Double lower central series and a double Johnson filtration for the Goeritz group of the sphere

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§1. Some basic group theory

§2. Applications to 2-dimensional topology

Group ⟶ « object » ⟶ define a property

K

- Derived subgroup: $[K, K]$
- Center: $Z(K)$
- Lower central series: $(\Gamma_i K)_{i \geq 1}$
- Derived series: $(D_i K)_{i \geq 1}$

- Abelian
- Nilpotent
- Solvable

cyclic $\subset$ Abelian $\subset$ Nilpotent $\subset$ Solvable

- $a, b \in K$: $[a, b] := aba^{-1}b^{-1}$
- $S, T \subset K$: $[S, T] := \left\langle\ [s, t] \mid s \in S, t \in T\ \right\rangle$

Lower central series

$$\Gamma_1 K \geq \Gamma_2 K \geq \Gamma_3 K \geq \cdots$$

$$\Gamma_1 K = K,$$

$$\Gamma_{n+1} K = [K, \Gamma_n K].$$

- K is **nilpotent** if $\exists i$ with $\Gamma_i K = \{e\}$

Derived series

$$D_1 K \geq D_2 K \geq D_3 K \geq \cdots$$

$$D_1 K = K,$$

$$D_{n+1} K = [D_n K, D_n K].$$

- K is **solvable** if $\exists i$ with $D_i K = \{e\}$

Idea : Generalize the indexing set

Good-ordered commutative monoid : commutative monoid $(\Lambda, +, 0)$ with a partial order $\leq$ compatible with addition and such that the zero element 0 is the smallest element.

Examples : $\mathbb{N}$ with usual order, $\mathbb{N}^2$ with component-wise order.

Λ-filtration : a Λ-filtration of K is a family $(K_\lambda)_{\lambda \in \Lambda}$ of normal subgroups of K such that

- $K_0 = K$
- $K_\lambda \supset K_{\lambda'}$ for $\lambda \leq \lambda'$
- $[K_\lambda, K_\mu] \subset K_{\lambda+\mu}$ for all $\lambda, \mu \in \Lambda$.

Examples : The lower central series $(\Gamma_i K)_{i \geq 0}$ is a $\mathbb{N}$-filtration (with $\Gamma_0 K = K$).

The derived series **is not** a $\mathbb{N}$-filtration.

$\mathbb{N}^2$-filtrations

A $\mathbb{N}^2$-filtration of K is a family $(K_{m,n})_{(m,n)\in\mathbb{N}^2}$ of normal subgroups of K such that

- $K_{0,0} = K$
- $K_{m,n} \supset K_{m',n'}$ for $(m,n) \leq (m',n')$
- $[K_{m,n}, K_{m',n'}] \subset K_{m+m',n+n'}$ for all $(m,n),(m',n') \in \mathbb{N}^2$

$$\begin{array}{ccccccc}
K_{0,0} & \supset & K_{0,1} & \supset & K_{0,2} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
K_{1,0} & \supset & K_{1,1} & \supset & K_{1,2} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
K_{2,0} & \supset & K_{2,1} & \supset & K_{2,2} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
K_{3,0} & \supset & K_{3,1} & \supset & K_{3,2} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

$$\bar{K}_{m,n} := \frac{K_{m,n}}{K_{m+1,n}\, K_{m,n+1}}$$ is an **Abelian group** for $(m,n) \neq (0,0)$

$$\bigoplus_{(m,n)\in\mathbb{N}^2\setminus\{(0,0)\}} \bar{K}_{m,n}$$ is a **bigraded Lie algabra.**

Double lower central series

Consider a triplet $(K, \bar{X}, \bar{Y})$ consisting of K and two normal subgroups $\bar{X}$ and $\bar{Y}$. The double lower central series of $(K, \bar{X}, \bar{Y})$ is the $\mathbb{N}^2$-filtration $(K_{m,n})_{(m,n)\in\mathbb{N}^2}$ given by

- $K_{0,0} = K$
- $K_{m,0} = \Gamma_m \bar{X}$ for $m \geq 1$
- $K_{0,n} = \Gamma_n \bar{Y}$ for $n \geq 1$
- $K_{m,n} = [K_{1,0}, K_{m-1,n}]\,[K_{0,1}, K_{m,n-1}]$ for $m, n \geq 1$

$$
\begin{array}{ccccccc}
K & \supset & \bar{Y} & \supset & \Gamma_2 \bar{Y} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
\bar{X} & \supset & K_{1,1} & \supset & K_{1,2} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
\Gamma_2 \bar{X} & \supset & K_{2,1} & \supset & K_{2,2} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
\Gamma_3 \bar{X} & \supset & K_{3,1} & \supset & K_{3,2} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

$K_{1,1} = [\bar{X}, \bar{Y}]$

$K_{2,1} = [\bar{X}, [\bar{X}, \bar{Y}]]$

$K_{2,2} = [\bar{Y}, [\bar{X}, [\bar{X}, \bar{Y}]]]\,[[\bar{X}, \bar{Y}], [\bar{X}, \bar{Y}]]$

Lemma

If $K = \bar{X}\bar{Y}$ and $(K_{i,j})_{(i,j)\in\mathbb{N}^2}$ is the double lower central series of $(K; \bar{X}, \bar{Y})$. Then, for $m \geq 1$, we have

$$\Gamma_m(K) = \prod_{i+j=m} K_{i,j}.$$

- $\Gamma_1 K = \bar{X}\bar{Y}$
- $\Gamma_2 K = K_{2,0}K_{1,1}K_{0,2}$
- $\Gamma_3 K = K_{3,0}K_{2,1}K_{1,2}K_{0,3}$

$$\begin{array}{ccccccc}
K = \bar{X}\bar{Y} & \supset & \bar{Y} & \supset & \Gamma_2\bar{Y} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
\bar{X} & \supset & K_{1,1} & \supset & K_{1,2} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
\Gamma_2\bar{X} & \supset & K_{2,1} & \supset & K_{2,2} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
\Gamma_3\bar{X} & \supset & K_{3,1} & \supset & K_{3,2} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

Johnson Filtration

Λ : good-ordered commutative monoid (e.g. $\mathbb{N}$ or $\mathbb{N}^2$).

$(K_\lambda)_{\lambda\in\Lambda}$ a group K endowed of a Λ-filtration.

Proposition

Let G be a group acting on K and such that $G(K_\lambda) \subset K_\lambda$. Set

$$G_\lambda := \{g \in G \mid [g, K_\mu] \subset K_{\lambda+\mu} \ \forall \mu \in \Lambda\}.$$

The the family $(G_\lambda)_{\lambda\in\Lambda}$ is a Λ-filtration of G. We call it Johnson filtration induced by the action of G on $(K_\lambda)_{\lambda\in\Lambda}$.

Here

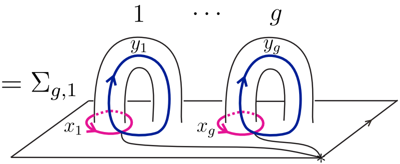
$$[g, K_\mu] = \{g(x)x^{-1} \mid x \in K_\mu\},$$

i.e., we take the commutator in $K \rtimes G$.

§2. Application to 2-dimensional topology

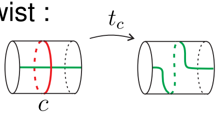
Mapping class group

$\Sigma = \Sigma_{g,1}$: Compact, connected oriented surface of genus g with one boundary component.



$$\mathcal{M} = \{h : \Sigma \xrightarrow{\sim} \Sigma \mid h_{\partial\Sigma} = \mathrm{Id}_{\partial\Sigma}\}/\text{isotopy}$$

Dehn Twist :



Dehn-Nielsen representation

$$\pi = \pi_1(\Sigma, *) = \langle x_1, \ldots, x_g, y_1, \ldots, y_g \rangle$$

The map

$$\rho : \mathcal{M} \longrightarrow \mathrm{Aut}(\pi), \quad h \longmapsto h_\#$$

is injective.

Lower central series of π

$$\Gamma_1\pi = \pi, \Gamma_2\pi = [\pi, \pi], \Gamma_3\pi = [\pi, [\pi, \pi]],$$

$$\Gamma_{n+1}\pi = [\pi, \Gamma_n\pi]$$

The action of $\mathcal{M}$ on π preserves the lower central series.

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \rho\ } & \mathrm{Aut}(\pi) \\ & {\scriptstyle \rho_n} \searrow & \downarrow \\ & & \mathrm{Aut}(\pi/\Gamma_{n+1}\pi) \end{array}$$

Torelli group

$$\mathcal{I} = \left\{ h \in \mathcal{M} \mid h_* = \mathrm{Id}_{H_1(\Sigma)} \right\}$$
$$= \ker(\rho_1)$$

Johnson filtration

$$\mathcal{I} = J_1\mathcal{M} \supset J_2\mathcal{M} \supset J_3\mathcal{M} \supset \cdots$$

$$J_n\mathcal{M} = \{h \in \mathcal{M} \mid \forall x \in \pi : \ h_\#(x)x^{-1} \in \Gamma_{n+1}\pi\}$$
$$= \{h \in \mathcal{M} \mid [h, \pi] \subset \Gamma_{n+1}\pi\}$$
(comm. in $\pi \rtimes \mathcal{M}$)
$$= \ker(\rho_n).$$

Notice that

$$[h, \pi] \subset \Gamma_{n+1}\pi \iff [h, \Gamma_m\pi] \subset \Gamma_{m+n}\pi \quad \forall m \geq 0$$

- The case $\Lambda = \mathbb{N}^2 = \{(m,n) \in \mathbb{Z} \mid m,n \geq 0\}$

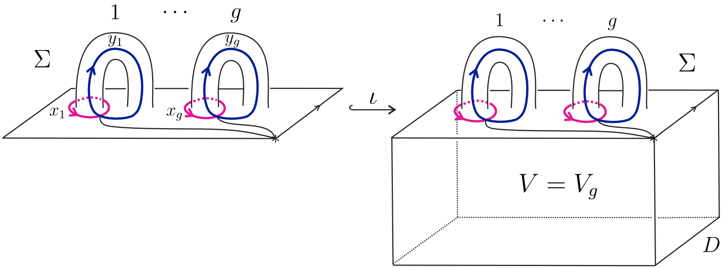
  with the order $(m,n) \leq (m',n')$ iff $m \leq m'$ and $n \leq n'$.

Goal

Define a **doubly indexed filtration** for the mapping class group $\mathcal{M}$ and for the Goeritz group $\mathcal{G} \leq \mathcal{M}$

$$
\begin{array}{ccccccccc}
\mathcal{M} = \mathcal{M}_{-1,-1} & \supset & \mathcal{M}_{-1,0} & \supset & \mathcal{M}_{-1,1} & \supset & \mathcal{M}_{-1,2} & \cdots \\
\cup & & \cup & & \cup & & \cup & \\
\mathcal{M}_{0,-1} & \supset & \mathcal{G} = \mathcal{M}_{0,0} & \supset & \mathcal{M}_{0,1} & \supset & \mathcal{M}_{0,2} & \cdots \\
\cup & & \cup & & \cup & & \cup & \\
\mathcal{M}_{1,-1} & \supset & \mathcal{M}_{1,0} & \supset & \mathcal{M}_{1,1} & \supset & \mathcal{M}_{1,2} & \cdots \\
\cup & & \cup & & \cup & & \cup & \\
\mathcal{M}_{2,-1} & \supset & \mathcal{M}_{2,0} & \supset & \mathcal{M}_{2,1} & \supset & \mathcal{M}_{2,2} & \cdots \\
\cup & & \cup & & \cup & & \cup & \\
\mathcal{M}_{3,-1} & \supset & \mathcal{M}_{3,0} & \supset & \mathcal{M}_{3,1} & \supset & \mathcal{M}_{3,2} & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

Goeritz group of $\mathbb{S}^3$



- $V = V_g$: **Handlebody of genus g standardly embedded** into $\mathbb{S}^3$, i.e., $V' = V'_g := \overline{\mathbb{S}^3 \setminus V_g}$ is also a handlebody.
- $\partial V = \partial V' = \Sigma \cup D$, where $D \in \partial V = \partial V'$ is a fixed disk.
- We have two embeddings $\iota : \Sigma \to V$ and $\iota' : \Sigma \to V'$.
- Consider the two normal subgroups of $K = \pi = \pi_1(\Sigma, *)$:

$$\bar{X} := \ker\big(K \xrightarrow{\iota_\#} \pi_1(V_g, *)\big) \quad \text{and} \quad \bar{Y} := \ker\big(K \xrightarrow{\iota'_\#} \pi_1(V'_g, *)\big).$$

Goeritz group of $\mathbb{S}^3$

Using generators :

- $K = \pi_1(\Sigma, *) = \langle x_1, \ldots, x_g, y_1, \ldots, y_g \rangle$ free group on $2g$ generators
- $\bar{X} = \langle\langle x_1, \ldots, x_g \rangle\rangle_K$ normal closure in K
- $\bar{Y} = \langle\langle y_1, \ldots, y_g \rangle\rangle_K$

Let $(K_{i,j})_{(i,j)\in\mathbb{N}^2}$ be the **double lower central series** of $(K; \bar{X}, \bar{Y})$.

The subgroup of $\mathcal{M}$:

$$\mathcal{G} = \mathcal{G}_{g,1} = \{ \ h \in \mathcal{M} \mid \quad h_\#(\bar{X}) \subset \bar{X}, \quad h_\#(\bar{Y}) \subset \bar{Y} \ \}$$

acts on K preserving the double lower central series of $(K; \bar{X}, \bar{Y})$

The group $\mathcal{G}$ is called **Goeritz group of $\mathbb{S}^3$** of genus g (relative to the disk D)

Goeritz group of $\mathbb{S}^3$

$K = \langle x_1, \ldots, x_g, y_1, \ldots, y_g \rangle$ $\quad\quad$ $\bar{X} = \langle\langle x_1, \ldots, x_g \rangle\rangle_K$ $\quad\quad$ $\bar{Y} = \langle\langle y_1, \ldots, y_g \rangle\rangle_K$

$$\mathcal{G} = \{ \ h \in \mathcal{M} \mid \quad h_\#(\bar{X}) \subset \bar{X}, \quad h_\#(\bar{Y}) \subset \bar{Y} \ \}$$

Equivalent definitions :

- $\mathcal{G}$ is the group of isotopy classes of orientation-preserving homeomorphisms $h : \mathbb{S}^3 \to \mathbb{S}^3$ such that $h(\partial V) = \partial V$ and $h_{|D} = \mathrm{Id}_D$.
- $\mathcal{G}$ is the subgroup of $\mathcal{M}$ consisting of the elements which extend to the two handlebodies V and V'.
- $\mathcal{G}$ is the subgroup of $\mathcal{M}$ consisting of the elements which preserve the standard Heegaard splitting of the 3-sphere.

Remark. We do not know if $\mathcal{G}$ is finitely generated for genus > 3.

Powell's conjecture : The group $\mathcal{G}$ is finitely generated. Moreover, Powell proposed a set of 5 generators.

Goeritz group of $\mathbb{S}^3$

$K = \langle x_1, \ldots, x_g, y_1, \ldots, y_g \rangle \qquad \bar{X} = \langle\langle x_1, \ldots, x_g \rangle\rangle_K \qquad \bar{Y} = \langle\langle y_1, \ldots, y_g \rangle\rangle_K$

$$\mathcal{G} = \{\ h \in \mathcal{M} \mid \quad h_\#(\bar{X}) \subset \bar{X}, \quad h_\#(\bar{Y}) \subset \bar{Y}\ \}$$

Some related groups :

Handlebody groups

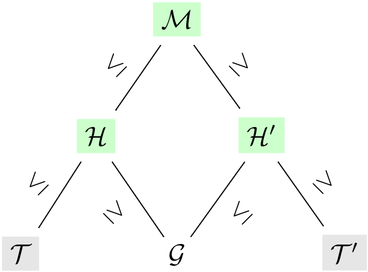
$$\mathcal{H} = \mathrm{MCG}(V, D) = \{h \in \mathcal{M} \mid \quad h_\#(\bar{X}) \subset \bar{X}\ \}$$

$$\mathcal{H}' = \mathrm{MCG}(V', D) = \{h \in \mathcal{M} \mid \quad h_\#(\bar{Y}) \subset \bar{Y}\ \}$$

Luft's groups (Twist groups)

$$\mathcal{T} = \ker\big(\mathcal{H} \longrightarrow \mathrm{Aut}(\pi_1(V, *))\big)$$

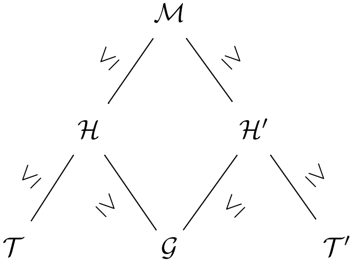
$$\mathcal{T}' = \ker\big(\mathcal{H}' \longrightarrow \mathrm{Aut}(\pi_1(V', *))\big)$$



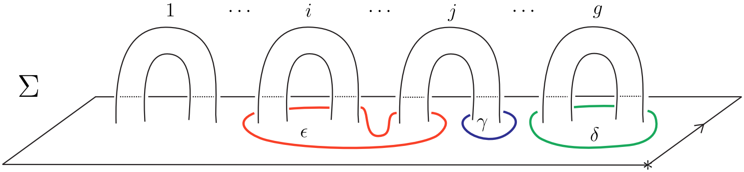
· finitely generated

· **non** finitely generated

Examples



- $t_\epsilon,\ t_\gamma,\ t_\delta \in \mathcal{T}$
- $t_\epsilon,\ t_\gamma \notin \mathcal{G}$
- $t_\epsilon t_\gamma^{-1},\ t_\delta \in \mathcal{G}$



Action on $H_1(\Sigma, \mathbb{Z})$

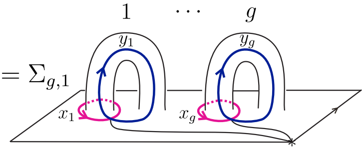
$K = \langle x_1, \ldots, x_g, y_1, \ldots, y_g \rangle$

$a_i = [x_i], \quad b_i = [y_i]$

$H = H_1(\Sigma; \mathbb{Z}) = \langle a_1, b_1, \ldots, a_g, b_g \rangle_{\mathrm{ab}}$

$\mathrm{Sp}(H, \omega) \simeq \mathrm{Sp}(2g, \mathbb{Z})$

$\Sigma = \Sigma_{g,1}$



$\omega : H \otimes H \longrightarrow \mathbb{Z}$
intersection form

Lemma (folklore)

Let $\sigma : \mathcal{M} \to \mathrm{Sp}(2g, \mathbb{Z})$ be the action on H. Then

- $\sigma(\mathcal{H}) = \left\{ \begin{pmatrix} P & R \\ 0 & (P^T)^{-1} \end{pmatrix} \;\middle|\; P^{-1}R \text{ is symmetric} \right\}$
- $\sigma(\mathcal{H}') = \left\{ \begin{pmatrix} P & 0 \\ R & (P^T)^{-1} \end{pmatrix} \;\middle|\; P^{T}R \text{ is symmetric} \right\}$
- $\sigma(\mathcal{G}) = \left\{ \begin{pmatrix} P & 0 \\ 0 & (P^T)^{-1} \end{pmatrix} \;\middle|\; P \in \mathrm{GL}(g, \mathbb{Z}) \right\} \simeq \mathrm{GL}(g, \mathbb{Z})$
- $\sigma(\mathcal{T}) = \left\{ \begin{pmatrix} \mathrm{Id}_g & R \\ 0 & \mathrm{Id}_g \end{pmatrix} \;\middle|\; R \text{ is symmetric} \right\} \simeq \mathbb{Z}^{\frac{1}{2}g(g+1)}$
- $\sigma(\mathcal{T}') = \left\{ \begin{pmatrix} \mathrm{Id}_g & 0 \\ R & \mathrm{Id}_g \end{pmatrix} \;\middle|\; R \text{ is symmetric} \right\} \simeq \mathbb{Z}^{\frac{1}{2}g(g+1)}$

Double Johnson filtration for the Goeritz group

$K = \langle x_1, \ldots, x_g, y_1, \ldots, y_g \rangle \qquad \bar{X} = \langle\langle x_1, \ldots, x_g \rangle\rangle_K \qquad \bar{Y} = \langle\langle y_1, \ldots, y_g \rangle\rangle_K$

The Goeritz group

$$\mathcal{G} = \{\ h \in \mathcal{M} \mid \quad h_\#(\bar{X}) \subset \bar{X}, \quad h_\#(\bar{Y}) \subset \bar{Y}\ \}$$

acts on K preserving the double lower central series $(K_{i,j})_{(i,j)\in\mathbb{N}^2}$ de $(K; \bar{X}, \bar{Y})$

For $(m, n) \in \mathbb{N}^2$

$$\mathcal{G}_{m,n} = \left\{ h \in \mathcal{G} \mid \quad [h, K_{i,j}] \subset K_{m+i,n+j} \quad \forall (i,j) \in \mathbb{N}^2 \right\} \qquad [h, z] = h_\#(z) z^{-1}$$

Proposition

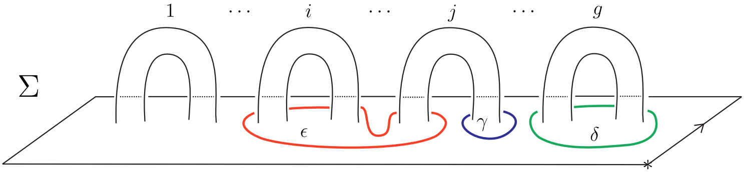
- $\mathcal{G}_{m,n} = \left\{ h \in \mathcal{G} \mid \quad [h, \bar{X}] \subset K_{m+1,n},\ [h, \bar{Y}] \subset K_{m,n+1} \right\}$
- $\mathcal{G}_{0,0} = \mathcal{G}$
- $\mathcal{G}_{m,n} \trianglelefteq \mathcal{G}$
- $[\mathcal{G}_{m,n}, \mathcal{G}_{a,b}] \subset \mathcal{G}_{m+a,n+b}$.

Examples

$t_\epsilon t_\gamma^{-1} \in \mathcal{G}_{1,0}$

$t_\delta \in \mathcal{G}_{1,1}$

$$
\begin{array}{ccccccc}
\mathcal{G}_{0,0} & \supset & \mathcal{G}_{0,1} & \supset & \mathcal{G}_{0,2} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
\mathcal{G}_{1,0} & \supset & \mathcal{G}_{1,1} & \supset & \mathcal{G}_{1,2} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
\mathcal{G}_{2,0} & \supset & \mathcal{G}_{2,1} & \supset & \mathcal{G}_{2,2} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
\mathcal{G}_{3,0} & \supset & \mathcal{G}_{3,1} & \supset & \mathcal{G}_{3,2} & \supset & \cdots \\
\cup & & \cup & & \cup & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$



Double Johnson filtration for the mapping class group $\mathcal{M}$

For $(m, n) \in \mathbb{Z}^2$ we set

$$\mathcal{M}_{m,n} := \left\{ h \in \mathcal{M} \mid \quad [h^{\pm 1}, K_{i,j}] \subset K_{m+i,m+j} \quad \forall (i,j) \in \mathbb{N}^2 \right\} \leq \mathcal{M}$$

Proposition

We have

- $\mathcal{M}_{m,n} = \left\{ h \in \mathcal{M} \mid [h^{\pm 1}, K_{1,0}] \subset K_{m+1,n} \text{ and } [h^{\pm 1}, K_{0,1}] \subset K_{m,n+1} \right\}$
- $\mathcal{M}_{m,n} = \mathcal{M}_{\max(-1,m),\max(-1,n)}$
- If $(m, n) \in \mathbb{N}^2$, then $\mathcal{M}_{m,n} = \mathcal{G}_{m,n}$
- $\mathcal{M}_{1,-1} = \mathcal{T}$ and $\mathcal{M}_{-1,1} = \mathcal{T}'$ (Luft's groups)
- $\mathcal{M}_{0,-1} = \mathcal{H}$ and $\mathcal{M}_{-1,0} = \mathcal{H}'$ (Handlebody groups)
- $\mathcal{M}_{-1,-1} = \mathcal{M}$ (Mapping class group)

Double Johnson filtration for the mapping class group $\mathcal{M}$

$$
\begin{array}{ccccccccc}
\mathcal{M} = \mathcal{M}_{-1,-1} & \supset & \mathcal{H}' = \mathcal{M}_{-1,0} & \supset & \mathcal{T}' = \mathcal{M}_{-1,1} & \supset & \mathcal{M}_{-1,2} & \cdots \\
\cup & & \cup & & \cup & & \cup & \\
\mathcal{H} = \mathcal{M}_{0,-1} & \supset & \mathcal{G} = \mathcal{M}_{0,0} & \supset & \mathcal{M}_{0,1} & \supset & \mathcal{M}_{0,2} & \cdots \\
\cup & & \cup & & \cup & & \cup & \\
\mathcal{T} = \mathcal{M}_{1,-1} & \supset & \mathcal{M}_{1,0} & \supset & \mathcal{M}_{1,1} & \supset & \mathcal{M}_{1,2} & \cdots \\
\cup & & \cup & & \cup & & \cup & \\
\mathcal{M}_{2,-1} & \supset & \mathcal{M}_{2,0} & \supset & \mathcal{M}_{2,1} & \supset & \mathcal{M}_{2,2} & \cdots \\
\cup & & \cup & & \cup & & \cup & \\
\mathcal{M}_{3,-1} & \supset & \mathcal{M}_{3,0} & \supset & \mathcal{M}_{3,1} & \supset & \mathcal{M}_{3,2} & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

Proposition

For $(m,n),(a,b) \in \{(k,l) \in \mathbb{Z}^2 \mid k,l \geq -1,\ k+l \geq 1\} \cup \{(0,0)\}$, we have

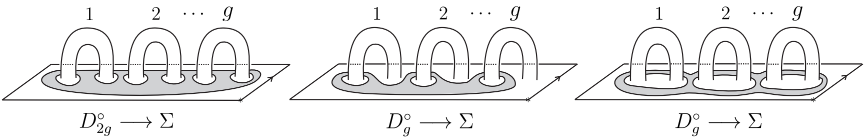
$$[\mathcal{M}_{m,n}, \mathcal{M}_{a,b}] \subset \mathcal{M}_{m+a,n+b}$$

Examples

D_n° : disk with n holes

$\mathrm{FPB}_n = \mathrm{MCG}(D_n^\circ)$ (framed) pure braid group in n strands

Any embedding $D_n^\circ \to \Sigma$ induces a map $\mathrm{FPB}_n \to \mathcal{M}$



Proposition

Consider an embedding $D_n^\circ \to \Sigma$ such that the images of the holes of D_n° bound mutually disjoing disks in V. Let $f : \mathrm{FPB}_n \to \mathcal{M}$ be the induced homomorphism. If $\beta \in \Gamma_k \mathrm{FPB}_n$, then $f(\beta) \in \mathcal{M}_{k,-1}$.

Relation with the usual Johnson filtration

$K = \pi_1(\Sigma, *)$

$$\underbrace{\mathcal{I} = J_1\mathcal{M}}_{\text{Torelli}} \supset \underbrace{\mathcal{K} = J_2\mathcal{M}}_{\textit{Johnson}} \supset J_3\mathcal{M} \supset \cdots$$

$$J_n\mathcal{M} = \{h \in \mathcal{M} \mid [h, K] \subset \Gamma_{n+1}K\}$$

$\forall m, n \geq -1$ with $m + n \geq 1$ we have

$$\mathcal{M}_{m,n} \subset J_{m+n}\mathcal{M}$$

$\mathcal{M}$	$\mathcal{H}'$	$\mathcal{T}'$	$\mathcal{M}_{-1,2}$	$\cdots$
$\mathcal{H}$	$\mathcal{G}$	$\mathcal{M}_{0,1}$	$\mathcal{M}_{0,2}$	$\cdots$
$\mathcal{T}$	$\mathcal{M}_{1,0}$	$\mathcal{M}_{1,1}$	$\mathcal{M}_{1,2}$	$\cdots$
$\mathcal{M}_{2,-1}$	$\mathcal{M}_{2,0}$	$\mathcal{M}_{2,1}$	$\mathcal{M}_{2,2}$	$\cdots$
$\mathcal{M}_{3,-1}$	$\mathcal{M}_{3,0}$	$\mathcal{M}_{3,1}$	$\mathcal{M}_{3,2}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

- $\mathcal{M}_{2,-1},\ \mathcal{M}_{1,0},\ \mathcal{M}_{0,1},\ \mathcal{M}_{-1,2} \subset J_1\mathcal{M} = \mathcal{I}$
- $\mathcal{M}_{3,-1},\ \mathcal{M}_{2,0},\ \mathcal{M}_{1,1},\ \mathcal{M}_{0,2},\ \mathcal{M}_{-1,3} \subset J_2\mathcal{M} = \mathcal{K}$

Theorem

$$\mathcal{I} = \mathcal{M}_{2,-1} \cdot \mathcal{M}_{1,0} \cdot \mathcal{M}_{0,1} \cdot \mathcal{M}_{-1,2} \cdot \mathcal{K}$$

Conjecture

We have $\mathcal{K} \subset \mathcal{M}_{2,-1} \cdot \mathcal{M}_{1,0} \cdot \mathcal{M}_{0,1} \cdot \mathcal{M}_{-1,2}$, therefore

$$\mathcal{I} = \mathcal{M}_{2,-1} \cdot \mathcal{M}_{1,0} \cdot \mathcal{M}_{0,1} \cdot \mathcal{M}_{-1,2}.$$

Theorem (case of automorphism groups of free groups)

Let $F = F_{p,q} = \langle x_1, \ldots, x_p, y_1, \ldots, y_q \rangle$ the free group on $p + q$ generators.
We can define a doubly indexed filtration $(\mathcal{A}_{i,j})_{i,j \geq -1}$ for $\mathrm{Aut}(F)$.
If $\mathcal{IA} = \ker\left(\mathrm{Aut}(F) \to \mathrm{Aut}(F/[F,F])\right)$, then

$$\mathcal{IA} = \mathcal{A}_{2,-1} \cdot \mathcal{A}_{1,0} \cdot \mathcal{A}_{0,1} \cdot \mathcal{A}_{-1,2}.$$

Double Johnson homomorphisms

$K = \langle x_1, \ldots, x_g, y_1, \ldots, y_g \rangle$ $\quad \bar{X} = \langle\langle x_1, \ldots, x_g \rangle\rangle_K$ $\quad \bar{Y} = \langle\langle y_1, \ldots, y_g \rangle\rangle_K$

$(K_{i,j})_{(i,j)\in\mathbb{N}^2}$: double lower central series of $(K; \bar{X}, \bar{Y})$

$a_i = [x_i], \quad b_i = [y_i]$ $\quad \mathbb{N}^2_+ = \{(m,n) \in \mathbb{N}^2 \mid m+n \geq 1\}$

$H = \langle a_1, b_1, \ldots, a_g, b_g \rangle_{\mathrm{ab}}$ $\quad A = \langle a_1, \ldots, a_g \rangle_{\mathrm{ab}}$ $\quad B = \langle b_1, \ldots, b_g \rangle_{\mathrm{ab}}$

$$\mathfrak{Lie}(A,B) = \bigoplus_{(m,n)\in\mathbb{N}^2_+} \mathfrak{Lie}_{m,n}(A,B)$$

$\mathbb{N}^2_+$-graded free Lie algebra generated by A in degree $(1,0)$ and B in degree $(0,1)$.

	$\mathfrak{Lie}_{0,1}(A,B)$	$\mathfrak{Lie}_{0,2}(A,B)\cdots$
$\mathfrak{Lie}_{1,0}(A,B)$	$\mathfrak{Lie}_{1,1}(A,B)$	$\mathfrak{Lie}_{1,2}(A,B)\cdots$
$\mathfrak{Lie}_{2,0}(A,B)$	$\mathfrak{Lie}_{2,1}(A,B)$	$\mathfrak{Lie}_{2,2}(A,B)\cdots$
$\mathfrak{Lie}_{3,0}(A,B)$	$\mathfrak{Lie}_{3,1}(A,B)$	$\mathfrak{Lie}_{3,2}(A,B)\cdots$
$\vdots$	$\vdots$	$\vdots$

Lemma

$$\bigoplus_{(m,n)\in\mathbb{N}^2_+} \frac{K_{m,n}}{K_{m+1,n}\cdot K_{m,n+1}} \simeq \bigoplus_{(m,n)\in\mathbb{N}^2_+} \mathfrak{Lie}_{m,n}(A,B) = \mathfrak{Lie}(A,B)$$

Proposition

For every $(m, n) \in \mathbb{N}_+^2$ there exist a subgroup

$$D_{m,n}(A, B) \leq \left(A \otimes \mathfrak{Lie}_{m,n+1}(A, B)\right) \oplus \left(B \otimes \mathfrak{Lie}_{m+1,n}(A, B)\right)$$

and a group homomorphism

$$\tau_{m,n} : \mathcal{M}_{m,n} \longrightarrow D_{m,n}(A, B)$$

such that

$$\mathcal{M}_{m+1,n} \cdot \mathcal{M}_{m,n+1} \subset \ker(\tau_{m,n}).$$

Moreover, these homomorphisms are compatible with the usual Johnson homomorphisms : the diagram

$$\begin{array}{ccc} \mathcal{M}_{m,n} & \xrightarrow{\subset} & J_{m+n}\mathcal{M} \\ \downarrow{\scriptstyle \tau_{m,n}} & & \downarrow{\scriptstyle \tau_{m+n}} \\ D_{m,n}(A, B) & \xrightarrow{j} & D_{m+n}(H) \end{array}$$

is commutative.

Remark

The above result can be extended to all the indices :

$$\Big(\tau_{m,n} : \mathcal{M}_{m,n} \longrightarrow D_{m,n}(A,B)\Big)_{m,n\geq -1}$$

We also get the following : If $h \in \mathcal{I}$, then there exist

$$h_1 \in \mathcal{M}_{2,-1}, \quad h_2 \in \mathcal{M}_{1,0}, \quad h_3 \in \mathcal{M}_{0,1}, \quad h_4 \in \mathcal{M}_{-1,2}$$

such that

$$\tau_1(h) = \tau_{2,-1}(h_1) + \tau_{1,0}(h_2) + \tau_{0,1}(h_3) + \tau_{-1,2}(h_4).$$

Thank you very much for your attention!