of genus g, with m boundary curves.

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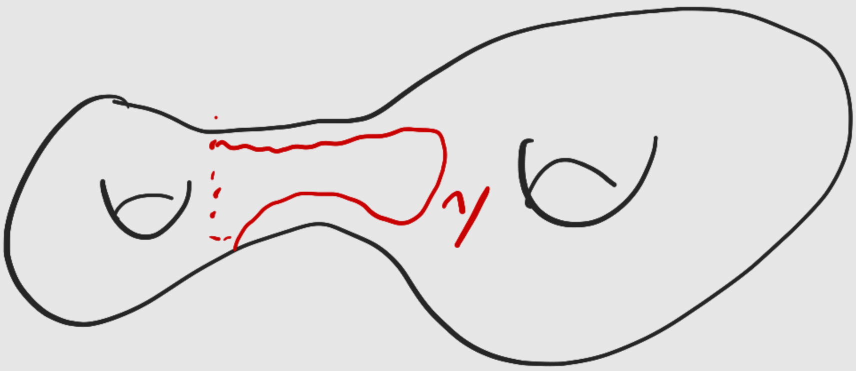
<u>Fact</u>: $\mathcal{T}(S)$ is a ball of dimension $6g-6+4m$

The <u>Fenchel-Nielsen</u> coordinates provide global coordinates on $\mathcal{T}(S)$:

Given a simple closed curve γ on S and a hyperbolic metric X, there exists a unique geodesic in the homotopy class of γ. Its length only depends on the isotopy class $[X]$, and is denoted $\ell_\gamma([X])$.

Additional comments



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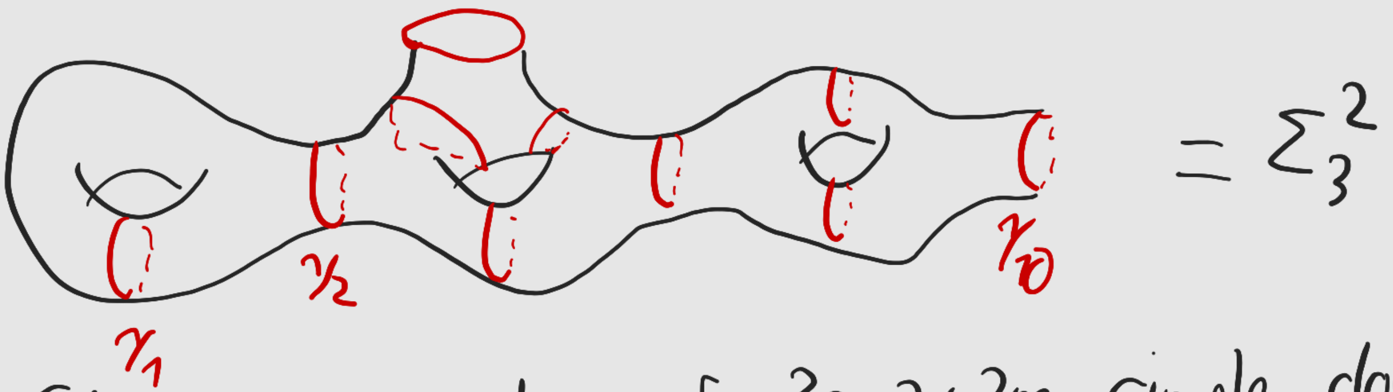
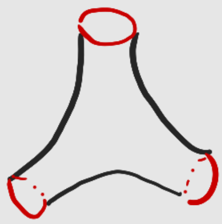


Choose a system of $3g-3+2m$ simple closed curves $(\gamma_1, \dots, \gamma_{3g-3+2m})$, including the m boundary curves, which decompose S into pairs of pants.



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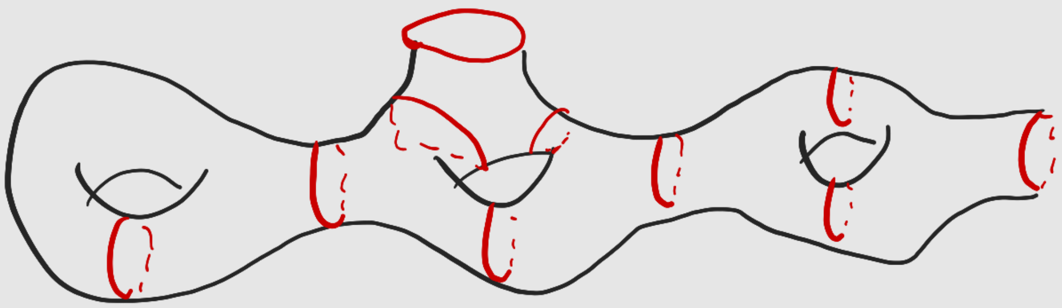

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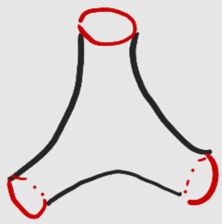

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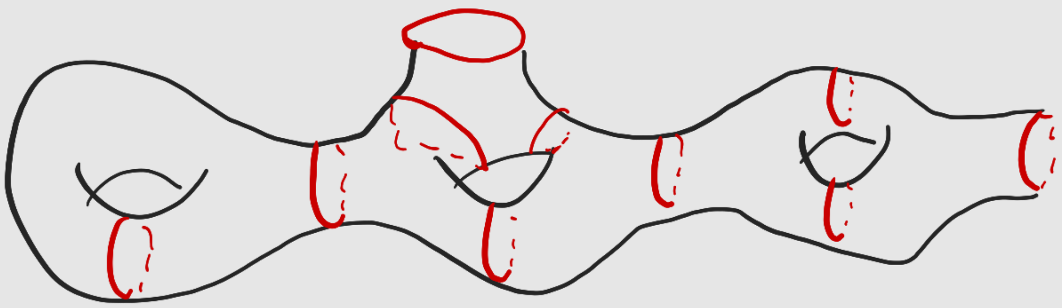

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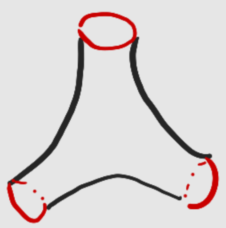

Their lengths $\ell_{\gamma_i}([X])$ produce half of the Fenchel-Nielsen coordinates. The remaining coordinates are twist parameters, one for each of the curves.

Why this works:

- There exists a unique pair of pants for any choice of lengths of its boundary curves.
- In order to pin down the metric [X] on S, we need to record how the pairs of pants are glued back together.
- This information is precisely given by the twist parameters.

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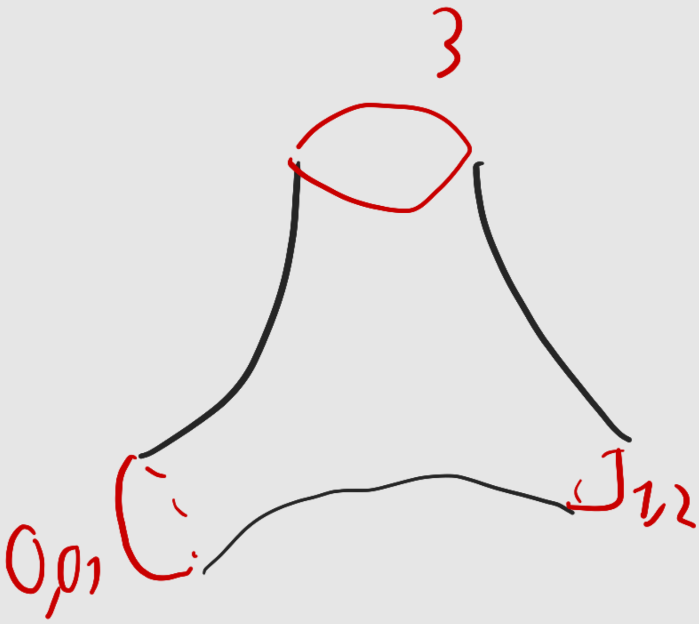
additional comments



We now turn to the groups $\mathrm{Mod}(V_g^m)$.

First recall some classical properties:

- $\mathrm{Mod}(V_g^m) \cong \mathrm{Mod}^{top}(V_g^m)$ [Cerf '67]
- $\mathrm{Mod}(V_g^m)$ injects into $\mathrm{Mod}(\Sigma_g^m)$ $\forall g, m$
- If $g=0$ $\mathrm{Mod}(V_0^m) \cong \mathrm{Mod}(\Sigma_0^m)$
- $\mathrm{Diff}(V_g^m) \simeq \mathrm{Mod}(V_g^m)$ [Hatcher] (maybe not quite explicit) $\quad \mathrm{Diff}(\Sigma_g^m) \simeq \mathrm{Mod}(\Sigma_g^m)$ Earle-Eells

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Additional comments

$$\begin{tikzcd}
\mathrm{Diff}(V_g \text{ rel } \partial V_g) \arrow[r] \arrow[d, "\simeq", "{[\text{Cerf}]}"'] & \mathrm{Diff}(V_g) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Diff}(\Sigma_g) \arrow[d, "\simeq"] \\
\mathrm{Homeo}(V_g \text{ rel } \partial V_g) \arrow[r] & \mathrm{Homeo}(V_g) \arrow[r] & \mathrm{Homeo}(\Sigma_g)
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- If $g=0$, $\mathrm{Mod}(V_0^m) \cong \mathrm{Mod}(\Sigma_0^m)$
- $\mathrm{Diff}(V_g^m) \simeq \mathrm{Mod}(V_g^m)$, $\mathrm{Diff}(\Sigma_g^m) \simeq \mathrm{Mod}(\Sigma_g^m)$

I will define an explicit open sublocus $\mathcal{H}\mathcal{M}_g^m \subset \mathcal{M}_g^m$ satisfying $\mathcal{H}\mathcal{M}_g^m \simeq B\mathrm{Mod}(V_g^m)$

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<u>Lemma</u>: (Collar lemma)

Let γ, γ' be two closed geodesics on a hyperbolic surface S. If $\ell(\gamma), \ell(\gamma') < \log(3+\sqrt{8})$, then $\gamma \cap \gamma' = \emptyset$.

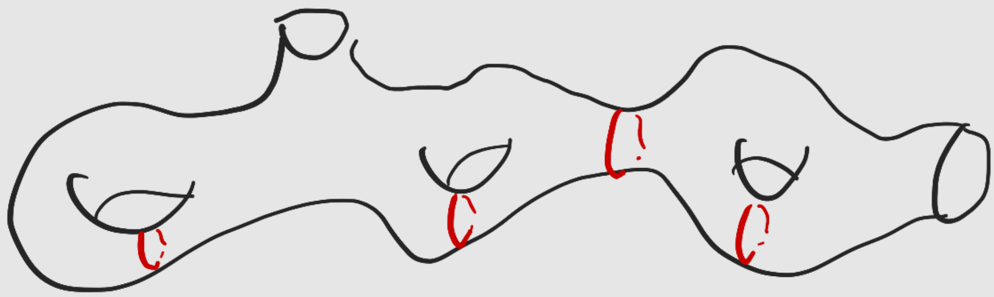
Fix $0 < \varepsilon < \log(3+\sqrt{8})$

A closed geodesic γ with $\ell(\gamma) \le \varepsilon$ is called <u>short</u>

<u>Definition:</u>

$\mathcal{H}\mathcal{M}_g^m$ is the open locus of $\mathcal{M}_g^m$ parametrizing hyperbolic surfaces S with the property that the collection of short closed geodesics decomposes S into genus 0 parts.

Additional comments



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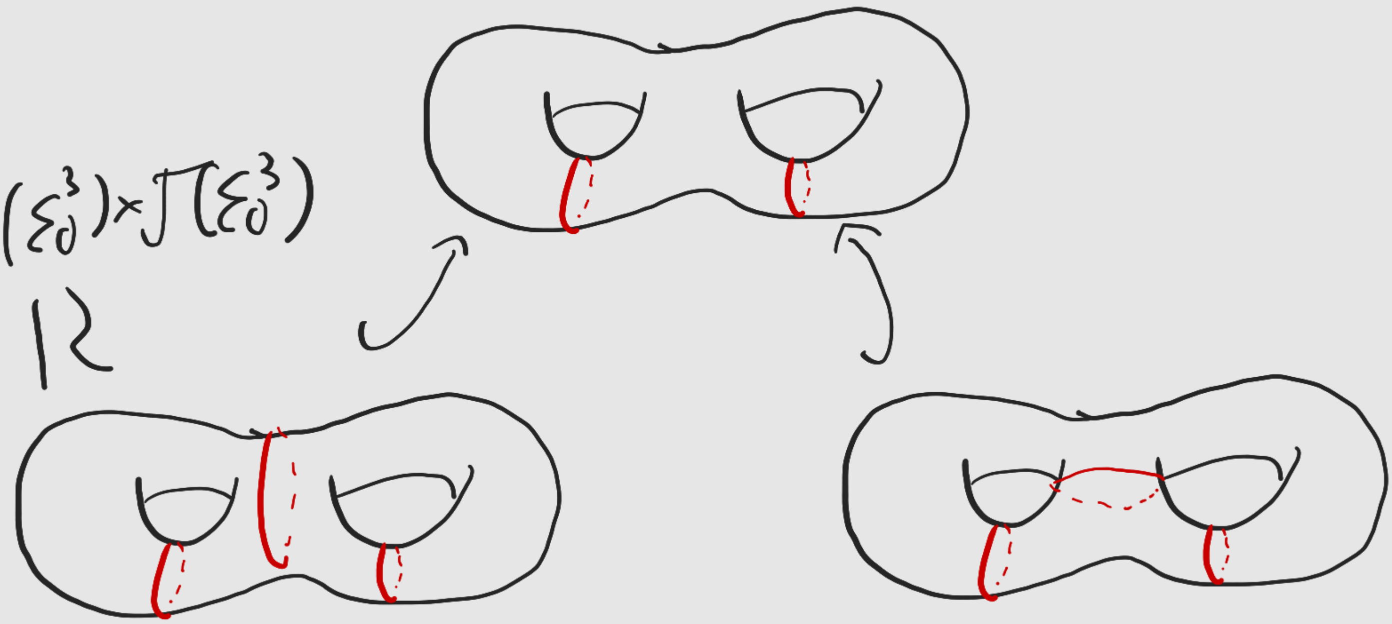
<u>Theorem</u>: [H.-Petersen]

$$\mathcal{H}\mathcal{M}_g^m \simeq B\mathrm{Mod}(V_g^m)$$

Idea of proof:

- Fix a handlebody V_g^m, and let $S = \partial V_g^m$.
- Define an open subset $U \subset \mathcal{T}(S)$, parametrizing hyperbolic metrics on S such that every short geodesic bounds a disk on V_g^m, and the collection of short geodesics decomposes S into genus 0 pieces
- U is contractible, and $U/\mathrm{Mod}(V_g^m) \simeq \mathcal{H}\mathcal{M}_g^m$.
  I will only explain the first statement.
- We can stratify U according to which curves are short geodesics

$\mathcal{T}(\Sigma_0^3) \times \mathcal{T}(\Sigma_0^3)$
$\cong$



Additional comments

If we lift $\mathcal{H}\mathcal{M}_g^m$ to $\mathcal{T}(S)$

$$\begin{tikzcd} \coprod U \arrow[r] \arrow[d] & \mathcal{T}(S) \arrow[d] \\ \mathcal{H}\mathcal{M}_g^m \arrow[r] & \mathcal{M}_g^m \end{tikzcd}$$

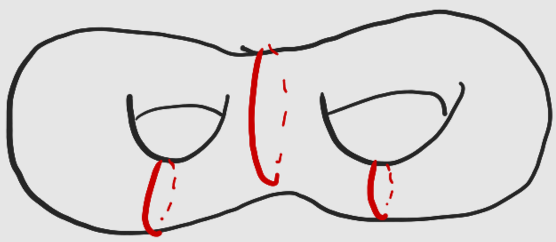
indexed by the cosets of $Mod(V_g^m) \to Mod(\Sigma_g^m)$

properly embedded disk:

$$\partial D_i = D_i \cap \partial V_g$$

$$S(V_g^m) \longrightarrow D(V_g^m)$$

- Each stratum is a product of Teichmuller spaces, hence contractible



$\simeq \mathcal{T}(\Sigma_0^3) \times \mathcal{T}(\Sigma_0^3)$

- The poset of strata is the simple disk system $S(V_g^m)$, whose elements are collections of properly embedded disks in V_g^m, having the property that removing them decomposes V_g^m into genus 0 pieces
- We also define $D(V_g^m)$, the poset of all disk systems
- McCullough and Hatcher-Wahl proved that the realization $|D(V_g^m)|$ is contractible, and from there Giansiracusa proved that the realization $|S(V_g^m)|$ is also contractible.
- Therefore $\mathcal{U}$ is indeed contractible.

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We now remove the assumption $n=0$.

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Remark:

There exist short exact sequences

$$0 \to \mathbb{Z} \to \mathrm{Mod}(\Sigma_{g,n}^{m+1}) \to \mathrm{Mod}(\Sigma_{g,n+1}^{m}) \to 0$$

$$0 \to \mathbb{Z} \to \mathrm{Mod}(V_{g,n}^{m+1}) \to \mathrm{Mod}(V_{g,n+1}^{m}) \to 0$$

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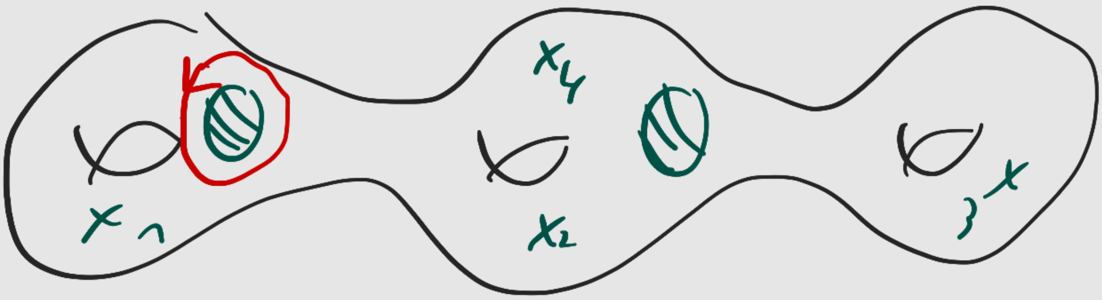

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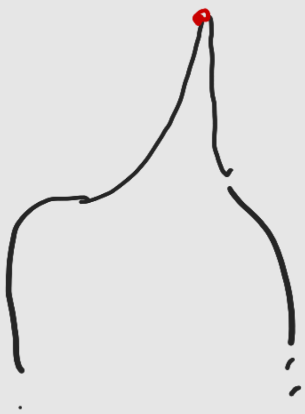
We define $\mathcal{HM}_{g,n}^{m}$ similarly as before, and we still have $\mathcal{HM}_{g,n}^{m} \simeq B\mathrm{Mod}(V_{g,n}^{m})$.

Additional comments

Remark: marked points correspond to hyperbolic cusps, which can be understood as boundary curves of length 0.



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One consequence of having this classifying space is the following:

Theorem: [Petersen-Wade]

The group $\mathrm{Mod}(V_{g,n}^{m})$ is a virtual duality group, with explicit dualizing module

Additional comments

G is a virtual duality group
if there exists a finite-index torsion-free subgroup G', and a G'-module M
such that $H_i(G, M) \cong H_{d-i}(G, \mathbb{Q})$

Poincaré duality: $M = \mathbb{Q}$

Relation to the moduli space of tropical curves

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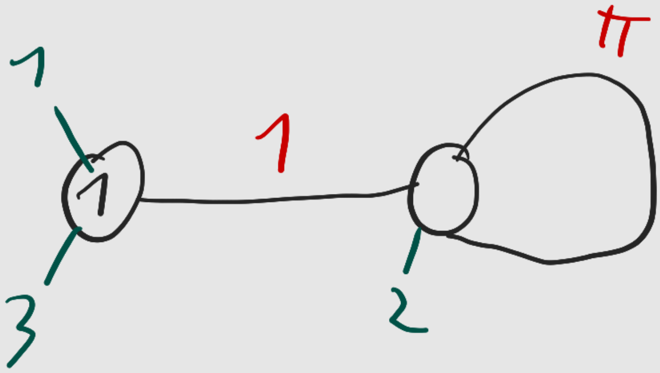
A tropical curve is a marked stable metric graph:



Additional comments

$$\mathrm{val}(v) = 2\cdot g(v) + \deg(v)$$

$2 \qquad 3 \qquad = 5$

stable: $\mathrm{val}(v) \geq 3$

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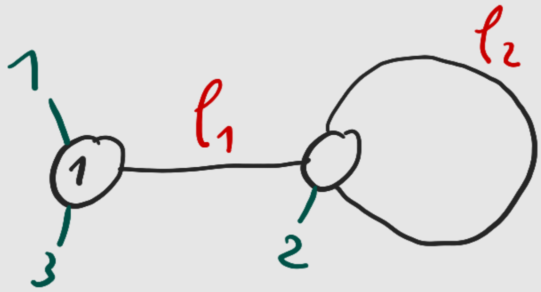

$\in \mathcal{M}_{2,3}^{trop}$

$\mathcal{M}_{g,n}^{trop}$ is the space whose points are isomorphism classes of such graphs.

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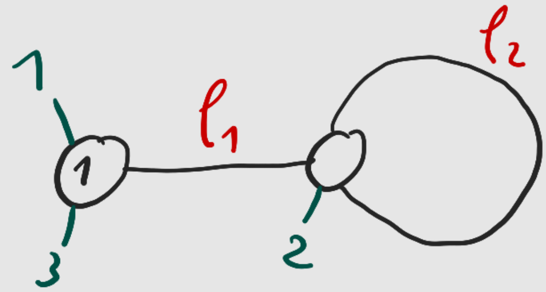

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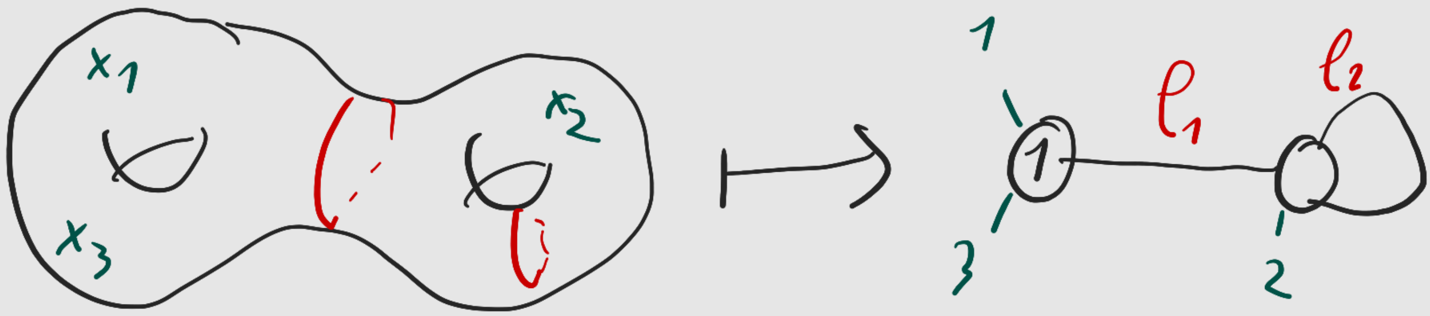
There is a map

$$\lambda_{g,n} : \mathcal{M}_{g,n} \longrightarrow \mathcal{M}^{trop}_{g,n}$$



$$\ell_i = \log(\varepsilon / \ell(\gamma_i))$$

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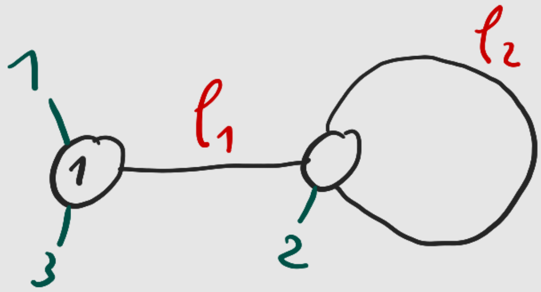

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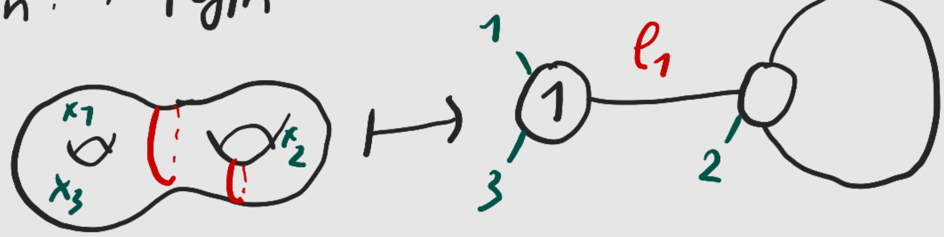

<u>Proposition:</u> [CGP, HP]

$\lambda_{g,n} : M_{g,n} \to M^{trop}_{g,n}$ is a proper surjective continuous map

$eV_{g,n}$ is the open locus inside $\mathcal{M}_{g,n}^{trop}$ of all graphs with trivial genus marking

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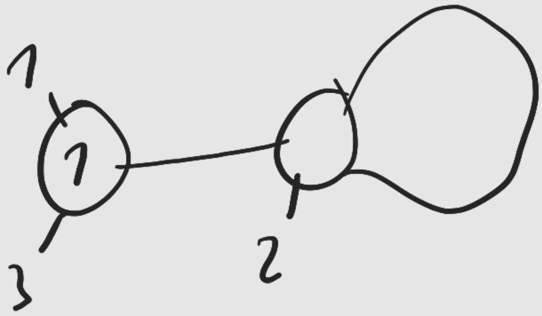
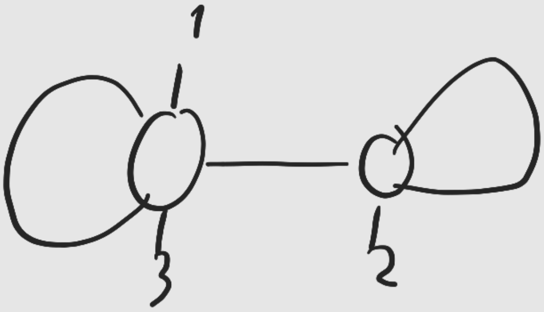
Additional comments



$\notin eV_{2,3}$



$\in eV_{2,3}$

$\mathcal{V}_{g,n}$ is the open locus inside $M^{trop}_{g,n}$ of all graphs with trivial genus marking

$$\begin{tikzcd} HM_{g,n} \arrow[r, hook, "open"] \arrow[d, "proper"] \arrow[dr, phantom, "\lrcorner", very near start] & M_{g,n} \arrow[d, "proper"] \\ CV_{g,n} \arrow[r, hook, "open"'] & M^{trop}_{g,n} \end{tikzcd}$$

Additional comments

$H^*_c(-;\mathbb{Q})$ is contravariant wrt. proper maps

Covariant wrt open embeddings

$CV_{g,n}$ is the open locus inside $M^{trop}_{g,n}$ of all graphs with trivial genus marking

$$\begin{tikzcd}
HM_{g,n} \arrow[r, hook, "\text{open}"] \arrow[d, "\text{proper}"'] & M_{g,n} \arrow[d, "\text{proper}"] \\
CV_{g,n} \arrow[r, hook, "\text{open}"'] & M^{trop}_{g,n}
\end{tikzcd}$$

$$\begin{tikzcd}
H_c^*(HM_{g,n};\mathbb{Q}) \arrow[r] & H_c^*(M_{g,n};\mathbb{Q}) \\
H_c^*(CV_{g,n};\mathbb{Q}) \arrow[u] \arrow[r, "\cong"', "{[CGP]}"'] & H_c^*(M^{trop}_{g,n};\mathbb{Q}) \arrow[u, "\text{injective}"']
\end{tikzcd}$$

isomorphism onto $gr_0^W H_c^*(M_{g,n})$

$\mathcal{CV}_{g,n}$ is the open locus inside $\mathcal{M}_{g,n}^{trop}$ of all graphs with trivial genus marking

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$\rightsquigarrow$

$$\begin{tikzcd} H_c^*(HM_{g,n};\mathbb{Q}) \arrow[r] & H_c^*(M_{g,n};\mathbb{Q}) \\ H_c^*(CV_{g,n};\mathbb{Q}) \arrow[u, "\text{injective}"] \arrow[r, "\cong"'] & H_c^*(M_{g,n}^{trop};\mathbb{Q}) \arrow[u, "\text{injective}"'] \end{tikzcd}$$

<u>Corollary</u>: $H_{4g-6}(\mathrm{Mod}(V_g);\mathbb{Q})$ grows at least exponentially in g.

Comparison of spectral sequences:

The map $\iota: \mathcal{M}_{g,n} \to \mathcal{M}_g$ induces a Leray-Serre spectral sequence

$$E_2^{p,q} = H_c^p(\mathcal{M}_g, H_c^q(\mathrm{Conf}_n(\Sigma_g);\mathbb{Q})) \Rightarrow H_c^{p+q}(\mathcal{M}_{g,n};\mathbb{Q})$$

Additional comments

$$H_c^p(\mathcal{M}_g, R^q\iota_! \underline{\mathbb{Q}}) \Rightarrow H_c^{p+q}(\mathcal{M}_{g,n};\underline{\mathbb{Q}})$$

is a local system, with stalks

$$(R^q\iota_!\underline{\mathbb{Q}})_{\Sigma_g} \cong H_c^q(\mathrm{Conf}_n(\Sigma_g);\mathbb{Q})$$

Comparison of spectral sequences:

The map $\mathcal{M}_{g,n} \to \mathcal{M}_g$ induces a Leray-Serre spectral sequence

$$E_2^{p,q} = gr_0^W H_c^p(\mathcal{M}_g, H_c^q(\mathrm{Conf}_n(\Sigma_g); \mathbb{Q})) \Rightarrow gr_0^W H_c^{p+q}(\mathcal{M}_{g,n}; \mathbb{Q})$$

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Bibby
Chan
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Yun

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[BCGY]

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[BCGY]

We can also define a spectral sequence

Conjecture: [H.-Petersen]

The two spectral sequences are isomorphic.

Our work provides a candidate map for this isomorphism:

$$\begin{tikzcd} HM_{g,n} \arrow[r, hook] \arrow[d] & M_{g,n} \arrow[d] \\ CV_{g,n} \arrow[r, hook] & M_{g,n}^{trop} \end{tikzcd}$$

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$$\begin{tikzcd} H\mathcal{M}_{g,n} \arrow[r, hook] \arrow[d] & \mathcal{M}_{g,n} \arrow[d] \\ CV_{g,n} \arrow[r, hook] & \mathcal{M}^{trop}_{g,n} \end{tikzcd}$$

$$H^p_c\left(CV_g, H^q_c\left(\mathrm{Conf}_n\left(\bigvee_{i=1}^{g} S^1\right); \mathbb{Q}\right)\right)$$
$$\downarrow$$
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- There exist natural filtrations on $H_c^*(\mathrm{Conf}_n(V_g);\mathbb{Q})$ and $H_c^*(\mathrm{Conf}_n(\Sigma_g);\mathbb{Q})$, whose associated graded pieces are representations of $\mathfrak{S}_n \times GL(H^1(V_g))$, resp. of $\mathfrak{S}_n \times Sp(H^1(\Sigma_g))$.

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<u>Theorem</u>: [H.-Petersen] The multiplicity of V_λ inside $gr_{|\lambda|} H_c^a(\mathrm{Conf}_n(V_g);\mathbb{Q})$ coincides with the multiplicity of $\mathbb{V}_\lambda$ inside $gr_{|\lambda|} H_c^a(\mathrm{Conf}_n(\Sigma_g);\mathbb{Q})$.

Additional comments

all pieces have genus 0

$$\begin{array}{ccc} \mathcal{H}\mathcal{M}_{g,n} & \hookrightarrow & \mathcal{M}_{g,n} \\ \| & & \| \\ B\mathrm{Mod}(V_{g,n}) & & B\mathrm{Mod}(\Sigma_{g,n}) \end{array}$$

$$\mathrm{Mod}(V_{g,n}) \hookrightarrow \mathrm{Mod}(\Sigma_{g,n})$$

[Palmer]

<u>Q</u>: What if we consider instead all pieces have genus $\leq \hat{g}$.