# Higher scissors congruence

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Moduli and Friends Seminar

IMAR

**Joint work with:**

Bohmann, Gerhardt, Merling, and Zakharevich (BGMMZ),

Kupers, Lemann, Miller, and Sroka (KLMMS),

Li (LM), and

Klang, Kuijper, Mehrle, Wittich (KKMMW).

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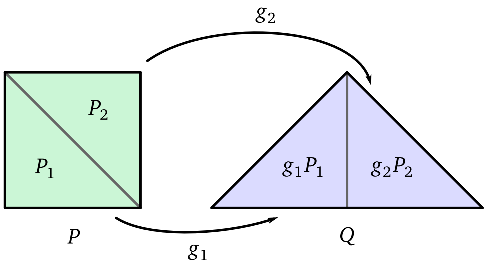

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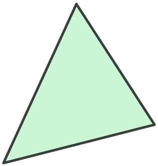
**Theorem.** (Wallace–Bolyai–Gerwien 1807)

$P$ and $Q$ are scissors congruent $\iff$ they have the same area.

Is this true in dimensions other than 2?

**Definition.** A **convex polytope** is a convex hull of finitely many points in Euclidean space $E^n$. (Must be nondegenerate.)



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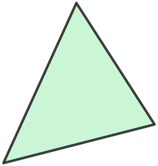
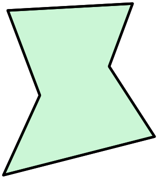
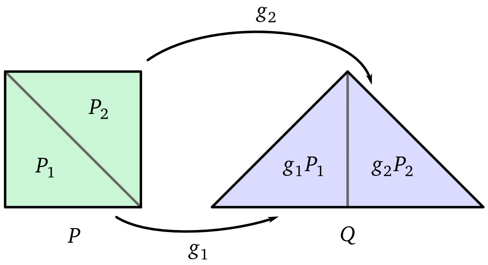


A **polytope** is a finite union of (nondegenerate) convex polytopes.



A **scissors congruence** from $P$ to $Q$ is

$$\begin{cases} P = \cup_{i=1}^{k} P_i & \text{interiors disjoint,} \\ Q = \cup_{i=1}^{k} Q_i & \text{interiors disjoint, and} \\ \text{isometries } g_i : P_i \cong Q_i, & i = 1, \dots, k. \end{cases}$$



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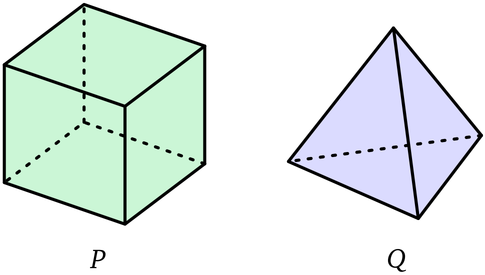
## Hilbert's 3rd Problem

Polyhedra in $E^3$ up to scissors congruence = volume?



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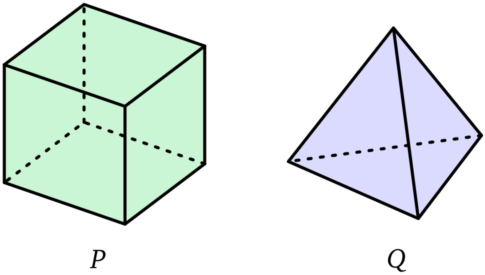


**Answer.** (Dehn 1901) No! Volume isn't enough.

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Volume $\in \mathbb{R}$ is an example. Are there more?

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**Theorem (Dehn 1901)**

A cube and a regular tetrahedron are never scissors congruent.

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Every other invariant factors through $K$-theory: $K_0(E^n) \to A$.

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## Theorem (Dehn–Sydler–Jessen, 1965, 1968)

Volume and Dehn invariant define an injective map

$$K_0(E^3) \to \mathbb{R} \times (\mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z}).$$

So this is everything in dimension 3.

In fact, there is an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow K_0(E^3) \longrightarrow (\mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z}) \longrightarrow \Omega^1_{\mathbb{R}/\mathbb{Z}} \longrightarrow 0.$$

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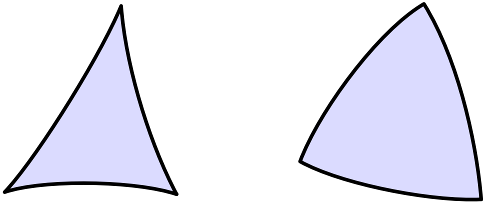
$K_0(E^5)$ has not been computed!

Generalization: consider other geometries!



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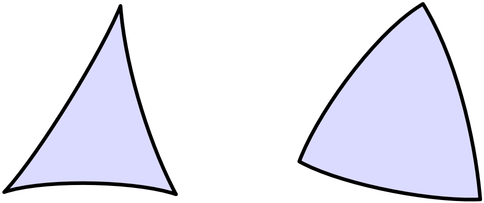

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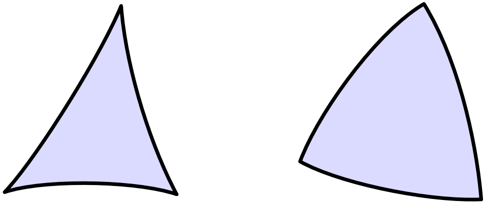

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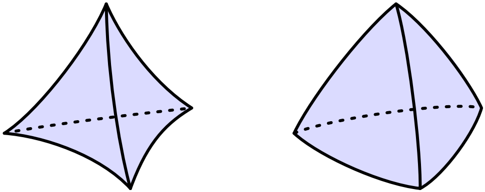
Next is $H^3$ and $S^3$:



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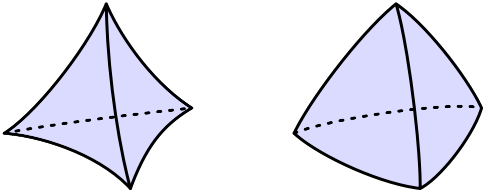

## Theorem (Dupont 1982)

There are exact sequences:

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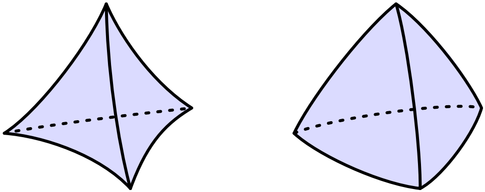

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Still open whether the volume and Dehn invariant are everything!

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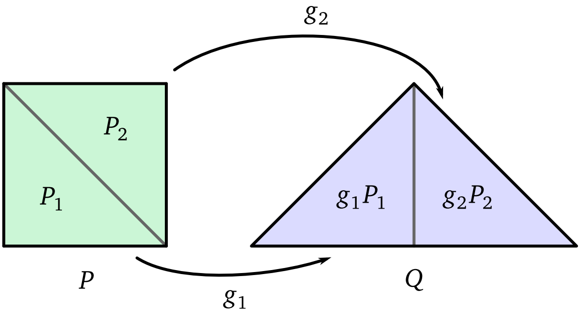
These form a groupoid. Each polytope $P$ has a **scissors automorphism group** $\mathrm{Aut}(P)$.

Again, a scissors congruence $P \to Q$ is:



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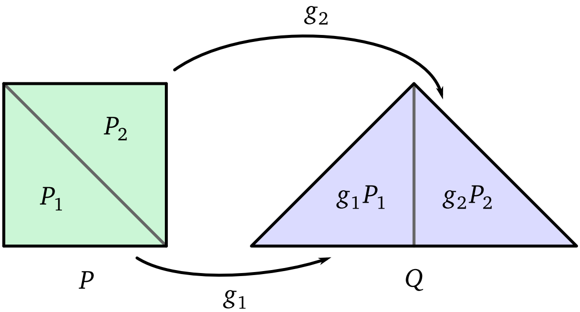


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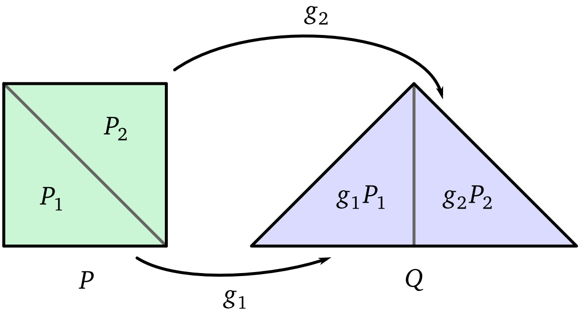


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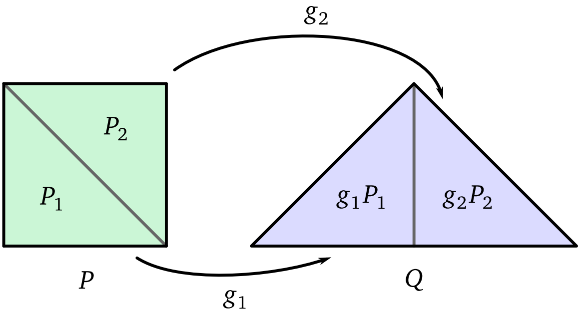


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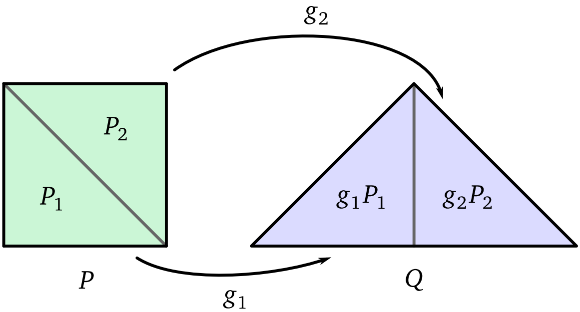


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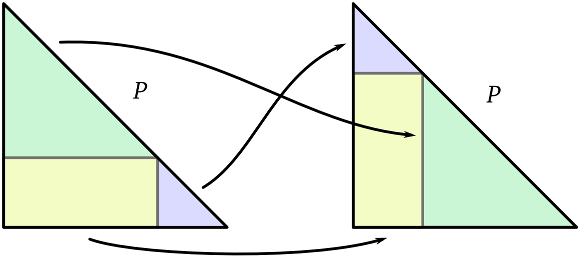
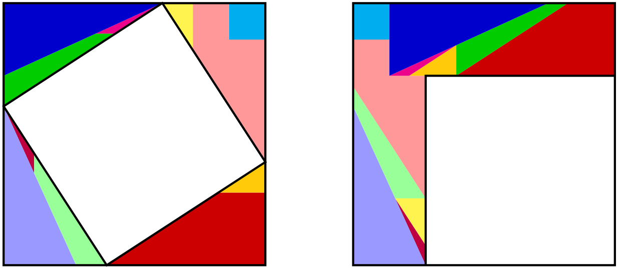
Cutting a piece $P_i$ into smaller pieces gives the same morphism.

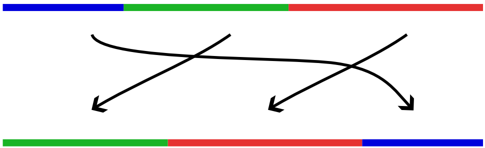
A scissors automorphism is a scissors congruence from $P$ to itself:





(image by Inna Zakharevich)

In $E^1$, if we don't allow reflections, $\mathrm{Aut}(P)$ is the group of **interval exchange transformations**:



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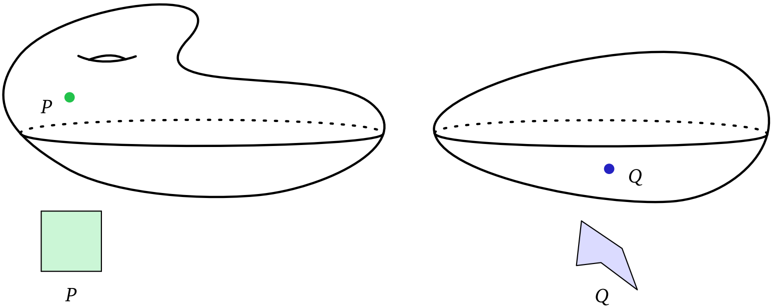

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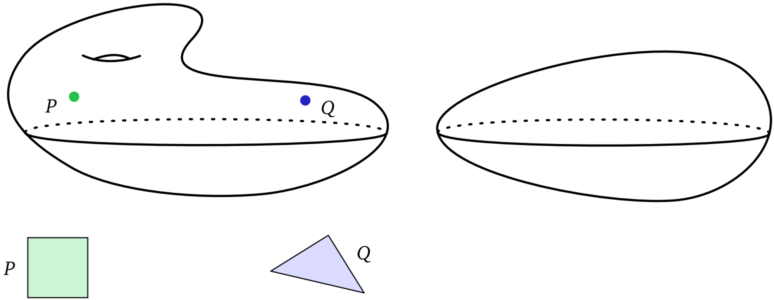

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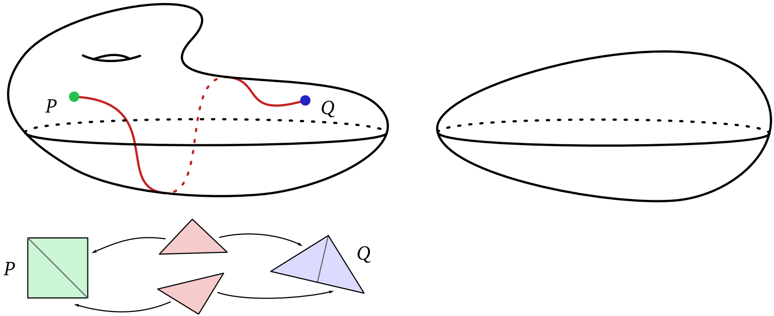

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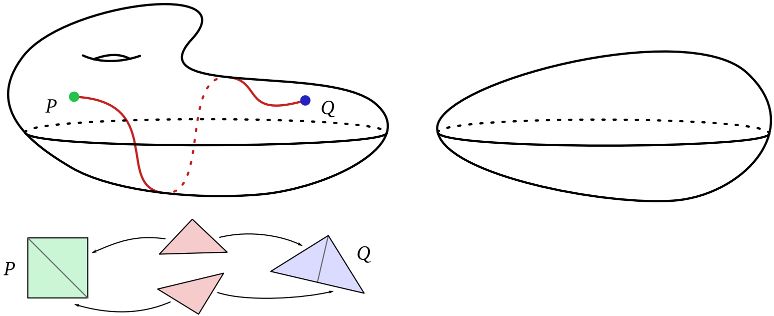


**Definition.** (Zakharevich) Scissors congruence $K$-theory is the group completion of this space. (Formally add negatives.)

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**Example.** The **SAF invariant** of an interval exchange transformation $f$:

$$\phi(f) = \sum_{\text{edges}} (\text{length}) \otimes (\text{translation distance}) \in \mathbb{R} \otimes \mathbb{R}.$$

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No other higher $K$-groups known! (As of 2022.)

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**Theorem (M 2022)**

$K_m(E^n)$ is always rational, and

$$K_m(E^n) \cong H_m(\mathrm{Isom}(E^n); St(E^n) \otimes \det).$$

Gives a general method!

Builds on joint work with Bohmann, Gerhardt, Merling, and Zakharevich.

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Ongoing work of Holley, Lemann, and others is drawing conclusions for $E^2$ and $H^2$!

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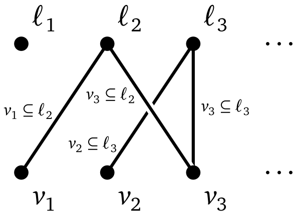

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Polytopes up to subdivision (but no moving around) gives $St(E^n)$.

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Can take $ST(E^n) \simeq \bigvee S^n$ and de-suspend $n$ times to get $\bigvee S^0$!

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$K(E^n)$ is a homotopy orbit spectrum,

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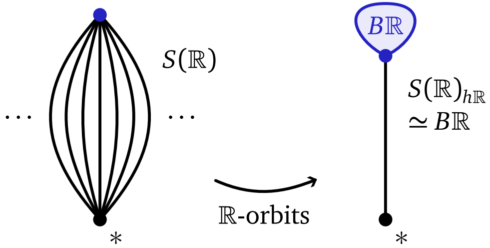

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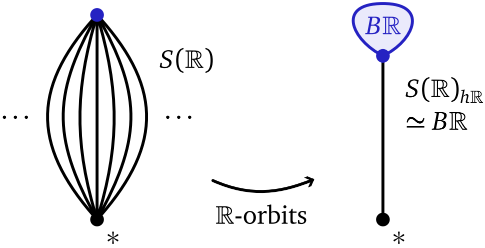


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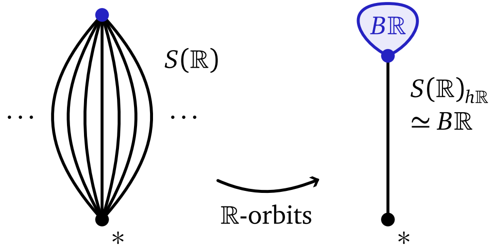


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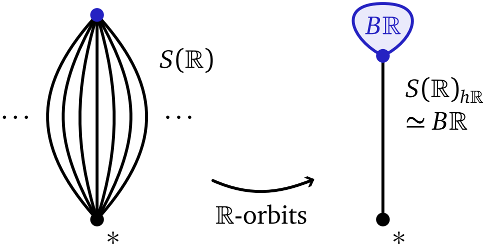


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Also get exact sequences for $K_*(E^3)$, higher Dehn-Sydler-Jessen theorem!

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So *stably*, $H_*(\mathrm{Aut}(P);\mathbb{Q})$ becomes free and the $K$-groups are the generators.

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**Open question:** Is $\mathrm{Aut}(P) \cong \mathrm{Aut}(Q)$?

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$\Lambda^*(-) =$ free graded-commutative algebra (polynomial $\otimes$ exterior)

## Theorem (Kupers, Lemann, M, Miller, Sroka 2024)

For any nonempty polytopes $P, Q \subseteq E^n$,

$$H_*(\mathrm{Aut}(P)) \cong H_*(\mathrm{Aut}(Q)).$$

**Corollary:**

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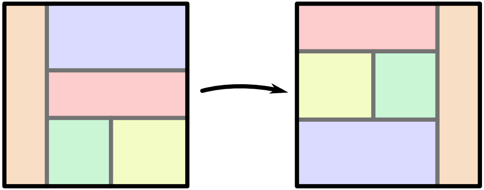
**Theorem (Kupers–Lemann–M–Miller–Sroka 2026)**

Solomon–Tits theorem when polytopes are cut out by a fixed collection of hyperplanes, or spanned by a fixed collection of points.

**Example.** "Rectangle exchange transformations" (Cornulier–Lacourte 2022)



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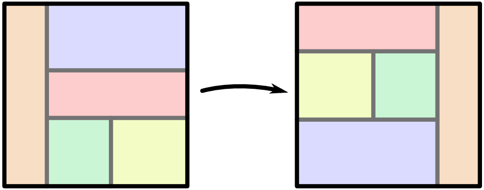

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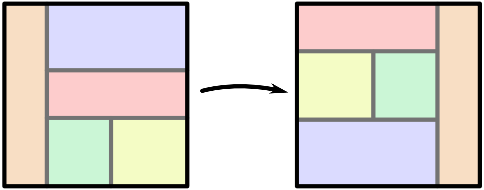

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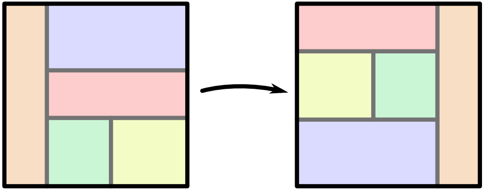

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$K_1 \cong H_1 \cong (\Lambda^2\mathbb{R} \otimes \mathbb{R}^{\otimes(n-1)})^{\oplus n}$ (Cornulier–Lacourte 2022)

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Can also do variants where the homology was not known before, e.g. the "irrational slope Thompson's group" (Burillo–Nucinkis–Reeves 2022).

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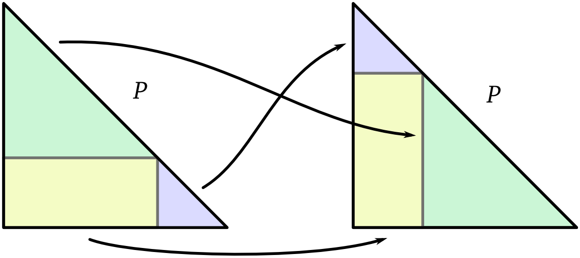
**Theorem (Li–M, in progress)**

Scissors congruence $K$-theory is $K$-theory of “ample groupoids,” $\mathrm{Aut}(P)$ is a “topological full group.”

Tantalizing open question: is $K_1(E^2) = 0$?



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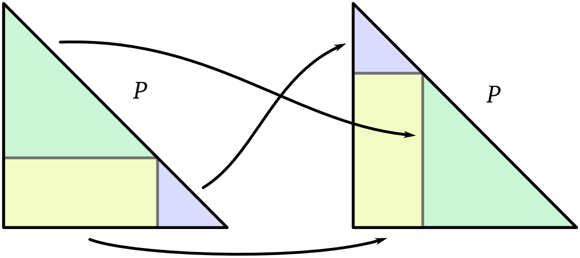

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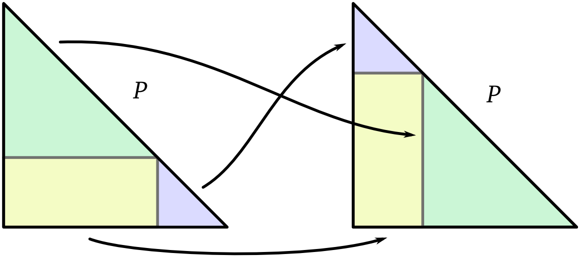


All you have to do is define a nonzero homomorphism $\mathrm{Aut}(P) \to A$ where $A$ is an abelian group...

Thank you!