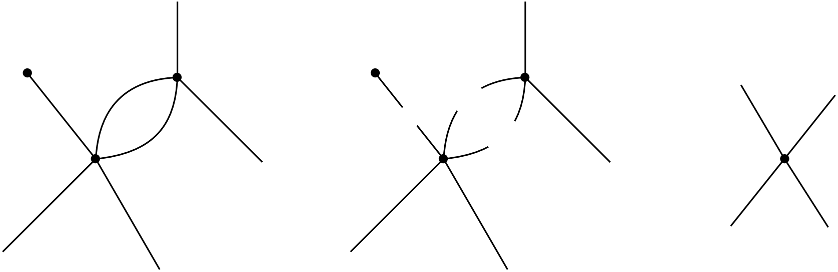
Open conformal field theories and ansular functors

Lukas Woike
Université Bourgogne Europe

Moduli and Friends Seminar
July 7, 2025
(building partially on joint work with L. Müller and A. Brochier)

A category of graphs [Costello 04]

We consider the following operations for graphs:



Γ $\qquad$ $\nu(\Gamma)$ $\qquad$ $\pi_0(\Gamma)$

A category of graphs

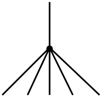
Definition [Costello 04]

We denote by Graphs the category

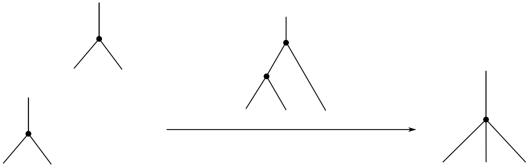
- whose objects are finite disjoint unions of corollas
- and whose morphisms $\Gamma : T \longrightarrow T'$ are equivalence classes of graphs Γ with identifications $T \cong \nu(\Gamma)$ and $T' \cong \pi_0(\Gamma)$.

Disjoint union endows Graphs with a symmetric monoidal structure.

an object:



a morphism:



Operads, cyclic operads and modular operads

The category Graphs has a subcategory Forests with the same objects, but only forests as morphisms. There is also a category RForests of rooted forests. We obtain symmetric monoidal functors

$$\mathsf{RForests} \longrightarrow \mathsf{Forests} \longrightarrow \mathsf{Graphs}\ .$$

Operads, cyclic operads and modular operads

The category Graphs has a subcategory Forests with the same objects, but only forests as morphisms. There is also a category RForests of rooted forests. We obtain symmetric monoidal functors

$$\mathsf{RForests} \longrightarrow \mathsf{Forests} \longrightarrow \mathsf{Graphs}\ .$$

Definition [following Costello building on Getzler-Kapranov]

Let $\mathcal{S}$ be a symmetric monoidal (higher) category. An *operad/cyclic operad/modular operad* with values in $\mathcal{S}$ is a symmetric monoidal functor

$$\mathcal{O} : \mathsf{RForests}/\mathsf{Forests}/\mathsf{Graphs} \longrightarrow \mathcal{S}$$

that is also equipped with operadic identities.

Operads, cyclic operads and modular operads

The category Graphs has a subcategory Forests with the same objects, but only forests as morphisms. There is also a category RForests of rooted forests. We obtain symmetric monoidal functors

$$\mathrm{RForests} \longrightarrow \mathrm{Forests} \longrightarrow \mathrm{Graphs}\ .$$

Definition [following Costello building on Getzler-Kapranov]

Let $\mathcal{S}$ be a symmetric monoidal (higher) category. An *operad/cyclic operad/modular operad* with values in $\mathcal{S}$ is a symmetric monoidal functor

$$\mathcal{O} : \mathrm{RForests}/\mathrm{Forests}/\mathrm{Graphs} \longrightarrow \mathcal{S}$$

that is also equipped with operadic identities.

If T_n is the corolla with legs numbered by $0, \ldots, n$, the object $\mathcal{O}(T_n) \in \mathcal{S}$ describes the n-ary operations (operations of total arity $n+1$).

Cyclic and modular algebras

Let $X \in \mathcal{S}$ be an object in a symmetric monoidal bicategory. Suppose that $\kappa : X \otimes X \longrightarrow I$ is a *non-degenerate symmetric pairing*, where

Let $X \in \mathcal{S}$ be an object in a symmetric monoidal bicategory. Suppose that $\kappa : X \otimes X \longrightarrow I$ is a *non-degenerate symmetric pairing*, where

- *non-degeneracy* means that κ exhibits X as its own dual in the homotopy category of $\mathcal{S}$ (in particular, there is a coevaluation $\Delta : I \longrightarrow X \otimes X$),

Cyclic and modular algebras

Let $X \in \mathcal{S}$ be an object in a symmetric monoidal bicategory. Suppose that $\kappa : X \otimes X \longrightarrow I$ is a *non-degenerate symmetric pairing*, where

- *non-degeneracy* means that κ exhibits X as its own dual in the homotopy category of $\mathcal{S}$ (in particular, there is a coevaluation $\Delta : I \longrightarrow X \otimes X$),
- and *symmetry* means that κ is a homotopy fixed point of the $\mathbb{Z}_2$-action on $\mathcal{S}(X \otimes X, I)$.

The symmetric monoidal bicategory $\mathrm{Lex}^{\mathrm{f}}$

Fix an algebraically closed field k. $\mathrm{Lex}^{\mathrm{f}}$ is the symmetric monoidal bicategory of *finite linear categories* in the sense of Etingof-Ostrik (linear abelian categories with finite-dimensional morphism spaces, enough projective objects, finitely many isomorphism classes of simple objects such that every object has finite length).
1-morphisms are left exact functors. 2-morphisms are linear natural transformations.

Cyclic and modular algebras

One checks that the assignment

$$\text{corolla } T \longmapsto \mathcal{S}(X^{\otimes \mathsf{Legs}(T)}, I)$$

extends to a symmetric monoidal functor

$$\mathsf{End}_{\kappa}^{X} : \mathsf{Graphs} \longrightarrow \mathsf{Cat}\ ,$$

the *modular endomorphism operad associated to (X, κ)*.

Cyclic and modular algebras

One checks that the assignment

$$\text{corolla } T \longmapsto \mathcal{S}(X^{\otimes \mathsf{Legs}(T)}, I)$$

extends to a symmetric monoidal functor

$$\mathsf{End}_{\kappa}^{X} : \mathsf{Graphs} \longrightarrow \mathsf{Cat} \ ,$$

the *modular endomorphism operad associated to* (X, κ).
The restriction of $\mathsf{End}_{\kappa}^{X}$ to Forests is the *cyclic endomorphism operad*.

Cyclic and modular algebras

One checks that the assignment

$$\text{corolla } T \longmapsto \mathcal{S}(X^{\otimes \mathsf{Legs}(T)}, I)$$

extends to a symmetric monoidal functor

$$\mathsf{End}_\kappa^X : \mathsf{Graphs} \longrightarrow \mathsf{Cat} \ ,$$

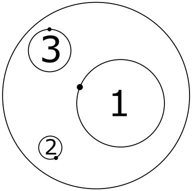
the *modular endomorphism operad associated to* (X, κ).
The restriction of End_κ^X to Forests is the *cyclic endomorphism operad*.

Definition [Müller-W., extending Costello]

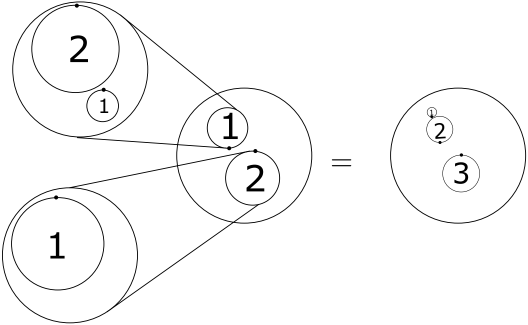
Let $\mathcal{O}$ be a Cat-valued cyclic (modular) operad. For any symmetric monoidal bicategory $\mathcal{S}$, an $\mathcal{S}$-valued *cyclic (modular) $\mathcal{O}$-algebra* is an object $X \in \mathcal{S}$, a non-degenerate symmetric pairing $\kappa : X \otimes X \longrightarrow I$ and a symmetric monoidal transformation $A : \mathcal{O} \longrightarrow \mathsf{End}_\kappa^X$.

Again, we need a version with operadic identities. This is omitted here.

Framed little disk operad



$fE_2(3)$



Theorem [Wahl 01, Salvatore-Wahl 03]

Framed little disks algebras in Cat *are equivalent to balanced braided categories.*

Reminder on balanced braided categories

- *braiding* on a monoidal category: natural isomorphism $X \otimes Y \longrightarrow Y \otimes X$ subject to the hexagon axioms. A braiding on a finite tensor category is called *non-degenerate* if the only objects that trivially double braid with all other objects are finite direct sums of the monoidal unit.

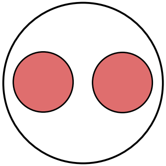
Reminder on balanced braided categories

- *braiding* on a monoidal category: natural isomorphism $X \otimes Y \longrightarrow Y \otimes X$ subject to the hexagon axioms. A braiding on a finite tensor category is called *non-degenerate* if the only objects that trivially double braid with all other objects are finite direct sums of the monoidal unit.
- *balancing* on a braided monoidal category: natural isomorphism $\theta_X : X \longrightarrow X$ subject to

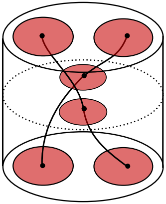
$$\begin{aligned} \theta_{X \otimes Y} &= c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y) \ , \\ \theta_I &= \mathrm{id}_I \ . \end{aligned}$$

How does this correspondence look like?

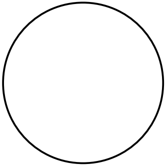
Textbook reference: [Fresse]



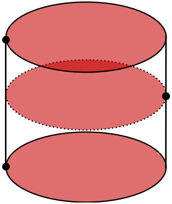
$\longmapsto \otimes : \mathcal{A} \boxtimes \mathcal{A} \longrightarrow \mathcal{A}$



$\longmapsto c : X \otimes Y \longrightarrow Y \otimes X$



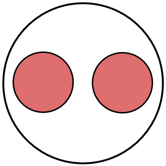
$\longmapsto I \in \mathcal{A}$



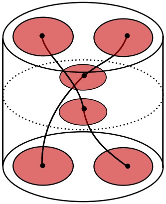
$\longmapsto \theta : \mathrm{id}_{\mathcal{A}} \Longrightarrow \mathrm{id}_{\mathcal{A}}$

How does this correspondence look like?

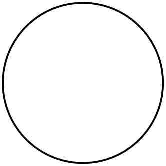
Textbook reference: [Fresse]



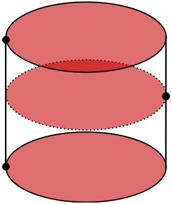
$\longmapsto \otimes : \mathcal{A} \boxtimes \mathcal{A} \longrightarrow \mathcal{A}$



$\longmapsto c : X \otimes Y \longrightarrow Y \otimes X$



$\longmapsto I \in \mathcal{A}$

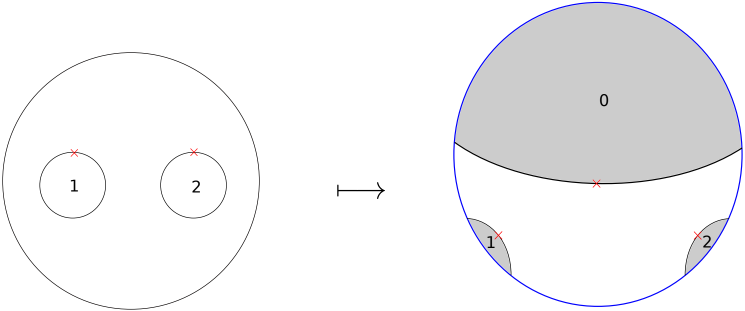


$\longmapsto \theta : \mathrm{id}_{\mathcal{A}} \Longrightarrow \mathrm{id}_{\mathcal{A}}$

Next critical observation: The framed E_2-operad is *cyclic*. What are the cyclic algebras?

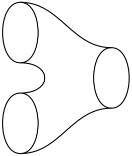
Cyclic structure on the framed little disks

The framed little disk operad is equivalent to the cyclic operad of genus zero surfaces. The latter has a cyclic structure by renumbering the boundary components.



Cyclic framed E_2-algebras (in $\mathrm{Lex}^{\mathrm{f}}$)

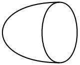
[Wahl 01, Salvatore-Wahl 03]



$\longmapsto \quad \otimes : \mathcal{A} \boxtimes \mathcal{A} \longrightarrow \mathcal{A}$

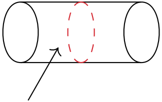
monoidal product

plus braiding $c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$



$\longmapsto \quad I \in \mathcal{A}$

monoidal unit



$\longmapsto \quad \theta : \mathrm{id}_{\mathcal{A}} \Longrightarrow \mathrm{id}_{\mathcal{A}}$

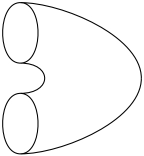
balancing

Dehn twist

$$\theta_{X \otimes Y} = c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y)$$
$$\theta_I = \mathrm{id}_I$$

[Müller-W. 20-22]

cyclic structure



ribbon Grothendieck-Verdier duality in the sense of Boyarchenko-Drinfeld

$$D : \mathcal{A} \xrightarrow{\cong} \mathcal{A}^{\mathrm{opp}}$$

$$\mathrm{Hom}_{\mathcal{A}}(- \otimes Y, K) \cong \mathrm{Hom}_{\mathcal{A}}(-, DY)$$

with $K := DI$

$$\theta_{DX} = D\theta_X$$

The Baez-Dolan microcosm principle

Baez-Dolan microcosm principle

An algebraic structure of a certain type (where the type is typically encoded by an operad) can be defined in a category carrying the same algebraic structure, but one categorical level higher.

The Baez-Dolan microcosm principle

Baez-Dolan microcosm principle

An algebraic structure of a certain type (where the type is typically encoded by an operad) can be defined in a category carrying the same algebraic structure, but one categorical level higher.

Example

An *algebra* can be defined in any *monoidal* category.

The Baez-Dolan microcosm principle

Baez-Dolan microcosm principle

An algebraic structure of a certain type (where the type is typically encoded by an operad) can be defined in a category carrying the same algebraic structure, but one categorical level higher.

Example

An *algebra* can be defined in any *monoidal* category.

Goal

Adapt this to cyclic and modular operads and use it in quantum topology.

Modular algebras in modular algebras

- Let $\mathcal{O} : \mathrm{Graphs} \longrightarrow \mathrm{Cat}$ be a category-valued modular operad (the same works for cyclic operads). Consider the Grothendieck construction $\int \mathcal{O}$, the category of all pairs (T, o) with $T \in \mathrm{Graphs}$ and $o \in \mathcal{O}(T)$. This is a symmetric monoidal category.

Modular algebras in modular algebras

- Let $\mathcal{O}$: Graphs $\longrightarrow$ Cat be a category-valued modular operad (the same works for cyclic operads). Consider the Grothendieck construction $\int \mathcal{O}$, the category of all pairs (T, o) with $T \in$ Graphs and $o \in \mathcal{O}(T)$. This is a symmetric monoidal category.
- Let $\mathcal{A}$ be a modular $\mathcal{O}$-algebra, for specificity in $\mathsf{Lex}^{\mathsf{f}}$. Denote by $\Delta = \int^{X \in \mathcal{A}} DX \boxtimes X$ its coevaluation object. Here $D : \mathcal{A}^{\mathsf{opp}} \longrightarrow \mathcal{A}$ is the equivalence induced by the pairing $\kappa : \mathcal{A} \boxtimes \mathcal{A} \longrightarrow \mathsf{vect}$.

Modular algebras in modular algebras

- Let $\mathcal{O} : \mathsf{Graphs} \longrightarrow \mathsf{Cat}$ be a category-valued modular operad (the same works for cyclic operads). Consider the Grothendieck construction $\int \mathcal{O}$, the category of all pairs (T, o) with $T \in \mathsf{Graphs}$ and $o \in \mathcal{O}(T)$. This is a symmetric monoidal category.
- Let $\mathcal{A}$ be a modular $\mathcal{O}$-algebra, for specificity in $\mathsf{Lex}^{\mathsf{f}}$. Denote by $\Delta = \int^{X \in \mathcal{A}} DX \boxtimes X$ its coevaluation object. Here $D : \mathcal{A}^{\mathrm{opp}} \longrightarrow \mathcal{A}$ is the equivalence induced by the pairing $\kappa : \mathcal{A} \boxtimes \mathcal{A} \longrightarrow \mathsf{vect}$.
- A self-dual object $X \in \mathcal{A}$ is an object equipped with an isomorphism $\psi : X \longrightarrow DX$ such that $D\psi$ agrees with ψ under the isomorphism $X \longrightarrow D^2X$ coming from the symmetry of the pairing.

Modular algebras in modular algebras

Let $\mathcal{O}$ be a Cat-valued modular operad and $X \in \mathcal{A}$ a self-dual object in a modular $\mathcal{O}$-algebra $\mathcal{A}$.

Theorem [W. 24] 'Flat vector bundles over the space of operations'

By sending a corolla T and $o \in \mathcal{O}(T)$ to the vector space $\mathcal{A}_o(X, \ldots, X)$, we obtain a symmetric monoidal functor $\mathbb{V}_X^{\mathcal{A}} : \int \mathcal{O} \longrightarrow \mathrm{vect}$.

Definition [W. 24] 'Modular microcosm principle'

A *modular $\mathcal{O}$-algebra in $\mathcal{A}$* is a self-dual object X together with a monoidal transformation $k \longrightarrow \mathbb{V}_X^{\mathcal{A}}$ of symmetric monoidal functors $\int \mathcal{O} \longrightarrow \mathrm{vect}$.

Again, all of of this works also for cyclic operads and their algebras.

Cyclic associative algebras

Frobenius algebras

Cyclic associative algebras in $\mathsf{Lex}^{\mathsf{f}}$ are equivalent to *pivotal Grothendieck-Verdier categories* [Müller-W. 20]; a class of examples are pivotal finite tensor categories. A cyclic associative algebra in a pivotal Grothendieck-Verdier category $\mathcal{A}$ is a symmetric Frobenius algebra F in $\mathcal{A}$. The corresponding transformation $k \longrightarrow \mathbb{V}_F^{\mathcal{A}}$ selects for a corolla with n legs and the 'standard' operation o of total arity n a vector in $\mathrm{Hom}_{\mathcal{A}}(K, F^{\otimes n})$, namely the total arity n operation of the Frobenius algebra F. In the rigid case, this is the standard notion. Beyond that, one recovers the definitions of [Fuchs-Schaumann-Schweigert-Wood 24]. For the cyclic framed E_2-operad, we obtain *symmetric braided commutative Frobenius algebras* in ribbon Grothendieck-Verdier categories.

This is very reassuring, but by no means impressive... we might as well have defined all this by hand, without the microcosm principle.

Modular extension

Let $\mathcal{O}$ be a category-valued cyclic operad. Denote by $\mathsf{U}_{\int}\mathcal{O}$ the derived modular envelope of $\mathcal{O}$ in the sense of [Costello], the smallest extension of $\mathcal{O}$ to a modular operad, in a homotopy coherent way. Then any cyclic $\mathcal{O}$-algebra extends uniquely to a modular $\mathsf{U}_{\int}\mathcal{O}$-algebra $\widehat{\mathcal{A}}$ [Müller-W. 22].

Modular extension

Let $\mathcal{O}$ be a category-valued cyclic operad. Denote by $\mathsf{U}_{\int}\mathcal{O}$ the derived modular envelope of $\mathcal{O}$ in the sense of [Costello], the smallest extension of $\mathcal{O}$ to a modular operad, in a homotopy coherent way. Then any cyclic $\mathcal{O}$-algebra extends uniquely to a modular $\mathsf{U}_{\int}\mathcal{O}$-algebra $\widehat{\mathcal{A}}$ [Müller-W. 22].

Theorem [W. 24] Modular extension

There is a canonical 'restriction-extension' equivalence

$$\mathsf{CycAlg}(\mathcal{O};\mathcal{A}) \simeq \mathsf{ModAlg}(\Pi|BU_{\int}\mathcal{O}|;\widehat{\mathcal{A}}) \ .$$

Open conformal field theories

- For the cyclic associative operad, $\Pi|BU_{\int}\mathrm{As}|$ is equivalent to the *open surface operad* O, the groupoid-valued operad whose operations of total arity $n \geq 0$ are connected compact oriented surfaces Σ with at least one boundary component and n parametrized intervals in $\partial\Sigma$; morphisms in the groupoids of operations are mapping classes [Costello,Giansiracusa, Müller-W.]. (The gluing is along intervals.)

Open conformal field theories

- For the cyclic associative operad, $\Pi|BU_{\int}\mathsf{As}|$ is equivalent to the *open surface operad* O, the groupoid-valued operad whose operations of total arity $n \geq 0$ are connected compact oriented surfaces Σ with at least one boundary component and n parametrized intervals in $\partial\Sigma$; morphisms in the groupoids of operations are mapping classes [Costello,Giansiracusa, Müller-W.]. (The gluing is along intervals.)
- $\mathsf{Lex}^{\mathsf{f}}$-valued modular O-algebras are categorified open two-dimensional topological field theories or simply *open modular functors* [Segal, Moore-Seiberg, Turaev, Tillmann, Bakalov-Kirillov, ...], the monodromy data of an *open conformal field theory*.

Open conformal field theories

- For the cyclic associative operad, $\Pi|BU_{\int}\mathsf{As}|$ is equivalent to the *open surface operad* O, the groupoid-valued operad whose operations of total arity $n \geq 0$ are connected compact oriented surfaces Σ with at least one boundary component and n parametrized intervals in $\partial\Sigma$; morphisms in the groupoids of operations are mapping classes [Costello, Giansiracusa, Müller-W.]. (The gluing is along intervals.)
- $\mathsf{Lex}^{\mathsf{f}}$-valued modular O-algebras are categorified open two-dimensional topological field theories or simply *open modular functors* [Segal, Moore-Seiberg, Turaev, Tillmann, Bakalov-Kirillov, ...], the monodromy data of an *open conformal field theory*.

Theorem [Müller-W. 20/24]

Cyclic associative algebras in Lex^f *are pivotal Grothendieck-Verdier categories, thereby giving us an equivalence between open modular functors and pivotal Grothendieck-Verdier categories.*

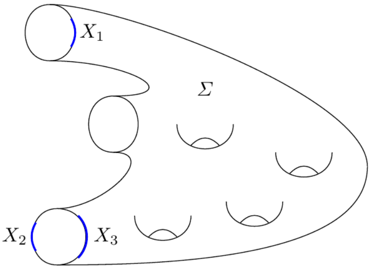
Open conformal field theories

Let $\mathcal{A}$ be a pivotal Grothendieck-Verdier category in $\mathsf{Lex}^{\mathsf{f}}$; this is a monoidal category $\mathcal{A}$ in $\mathsf{Lex}^{\mathsf{f}}$ with a Grothendieck-Verdier duality $D : \mathcal{A}^{\mathsf{opp}} \longrightarrow \mathcal{A}$ with a monoidal trivialization $\omega : \mathrm{id}_{\mathcal{A}} \cong D^2$ whose component $\omega_K : K \cong D^2K$ at the dualizing object K is the 'canonical map'.

Denote by $\mathcal{A}_{\mathbf{!}}$ the unique extension to an open modular functor.

Open conformal field theories

Let $\mathcal{A}$ be a pivotal Grothendieck-Verdier category in $\mathsf{Lex}^{\mathsf{f}}$; this is a monoidal category $\mathcal{A}$ in $\mathsf{Lex}^{\mathsf{f}}$ with a Grothendieck-Verdier duality $D : \mathcal{A}^{\mathrm{opp}} \longrightarrow \mathcal{A}$ with a monoidal trivialization $\omega : \mathrm{id}_{\mathcal{A}} \cong D^2$ whose component $\omega_K : K \cong D^2 K$ at the dualizing object K is the 'canonical map'.
Denote by $\mathcal{A}_!$ the unique extension to an open modular functor.



$$\longmapsto \quad \mathcal{A}_!(\Sigma; X_1, X_2, X_3) \curvearrowleft \mathsf{Map}(\Sigma)$$

(These mapping class group representations are called *spaces of conformal blocks* for the open conformal field theory.)

Open correlators

By the modular extension theorem modular O-algebras in $\mathcal{A}_{!}$ are equivalent to symmetric Frobenius algebra $F \in \mathcal{A}$. These give us linear maps $k \longrightarrow \mathcal{A}_{!}(\Sigma; F, \ldots, F)$ for all surfaces, i.e. vectors in the spaces of conformal blocks. By naturality of the assignment, these are mapping class group invariant and compatible with gluing. These are exactly the *correlators of the open conformal field theory* with monodromy data $\mathcal{A}$.

Open correlators

By the modular extension theorem modular O-algebras in $\mathcal{A}_{!}$ are equivalent to symmetric Frobenius algebra $F \in \mathcal{A}$. These give us linear maps $k \longrightarrow \mathcal{A}_{!}(\Sigma; F, \ldots, F)$ for all surfaces, i.e. vectors in the spaces of conformal blocks. By naturality of the assignment, these are mapping class group invariant and compatible with gluing. These are exactly the *correlators of the open conformal field theory* with monodromy data $\mathcal{A}$. We summarize:

Theorem [W. 24] 'Classification of open correlators'

Consistent systems of correlators for an open conformal field theories with monodromy data $\mathcal{A}$ (a pivotal Grothendieck-Verdier category) are equivalent to symmetric Frobenius algebras in $\mathcal{A}$.

Open correlators

- Let $\mathcal{A}$ be a pivotal finite tensor category and $L : \mathcal{A} \longrightarrow Z(\mathcal{A})$ the left adjoint to the forgetful functor $U : Z(\mathcal{A}) \longrightarrow \mathcal{A}$ from the Drinfeld center. Assume for simplicity that $\mathcal{A}$ is unimodular and spherical.

Open correlators

- Let $\mathcal{A}$ be a pivotal finite tensor category and $L : \mathcal{A} \longrightarrow Z(\mathcal{A})$ the left adjoint to the forgetful functor $U : Z(\mathcal{A}) \longrightarrow \mathcal{A}$ from the Drinfeld center. Assume for simplicity that $\mathcal{A}$ is unimodular and spherical.
- The open modular functor $\mathcal{A}_!$ describes the Lyubashenko modular functor for the modular category $Z(\mathcal{A})$, or rather its restriction along L [Müller-Schweigert-W.-Yang 23, Müller-W. 24]. Therefore, the above construction produces mapping class group invariants in the spaces of conformal blocks for $Z(\mathcal{A})$.

Open correlators

- Let $\mathcal{A}$ be a pivotal finite tensor category and $L : \mathcal{A} \longrightarrow Z(\mathcal{A})$ the left adjoint to the forgetful functor $U : Z(\mathcal{A}) \longrightarrow \mathcal{A}$ from the Drinfeld center. Assume for simplicity that $\mathcal{A}$ is unimodular and spherical.
- The open modular functor $\mathcal{A}_!$ describes the Lyubashenko modular functor for the modular category $Z(\mathcal{A})$, or rather its restriction along L [Müller-Schweigert-W.-Yang 23, Müller-W. 24]. Therefore, the above construction produces mapping class group invariants in the spaces of conformal blocks for $Z(\mathcal{A})$.
- But careful: A priori, these just glue along intervals!

Open correlators

- Let $\mathcal{A}$ be a pivotal finite tensor category and $L : \mathcal{A} \longrightarrow Z(\mathcal{A})$ the left adjoint to the forgetful functor $U : Z(\mathcal{A}) \longrightarrow \mathcal{A}$ from the Drinfeld center. Assume for simplicity that $\mathcal{A}$ is unimodular and spherical.
- The open modular functor $\mathcal{A}_!$ describes the Lyubashenko modular functor for the modular category $Z(\mathcal{A})$, or rather its restriction along L [Müller-Schweigert-W.-Yang 23, Müller-W. 24]. Therefore, the above construction produces mapping class group invariants in the spaces of conformal blocks for $Z(\mathcal{A})$.
- But careful: A priori, these just glue along intervals!
- In summary, we obtain a vast generalization of the open part of the correlator construction of [Fuchs-Runkel-Schweigert] from the early 2000s.

Ansular functors

Recall from above:

Theorem [Müller-W. 20]

Cyclic framed little disks algebras in Lex^f *are equivalent to ribbon Grothendieck-Verdier categories in* Lex^f.

Ansular functors

Recall from above:

Theorem [Müller-W. 20]

Cyclic framed little disks algebras in Lex^f *are equivalent to ribbon Grothendieck-Verdier categories in* Lex^f.

We denote by Hbdy the (groupoid-valued) *modular operad of handlebodies*. For a corolla T, its groupoid $\mathsf{Hbdy}(T)$ of operations has as objects connected compact oriented three-dimensional handlebodies H together with an embedding of $\bigsqcup_{\mathsf{Legs}(T)} \mathbb{D}^2$ into ∂H (called parametrization) and as morphisms isotopy classes of parametrization and orientation preserving diffeomorphisms.

Ansular functors

Recall from above:

Theorem [Müller-W. 20]

Cyclic framed little disks algebras in Lex^f *are equivalent to ribbon Grothendieck-Verdier categories in* Lex^f.

We denote by Hbdy the (groupoid-valued) *modular operad of handlebodies*. For a corolla T, its groupoid $\mathsf{Hbdy}(T)$ of operations has as objects connected compact oriented three-dimensional handlebodies H together with an embedding of $\bigsqcup_{\mathsf{Legs}(T)} \mathbb{D}^2$ into ∂H (called parametrization) and as morphisms isotopy classes of parametrization and orientation preserving diffeomorphisms.

Definition

For any symmetric monoidal bicategory $\mathcal{S}$, we call a modular $\mathcal{S}$-valued Hbdy-algebra an *ansular functor* with values in $\mathcal{S}$.

Ansular functors

We use a result of Giansiracusa on the *derived modular envelope* of framed E_2 (and several additional results) to prove:

Theorem [Müller-W. 22]

Genus zero restriction provides an equivalence between ansular functors and cyclic framed E_2-algebras.
In Lex^f, the ansular functor associated to a ribbon Grothendieck-Verdier category $\mathcal{A}$ sends a handlebody of genus g and n boundary components labeled with $X_1, \ldots, X_n$ to the hom space

$$\mathcal{A}(K, X_1 \otimes \cdots \otimes X_n \otimes \mathbb{F}^{\otimes g})$$

defined using the canonical coend $\mathbb{F} = \otimes \left(\int^{X \in \mathcal{A}} X \otimes DX \right)$ (D is the duality functor of $\mathcal{A}$).

This is a vast generalization of the Lyubashenko construction.

Classification of ansular correlators

Theorem [W. 24]

Consistent systems of correlators for an ansular functor based on the ribbon Grothendieck-Verdier category $\mathcal{A}$ are equivalent to symmetric commutative Frobenius algebras F in $\mathcal{A}$. In more detail, F produces for each handlebody H with n embedded disks mapping class group invariant vectors $\xi_H^F \in \widehat{\mathcal{A}}(H; F, \ldots, F)$. If $\mathcal{A}$ is a finite ribbon category and $F = I$, these are non-zero.

This is an entirely categorical construction of the *empty skein*.