Arrangements Close to Free: Two Deletions, Resolutions, and a Saito-Type Relation

Seminar "Moduli and Friends II"
Mathematical Institute of the Romanian Academy, online

Junyan Chu

Kyoto University

16 Feb. 2026

Roadmap

Background & setup

Deleting one hyperplane

Deleting two hyperplanes

Generalized Saito-type criterion

Outline

Setup & notation

- $V = \mathbb{K}^\ell$, $\ell \in \mathbb{N}$
- $S = \mathrm{Sym}(V^*) = \mathbb{K}[x_1, \ldots, x_\ell]$
- $\mathcal{A} = \{H_i = \ker(\alpha_i) \mid \alpha_i \in S, \deg(\alpha_i) = 1\}$: a *central arrangement*
- $Q(\mathcal{A}) = \prod_i \alpha_i$: *defining polynomial* of $\mathcal{A}$
- $\mathcal{A}^{H_i} = \{H_i \cap H_j \mid H_j \in \mathcal{A} \setminus \{H_i\}\}$: *restriction* on $H_i \in \mathcal{A}$
- $\mathcal{A}_{H_i \cap H_j} = \{H \in \mathcal{A} \mid H_i \cap H_j \subset H\}$: *localization* at $H_i \cap H_j$

Setup & notation

Definition 1

Logarithmic derivation module $D(\mathcal{A})$:

$$D(\mathcal{A}) = \left\{ \theta \in \oplus_{i=1}^{\ell} S \frac{\partial}{\partial x_i} \mid \theta(\alpha_i) \in S\,\alpha_i \ \forall i \right\}.$$

- $D(\mathcal{A})$ is graded and reflexive, but often not free.
- We measure complexity via the *minimal free resolution* of $D(\mathcal{A})$.

Free arrangements

Definition 2
If $D(\mathcal{A}) \cong \bigoplus_{i=1}^{\ell} S[-d_i]$, then we say $\mathcal{A}$ is free with exponent $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$.

Free arrangement has nice properties and has been extensively studied.

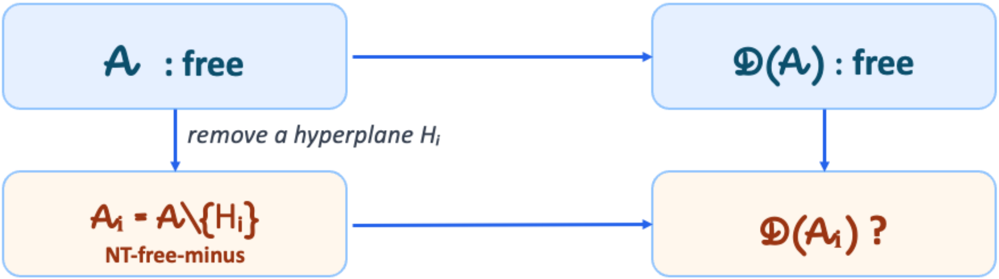
- Homological simplicity (no higher syzygies)
- Strong constraints on characteristic polynomial
- Addition–Deletion helps recognize/build freeness

Outline

Definition 3 (Abe, 2021)

Arrangement $\mathcal{B}$ is *next to free minus one (NT-free-minus)* if $\exists$ free arrangement $\mathcal{A}$ and $H_i \in \mathcal{A}$ such that $\mathcal{B} = \mathcal{A} \setminus \{H_i\} := \mathcal{A}_i$.

- Goal: understand $D(\mathcal{A}_i)$ via its minimal free resolution.



Guiding questions

1. How many new generators appear in $D(\mathcal{A}_i)$?
2. Where do the first syzygies appear in the minimal resolution?
3. Which features are combinatorial?

Theorem (Abe, 2021)

Theorem 4 (Abe, 2021)

Let $\mathcal{A}$ be free with $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$*. Then the deletion* $\mathcal{A}_i = \mathcal{A} \setminus \{H_i\}$ *is either*

- *free, or*
- *SPOG with level* $d = |\mathcal{A}_i| - |\mathcal{A}^{H_i}|$.

Definition 5

$\mathcal{A}$ is *strictly plus-one generated (SPOG)* with $\mathrm{POexp}(\mathcal{A}) = (d_1, \ldots, d_\ell)$ and level d if D(A) has a minimal free resolution:

$$0 \to S[-d-1] \to S[-d] \oplus \bigoplus_{i=1}^{\ell} S[-d_i] \to D(\mathcal{A}) \to 0.$$

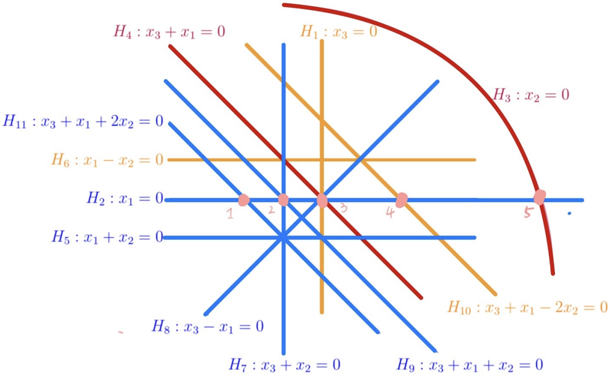
Equivalent viewpoint:

$$D(\mathcal{A}) = S\theta_1 + \cdots + S\theta_\ell + S\theta_{\ell+1},$$
$$f_1\theta_1 + \cdots + f_\ell\theta_\ell + f_{\ell+1}\theta_{\ell+1} = 0,$$

where $f_{\ell+1} \in S_1 \setminus \{0\}$.

Example: different deletions, different outcomes

This is a free arrangent with exp(𝒜) = (1,5,5)



$\mathcal{A}_i$: SPOG with level $c_i = 5$

$\mathcal{A}_j$: SPOG with level $c_j = 6$

$\mathcal{A}_k$: free with exp = (1,4,5)

Outline

Background & setup

Deleting one hyperplane

Deleting two hyperplanes

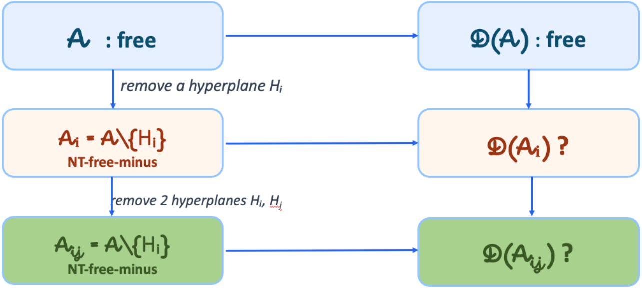
Generalized Saito-type criterion

Deleting two hyperplanes

- Setup: start with free $\mathcal{A}$, delete H_i and H_j:

$$\mathcal{A}_{i,j} = \mathcal{A} \setminus \{H_i, H_j\}.$$

- Goal: describe the minimal free resolution of $D(\mathcal{A}_{i,j})$.



Theorem 6 (Chu, 2025)

Assume $\mathcal{A}$ is free and $\mathcal{A}_1, \mathcal{A}_2$ are SPOG with levels $c_1 \leq c_2$.

- *If $\beta_0(D(\mathcal{A}_{1,2})) \leq \ell + 2$, then we may give a case-by-case minimal free resolution of $D(\mathcal{A}_{1,2})$.*
- *If $\beta_0(D(\mathcal{A}_{1,2})) > \ell + 2$, then $\mathrm{pd}_S(D(\mathcal{A}_{1,2})) \geq 2$.*

Example: projective dimension can jump

Example 7

The case pdim $D(\mathcal{A}) \geq 2$.

$Q(\mathcal{A}) = x_1x_2x_3x_4(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_3 - x_4)(x_1 - x_3 + x_4)(x_1 - x_2 + x_3 - x_4)$.

The arrangement $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, 3, 3, 3)$.
Let $H_1 = \{x_1 = 0\}, H_2 = \{x_3 = 0\}$, and $\mathcal{A}_{1,2} = \mathcal{A} \setminus \{H_1, H_2\}$.
The minimal free resolution of $D(\mathcal{A}_{1,2})$ is

$$0 \longrightarrow S(-5) \longrightarrow S(-4)^4 \longrightarrow S(-3)^6 \oplus S(-1) \longrightarrow D(\mathcal{A}_{1,2}) \longrightarrow 0.$$

Remark: This is a counterexample to Orlik's conjecture from [DiPasquale, 2023]

Key bound on $\mathrm{pd}_S(D(\mathcal{B}))$

Corollary 8

Let $\mathcal{A}$ be free. Then

$$\beta_0(D(\mathcal{A}_{i,j})) \leq \ell + 2 \quad \Longleftrightarrow \quad \mathrm{pd}_S(D(\mathcal{A}_{i,j})) \leq 1.$$

- Recall that for any arrangement $\mathcal{B}$ in $V = \mathbb{K}^\ell$:

$$\mathrm{pd}_S(D(\mathcal{B})) \leq \ell - 2.$$

- In particular, when $\ell = 3$ we always have $\mathrm{pd}_S(D(\mathcal{B})) \leq 1$.

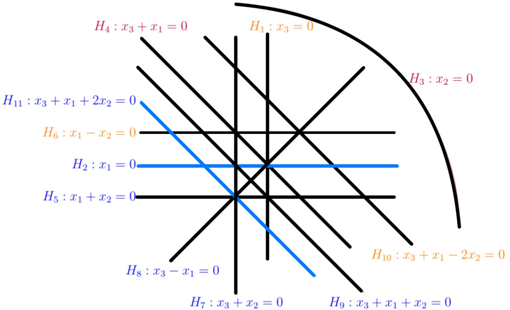
Theorem in dimension three $\ell = 3$ (Chu, 2025)

Theorem 9

Assume that $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, d_2, d_3)$, and the deletions $\mathcal{A}_i$ and $\mathcal{A}_j$ are SPOG with levels c_j and c_j. Then the minimal free resolution of $D(\mathcal{A}_{i,j})$ is determined by

- *the local intersection multiplicity $|\mathcal{A}_{H_i \cap H_j}|$, and*
- *how the SPOG levels c_i and c_j compare.*

Case (1): $|\mathcal{A}_{H_i \cap H_j}| = 2$



$\mathcal{A}$: free with $(1,5,5)$

$\mathcal{A}_i$: SPOG with level $c_i = 5$

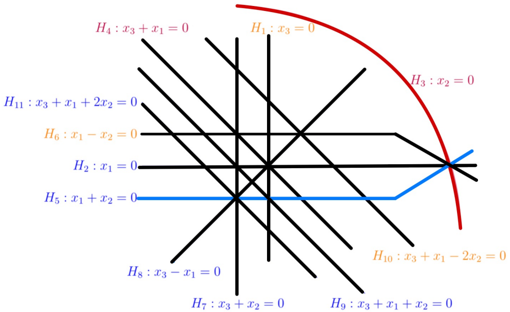
$$|\mathcal{A}_{H_2 \cap H_{11}}| = 2$$

$$0 \to S[-6]^2 \to S[-5]^4 \oplus S[-1] \to D(\mathcal{A}_{2,11}) \to 0$$

$$S[-6]^2 = S[-c_2 - 1] \oplus S[-c_{11} - 1]$$

$$S[-5]^4 \oplus S[-1] = S[-c_2] \oplus S[-c_{11}] \oplus \left(\bigoplus_{i=1}^{3} S[-d_i] \right)$$

Case (2): $|\mathcal{A}_{H_i \cap H_j}| > 2$ and $c_i \neq c_j$



$\mathcal{A}$: free with $(1,5,5)$

$\mathcal{A}_5$: SPOG with level $c_5 = 5$

$\mathcal{A}_3$: SPOG with level $c_3 = 6$

$|\mathcal{A}_{H_3 \cap H_5}| > 2$

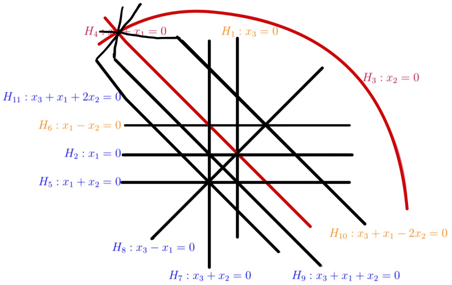
$c_3 > c_5$

$$0 \to S[-6]^2 \to S[-5]^4 \oplus S[-1] \to D(\mathcal{A}_{3,5}) \to 0$$

$$S[-6]^2 = S[-c_5 - 1] \oplus S[-c_3]$$

$$S[-5]^4 \oplus S[-1] = S[-c_5] \oplus S[-(c_3 - 1)] \oplus \left(\bigoplus_{i=1}^{3} S[-d_i] \right)$$

Case (3): $|\mathcal{A}_{H_i \cap H_j}| > 2$ and $c_i = c_j$



$\mathcal{A}$: free with $(1,5,5)$

$\mathcal{A}_i$: SPOG with level $c_i = 6$

$|\mathcal{A}_{H_3 \cap H_4}| > 2$

$C_3 = C_4 = 6$

$$0 \to S[-7] \to S[-5]^3 \oplus S[-1] \to D(\mathcal{A}_3, 4) \to 0$$

$$S[-7] = S[-c_3 - 1], \qquad S[-5]^3 \oplus S[-1] = S[-(c_3-1)] \oplus \left(\bigoplus_{i=1}^{3} S[-d_i]\right)$$

Theorem 10 (Addition-Deletion)

Let $H_i \in \mathcal{A}$, $\mathcal{A}_i := \mathcal{A} \setminus \{H_i\}$ and $\mathcal{A}^i := \mathcal{A}^{H_i}$. Then two of the following imply the third:

1. *$\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$.*
2. *$\mathcal{A}_i$ is free with $\exp(\mathcal{A}_i) = (d_1, \ldots, d_{\ell-1}, d_\ell - 1)$.*
3. *$\mathcal{A}^i$ is free with $\exp(\mathcal{A}^i) = (d_1, \ldots, d_{\ell-1})$.*

Moreover, all the three above hold true if $\mathcal{A}$ and $\mathcal{A}_i$ are free.

Problem 11 (Abe)

Can we have a similar theory for SPOG arrangements?

Addition–deletion problems for SPOG

Problem 12 (same level)

Assume that

1. $\mathcal{A}$ *is SPOG with*

$$POexp(\mathcal{A}) = (d_1, \ldots, d_\ell)_\leq \quad \text{and level } d,$$

2. $\mathcal{A}_i$ *is SPOG with*

$$POexp(\mathcal{A}_i) = (d_1, \ldots, d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_\ell) \quad \text{and level } d.$$

then $\mathcal{A}^{H_i}$ *is SPOG with* $POexp(\mathcal{A}^{H_i}) = (d_1, \ldots, d_{i-1}, \hat{d}_i, d_{i+1}, \ldots, d_\ell)$ *and level* d*.*

Addition–deletion problems for SPOG

Problem 13 (level drops by one)

Assume that

1. $\mathcal{A}$ *is SPOG with*
$$POexp(\mathcal{A}) = (d_1, \ldots, d_\ell)_\leq \quad \text{and level } d,$$
2. $\mathcal{A}_i$ *is SPOG with*
$$POexp(\mathcal{A}_i) = (d_1, \ldots, d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_\ell) \quad \text{and level } d - 1.$$

then $\mathcal{A}^{H_i}$ *is free with* $\exp(\mathcal{A}^{H_i}) = (d_1, \ldots, d_{i-1}, \hat{d}_i, d_{i+1}, \ldots, d_\ell)$.

Outline

Theorem 14 (Saito's criterion)

For homogeneous derivations $\{\theta_1, \ldots, \theta_\ell\} \in D(\mathcal{A})$, *define*

$$\Delta = \det(\theta_i(x_j)).$$

Then, $\mathcal{A}$ *is free with basis* $\{\theta_1, \ldots, \theta_\ell\}$ *iff* $\Delta/Q(\mathcal{A}) \in \mathbb{K} \setminus \{0\}$.

Some generalizations

- Higher-order $\mathcal{A}$-differential operators [Holm, 2002],
- Freeness of toric logarithmic sheaves [Daniele–Marcos–William, 2024].

Minors attached to generators

Definition 15

Let $\theta_1, \ldots, \theta_{\ell+1} \in D(\mathcal{A})$ be homogeneous derivations. Define the $(\ell+1) \times \ell$ matrix

$$M = M[\theta_1, \ldots, \theta_{\ell+1}], \qquad M(i,j) := \theta_i(x_j).$$

For each i, let M_i be the $\ell \times \ell$ matrix obtained by deleting the i-th row of M, and set

$$\Delta_i := (-1)^i \det(M_i).$$

Since $\Delta_i \in Q(\mathcal{A})S$, we may write $\Delta_i = g_i\, Q(\mathcal{A})$ with $g_i \in S$.

$*$ Starting from this slide, the detailed arguments and proofs are presented in Chapter 3 of my PhD thesis.

Example

Example 16

Let $\ell = 4$, and $Q := Q(\mathcal{A}) = x_1x_2x_3x_4(x_2 - x_3 + x_4)(-x_1 + x_2 - x_3 + x_4)$.
Then $D(\mathcal{A}) \cong S(-1) \oplus S(-2)^4$. A minimal set of generators of $D(\mathcal{A})$ is given by

$$M = \begin{pmatrix} x_1 & x_1x_2 & x_1x_3 & x_1x_4 & x_1^2 - x_1x_2 + x_1x_3 - x_1x_4 \\ x_2 & x_2^2 + x_2x_4 & x_2x_3 & 0 & 0 \\ x_3 & x_2x_3 & x_3^2 - x_3x_4 & -x_2x_3 + x_3^2 - x_3x_4 & 0 \\ x_4 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\Delta_1 = (-1)^1 \det M_0 = 0 = g_1 Q, \qquad \Delta_2 = (-1)^2 \det M_1 = -x_3 Q = g_2 Q,$$
$$\Delta_3 = (-1)^3 \det M_2 = (x_2 + x_4) Q = g_3 Q, \qquad \Delta_4 = (-1)^4 \det M_3 = -x_4 Q = g_4 Q,$$
$$\Delta_5 = (-1)^5 \det M_4 = 0 = g_5 Q.$$

$$\Delta_1 \theta_1 + \Delta_2 \theta_2 + \Delta_3 \theta_3 + \Delta_4 \theta_4 + \Delta_5 \theta_5 = Q(\sum_{i=1}^{5} g_i \theta_i) = 0.$$

Question 1

Suppose homogeneous derivations $\theta_1, \ldots, \theta_{\ell+1} \in D(\mathcal{A})$ *satisfy a relation*

$$g_1 \theta_1 + \cdots + g_{\ell+1} \theta_{\ell+1} = 0,$$

where $g_i = \Delta_i / Q(\mathcal{A})$.
Can we conclude that $\theta_1, \ldots, \theta_{\ell+1}$ *generate* $D(\mathcal{A})$*?*

Lemma 17

For any homogeneous derivations $\theta_1, \ldots, \theta_{\ell+1} \in D(\mathcal{A})$, we have the relation

$$g_1\,\theta_1 + \cdots + g_{\ell+1}\,\theta_{\ell+1} = 0,$$

where $g_i = \Delta_i / Q(\mathcal{A})$.

Question 2

Can we determine the structure of $D(\mathcal{A})$ from additional relations among the Δ_i?

Definition 18

We say that the relation in $D(\mathcal{A})$

$$f_1\theta_1 + \cdots + f_{\ell+1}\theta_{\ell+1} = 0$$

is of minimal degree (or primitive) if the polynomials $f_1, \ldots, f_{\ell+1}$ have no nonunit common divisor.

Conjecture 1

$\mathcal{A}$ is SPOG, and the set $\{\theta_1, \ldots, \theta_{\ell+1}\}$ forms a minimal generating set for $D(\mathcal{A})$, satisfying the unique relation

$$g_1\theta_1 + \cdots + g_{\ell+1}\theta_{\ell+1} = 0,$$

of minimal degree if and only if $g_{\ell+1} \in S_1 \setminus \{0\}$, and $g_1, \ldots, g_\ell \in S_{>0}$ have no nonunit common divisor modulo $g_{\ell+1}$.

Proposition: minors recover the relation

Proposition 1

If $\theta_1, \ldots, \theta_{\ell+1}$ form a minimal generating set for $D(\mathcal{A})$ and satisfy a unique relation

$$f_1\theta_1 + \cdots + f_{\ell+1}\theta_{\ell+1} = 0 \tag{1}$$

of minimal degree, then there exists $c \in \mathbb{K}^*$ such that $\Delta_i = cf_i Q(\mathcal{A})$ for all $i = 1, \ldots, \ell + 1$.
Moreover, if $f_{\ell+1} \neq 0$, then $f_1, \ldots, f_\ell$ have no nonunit common divisor modulo $f_{\ell+1}$.

Main Theorem

Theorem 19

Assume that $g_{\ell+1} \in S_1 \setminus \{0\}$, and that $g_1, \ldots, g_\ell \in S_{>0}$ have no nonunit common divisor modulo $g_{\ell+1}$.
Suppose $\mathrm{pd}_S(D(\mathcal{A})) \leq 1$, then $\theta_1, \ldots, \theta_{\ell+1}$ form a minimal generating set for $D(\mathcal{A})$, and $\mathcal{A}$ is SPOG with a relation

$$g_1\theta_1 + \cdots + g_{\ell+1}\theta_{\ell+1} = 0.$$

- Corollary: Conjecture 1 holds for $\ell = 3$.

Conjecture 2

When $\mathrm{pd} D(\mathcal{A}) = 1$, *there is a minimal free resolution of the following form:*

$$0 \to \bigoplus_{i=\ell+1}^{p} S[-d_i - 1] \to \bigoplus_{i=1}^{p} S[-d_i] \to D(\mathcal{A}) \to 0.$$

Conjecture 3

Let $G = \{\theta_i \in D(\mathcal{A}) \mid i \in I\}$. *The ideal* Δ_I *equals* $S_{\geq k} Q(\mathcal{A})$, *where*

$$k = (\ell - 1)(|\mathcal{A}| - \ell - 1)$$

if and only if $\mathcal{A}$ *is generic and* G *forms a minimal generating set of* $D(\mathcal{A})$.

Thank you!

References

T. Abe, Plus-one generated and next to free arrangements of hyperplanes, *Int. Math. Res. Not.* 2021 (12) (2019), 9233–9261. doi:10.1093/imrn/rnz099.

J. Chu, Free resolution of the logarithmic derivation modules of close to free arrangements, *J. Algebraic Combin.* 61 (2025), no. 2, Paper No. 26. doi:10.1007/s10801-025-01394-7.

J. Chu, Free resolution of the logarithmic derivation modules and its application, PhD thesis, Kyushu University, 2025. PDF

P. Holm, *Differential Operators on Arrangements of Hyperplanes*, PhD thesis, Stockholm University, 2002.

D. Faenzi, M. Jardim, W. D. Montoya, Logarithmic vector fields and foliations on toric varieties, arXiv:2408.10671 (2024).

M. DiPasquale, A homological characterization for freeness of multi-arrangements, *Math. Ann.* 385 (2023), 745–786. doi:10.1007/s00208-021-02357-6.