Towards a finite presentation of the 3-dim bordism bicategory

Background

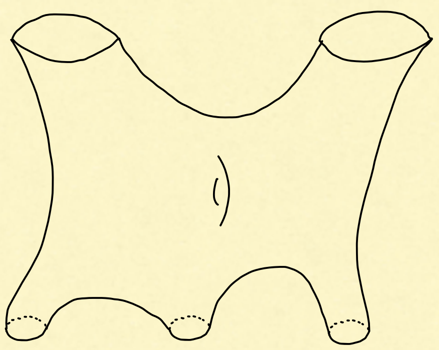
TQFTs

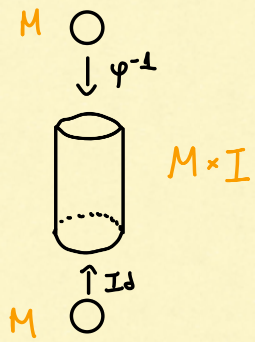
Definition A cobordism $\overline{W}$ between closed $(d-1)$-dim manifolds Y_0, Y_1 is a 5-tuple $(W; \partial_{in}W, \partial_{out}W; \iota_0, \iota_1)$:

- W is a compact smooth manifold with a decomposition of its boundary as a disjoint union $\partial_{in}W \sqcup \partial_{out}W$

- $\iota_0 : Y_0 \xrightarrow{\cong} \partial_{in}W$
  $\iota_1 : Y_1 \xrightarrow{\cong} \partial_{out}W$

Examples

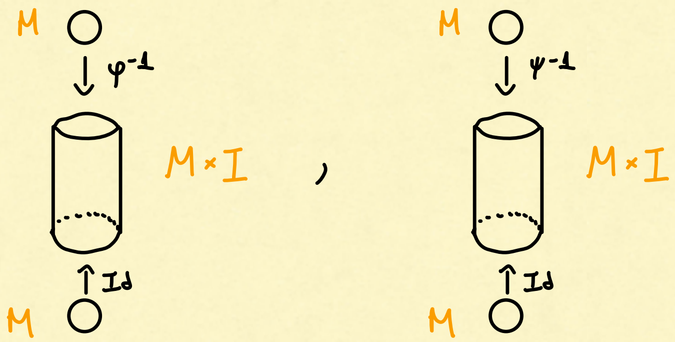




(Mapping cylinder for $\varphi : M \xrightarrow{\cong} M$)

Definition Let $\overline{W}$, $\overline{W}'$ be two cobordisms between Y_0 and Y_1. An *equivalence* between $\overline{W}$ and $\overline{W}'$ is a diffeomorphism $D: W \longrightarrow W'$ that respects all extra structure.

Example Let



be two mapping cylinders on M. An equivalence of cobordisms between them is a *pseudoisotopy*.

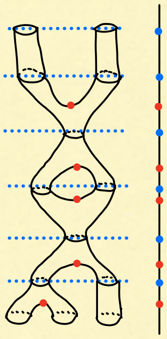
Definition The *bordism category* $\mathrm{Bord}_{(d-1;d)}$ is the category that has

- Objects: closed, oriented, $(d-1)$-dim manifolds
- Morphisms: equivalence classes of oriented, d-dim cobordisms

This is a symmetric monoidal category, as witnessed by disjoint union.

Definition An oriented d-dim topological quantum field theory is a symmetric monoidal functor

$$Z: \mathrm{Bord}_{(d-1;d)} \longrightarrow \mathrm{Vect}_{\mathbb{k}}$$



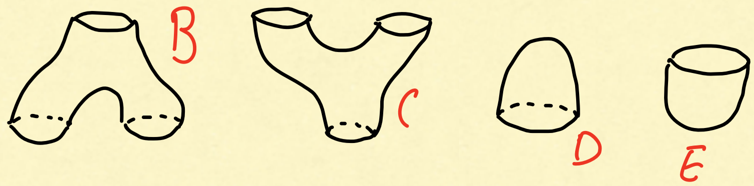
Goal: Understand how to construct and classify TQFTs in as many cases as possible.

Analogy with abelian group presentations

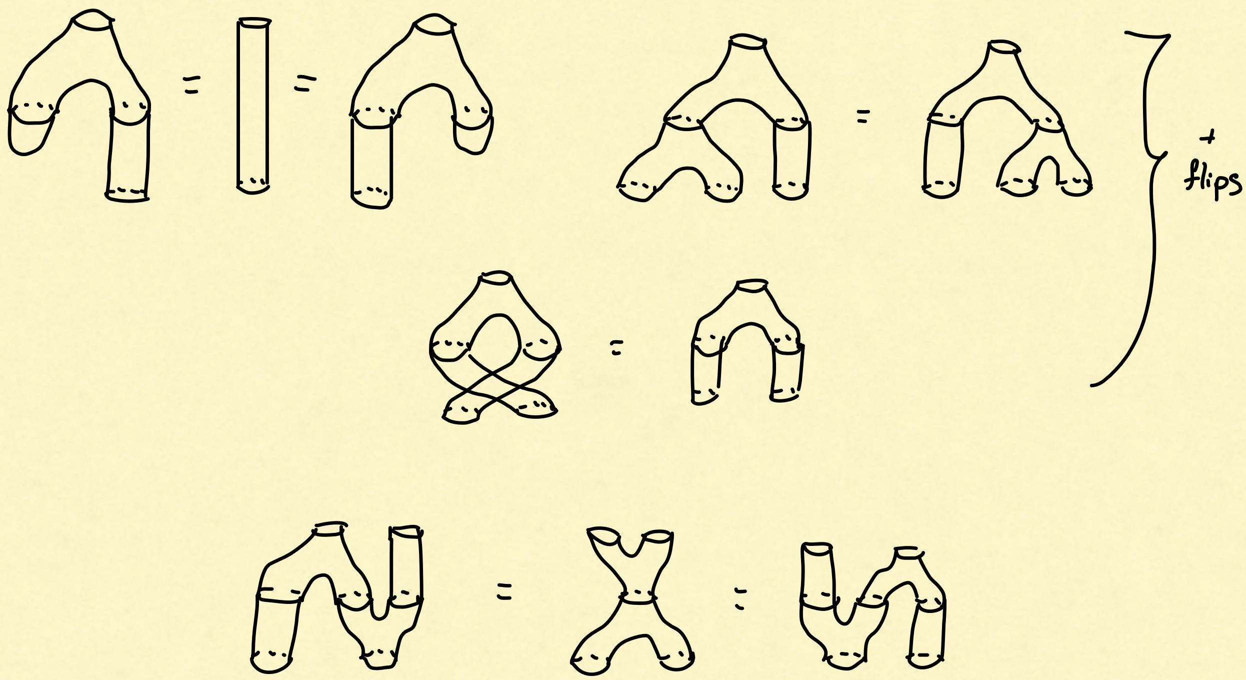
$\rightsquigarrow$ Presentations of bordism category with generators and relations

Example Presentation of Bord_{12} with generators and relations as symmetric monoidal category

1 Generating object: 

4 Generating morphisms: 

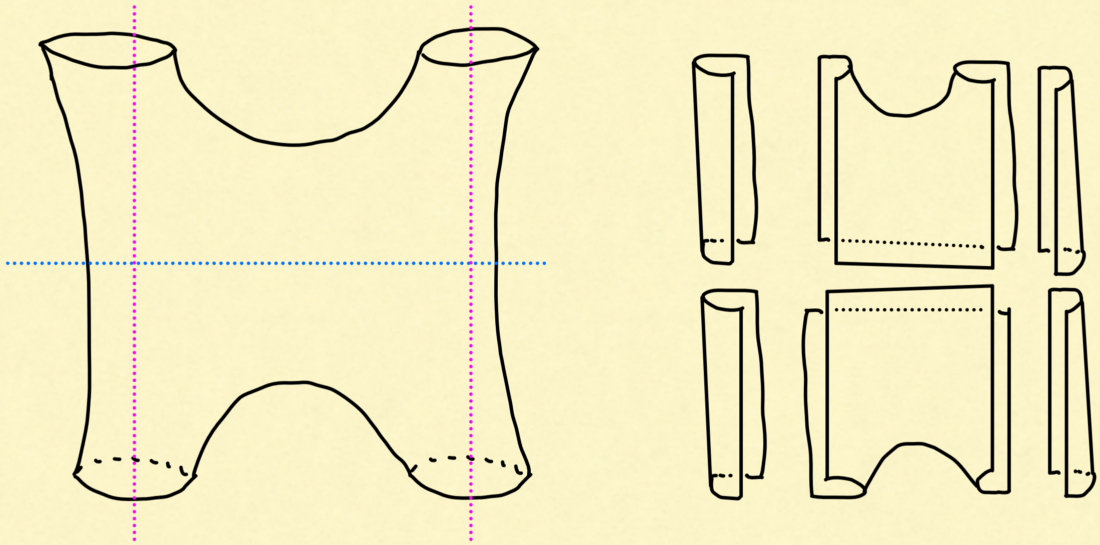
10 Generating relations:



Using this presentation, we can conclude that oriented 2-dim TQFTs are classified by commutative Frobenius algebras.

Bordism bicategories / Extended TQFTs

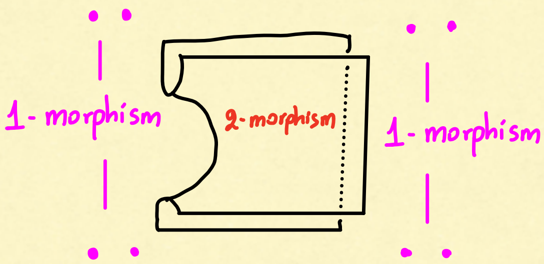
Extending down:



Definition The bordism bicategory $\mathrm{Bord}_{(d-1;d;d+1)}$ has

- Objects: closed, oriented $(d-1)$-dim manifolds
- 1-Morphisms: oriented d-dim cobordisms
- 2-Morphisms: equivalence classes of oriented $(d+1)$-dim cobordisms between cobordisms

Example



Definition An once extended oriented $(d+1)$-dim TQFT is a symmetric monoidal functor

$$Z : \mathrm{Bord}_{(d-1;d;d+1)} \longrightarrow \mathcal{C}$$

C. Shommer-Pries constructed a finite presentation of Bord_{012} and used it to classify once extended 2-dim TQFTs for some chosen target symm. monoidal bicategories.

B. Bartlett, C. Douglas, C. Shommer-Pries and J. Vicary conjectured a finite presentation of Bord_{123} and used it to classify once extended 3-dim TQFTs for some chosen targets in terms

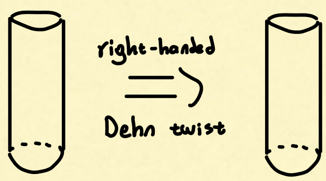
Goal: Obtain a finite presentation of Bord_{123}

Extending up: $\mathrm{Bord}_{12\sim} \longrightarrow \mathrm{Bord}_{123}$

Definition The bordism (2,1)-category $\mathrm{Bord}_{(d-1;d;\sim)}$ has

- Objects: closed, oriented (d-1)-dim manifolds
- 1-Morphisms: oriented d-dim cobordisms
- 2-Morphisms: isotopy classes of orientation preserving diffeomorphisms between cobordisms

Example



Definition A topological modular functor is a symmetric monoidal functor

$$Z : \mathrm{Bord}_{12\sim} \longrightarrow \mathcal{C}$$

$\mathrm{Bord}_{12\sim}$ is interesting
- "Contains" all mapping class groups of surfaces
- Stepping stone towards Bord_{123}

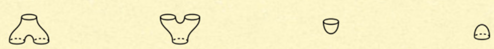
Goal: Obtain a finite presentation of $\mathrm{Bord}_{12\sim}$

Presentation of $\mathrm{Bord}_{12\sim}$

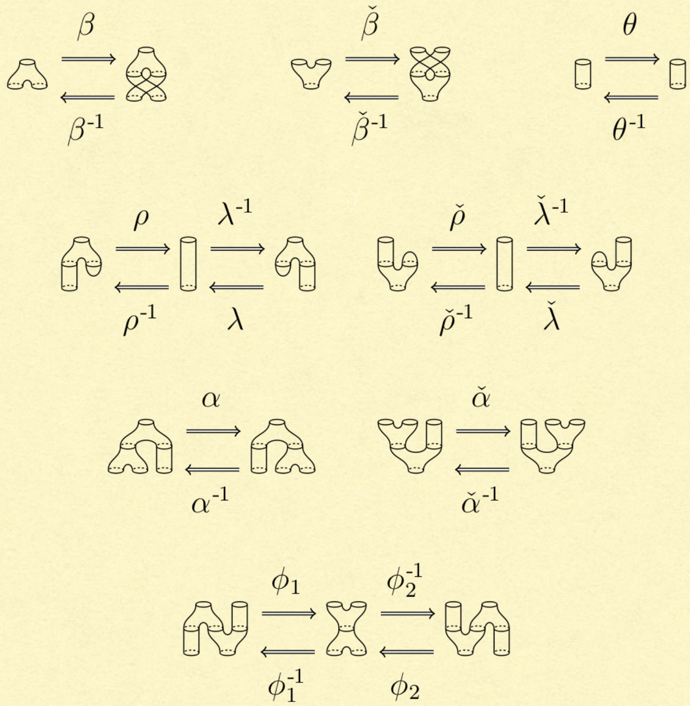
Theorem (B. Bartlett, C. Douglas, F-S)

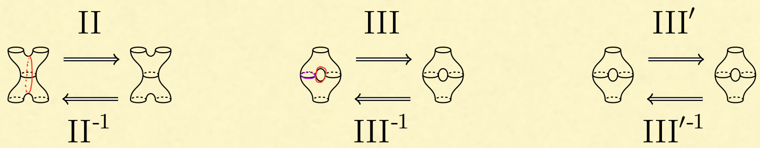
$\mathrm{Bord}_{12\sim}$ admits a finite presentation, called the $\mathcal{F}^2$-presentation, with 1 generating object, 4 generating 1-morphisms, 28 generating 2-morphisms, 66 generating relations, defined as follows:

- Generating 0-cell:
- Generating 1-cells:



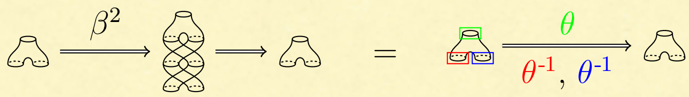
- Generating 2-cells:





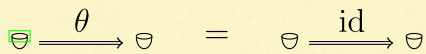
- Generating relations:

1. **Inverses.**
   Each of the generating 2-cells ω satisfies $\omega\omega^{-1} = \text{id}$ and $\omega^{-1}\omega = \text{id}$.
2. **Naturality - Frame flip.**



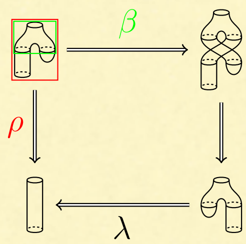
(N-F) x

3. **Isotopy of Morse data invariance.**



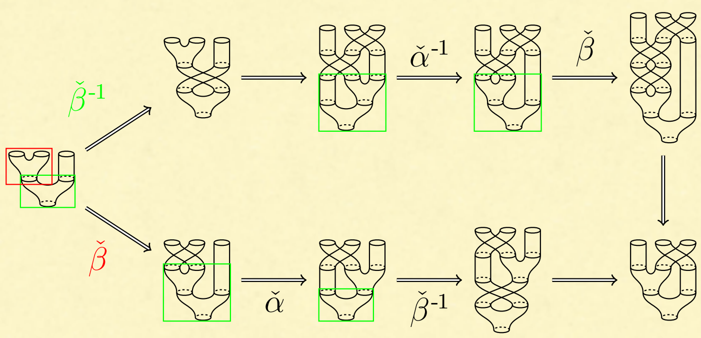
(IMD) x

4. **Naturality - Frame flip - Cusp.**

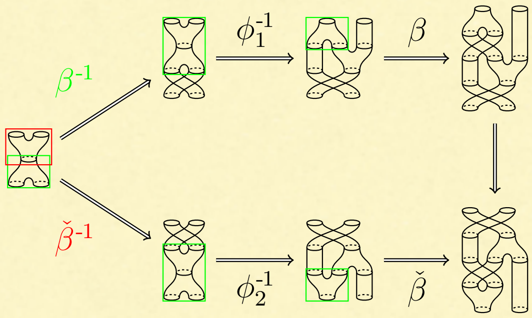


(N-F-Cusp) xz

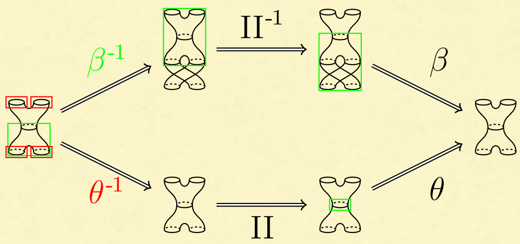
5. **Naturality - Frame flip - Crossing.**



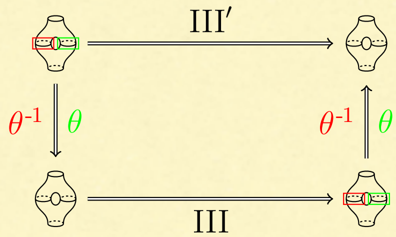
(N-F-Cross-1) x



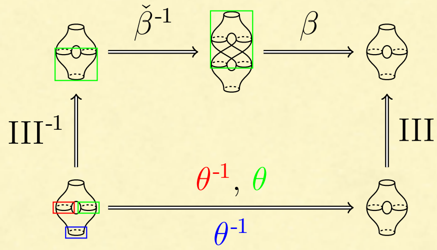
(N-F-Cross-2) z



(N-F-Cross-3) x

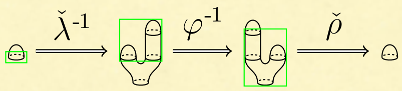


(N-F-Cross-4)

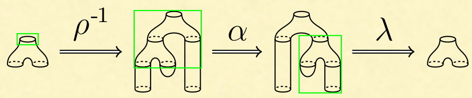


(N-F-Cross-5)

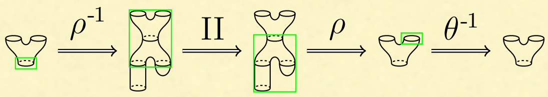
6. **Swallowtail.**



$=$ id (ST1) x

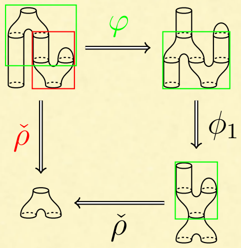


$=$ id (ST2) x

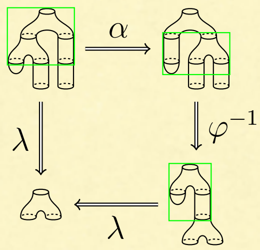


$=$ id (ST3) xz

7. **Beak.**

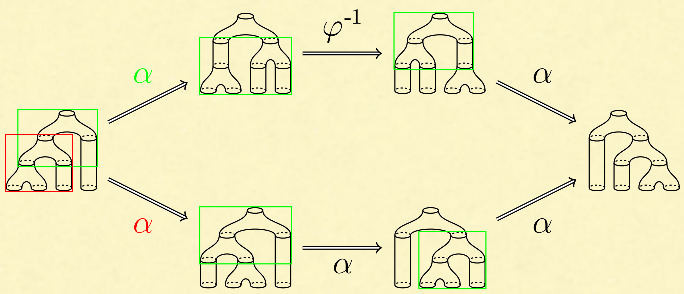


(B1) xz

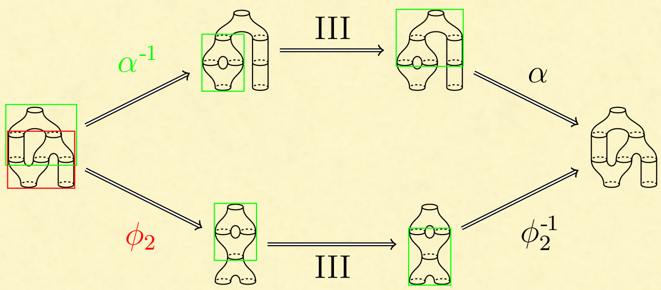


(B2) xz

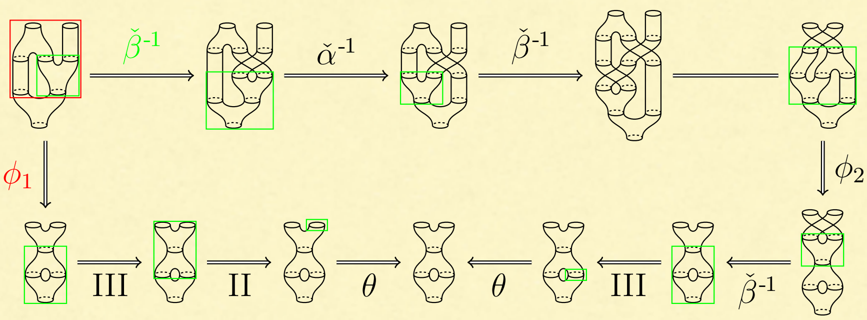
8. **Triple point.**



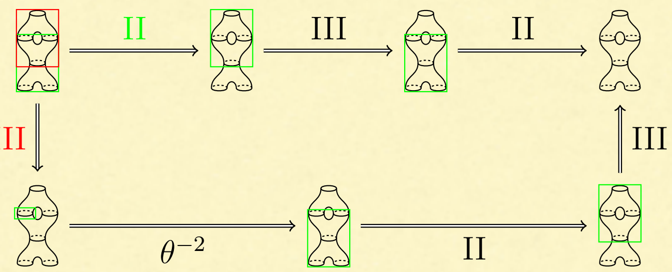
(T1) x



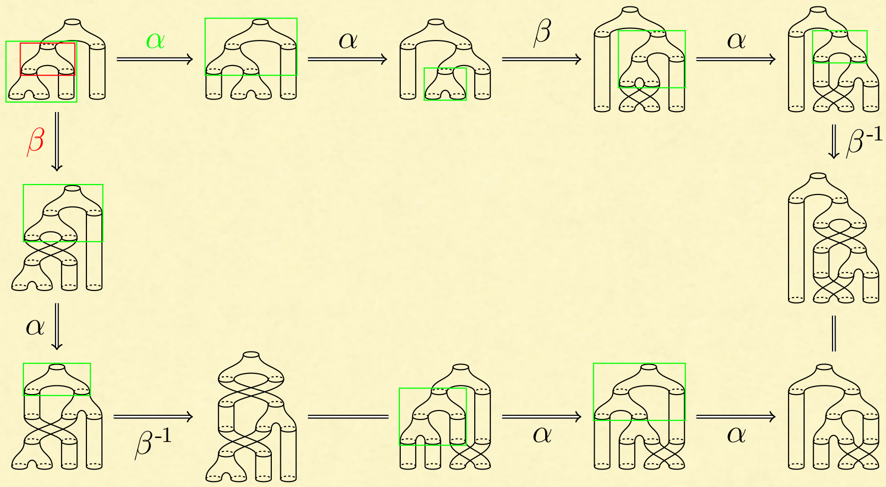
(T2) x



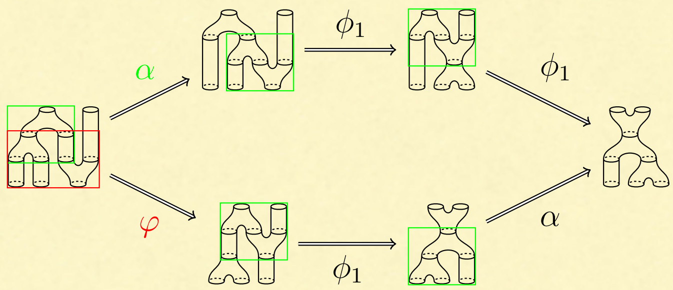
(T3) x



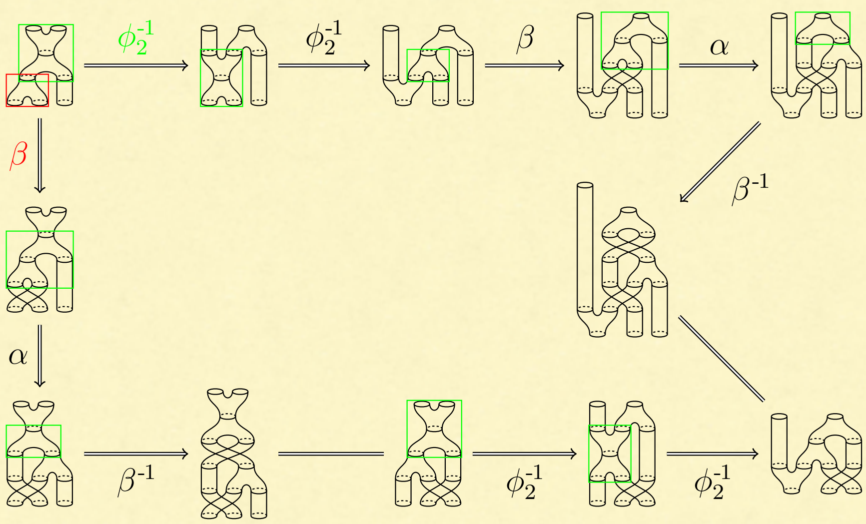
(T4) x



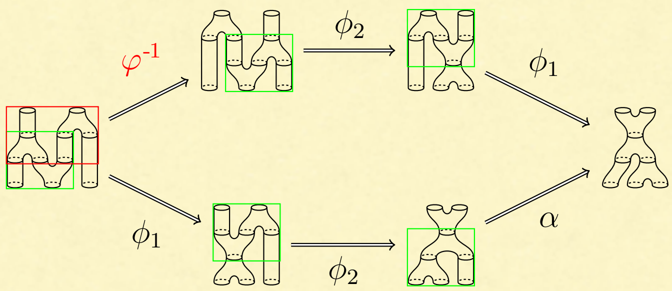
(T5) x



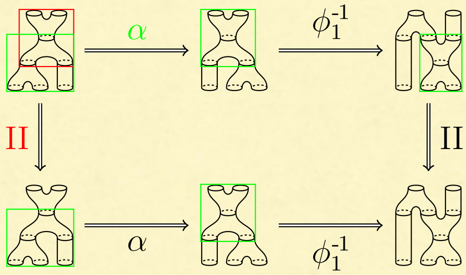
(T6) x



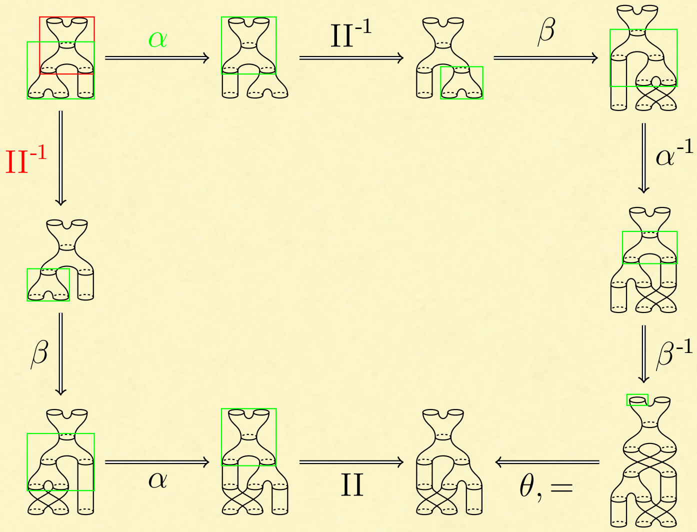
(T7) x


φ^{-1}
ϕ_2
ϕ_1
ϕ_1
ϕ_2
α

(T8) x


α
ϕ_1^{-1}
II
II
α
ϕ_1^{-1}

(T9) x

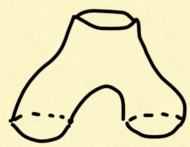

α
II^{-1}
β
II^{-1}
α^{-1}
β
β^{-1}
α
II
$\theta, =$

(T10) x

Goal: Describe a very general way of obtaining presentations of bordism categories using surgery.

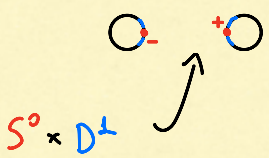
Surgery diagrams

Let's look a little closer at one of the generating morphisms,

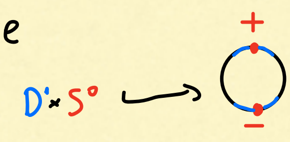


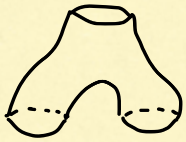
We note the following:

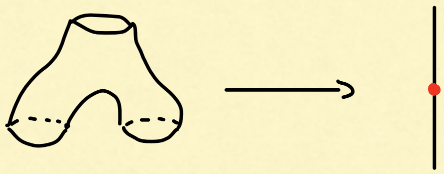
- Consider the surgery on ○ ○ associated with the framed attaching sphere

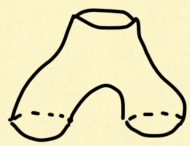


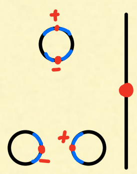
The outcome of this surgery is ○ with framed attaching sphere



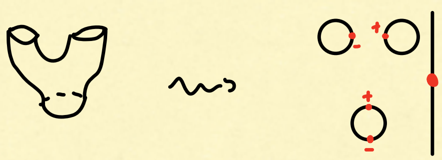
 comes equipped with a Morse function that has a single critical point of index 1.

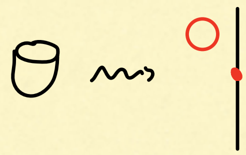


These are two key properties we would like to emphasize. Hence, we will denote  by

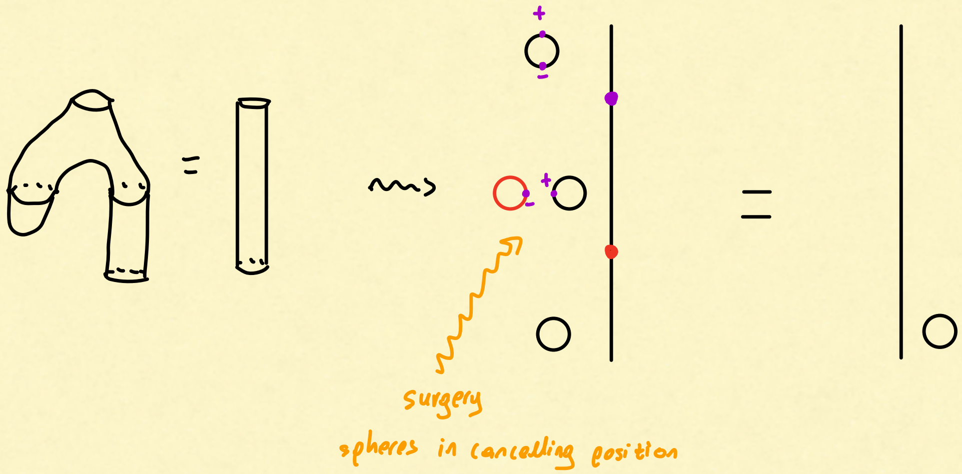


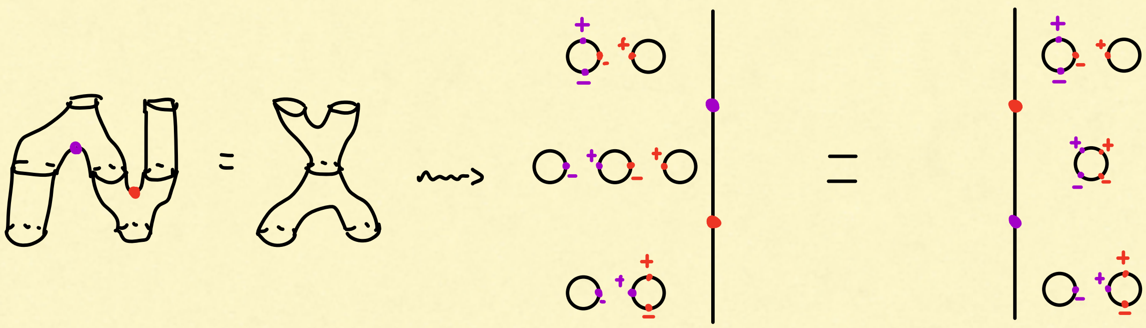
Likewise we have





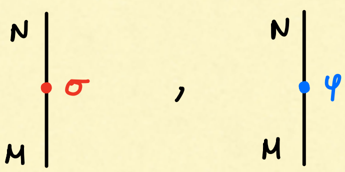
These are examples of surgery diagrams.
Let's see what some of the relations look like





We also have higher-dimensional analogues of surgery diagrams as well

In general,



will represent the trace of the surgery and mapping cylinders, respectively

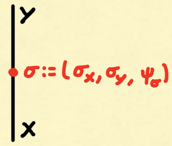
$\boxed{Sur_{(d-1;d)}}$ a presentation of $Bord_{(d-1;d)}$

- Gen. <u>0-cells</u> closed, oriented, connected, $(d-1)$-dim manifolds

- <u>Generating 1-cells</u>

  - Diffeomorphism type gen. 1-cells

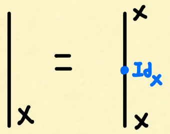
  - Surgery type gen. 1-cells



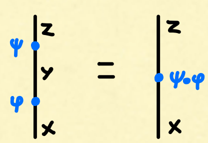
- <u>Relations</u>

Diffeomorphism type

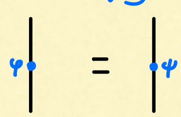
<u>Unit</u>



<u>Composition</u>

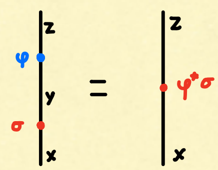


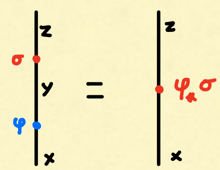
<u>Isotopy</u>



if φ and ψ are isotopic

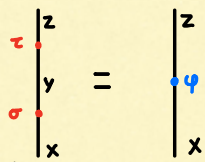
Naturality type





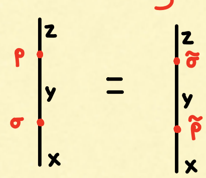
Surgery type

<u>Cusp</u>



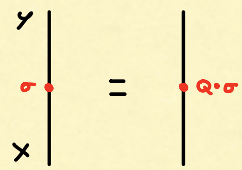
when the attaching spheres of σ, τ are in cancelling position and φ is a compatible cancelling diffeomorphism

<u>Crossing</u>



when (σ,ρ), $(\tilde{\rho},\tilde{\sigma})$ are in compatible crossing positions

Frame change type



where σ is of index k and

$Q \in O(k)\times O(n-k) \cap SO(n)$

$S^{k-1}\times D^{n-k} \hookrightarrow X$

$D^{k}\times S^{n-k-1} \hookrightarrow Y$

Theorem (Juhasz) $\mathcal{F}(Sur_{(d-1;d)})$ is equivalent (in fact, isomorphic) to $Bord_{(d-1;d)}$

Sketch Construct intermediate categories

$$F(Sur_{(d-1;d)}) \rightleftarrows Bord^{\text{Morse data}}_{(d-1;d)} \longrightarrow Bord^{\text{Morse}}_{(d-1;d)} \longrightarrow Bord_{(d-1;d)}$$

Interlude

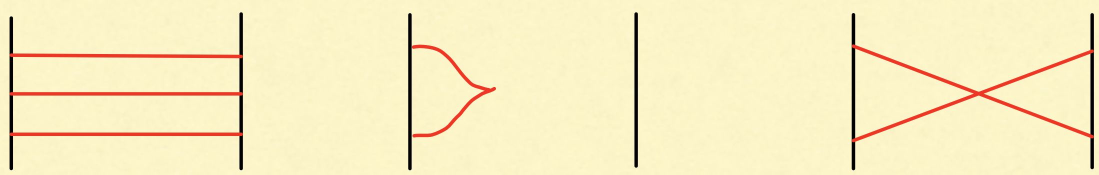
$\mathcal{F}(W) := C^\infty(W;\mathbb{R})$

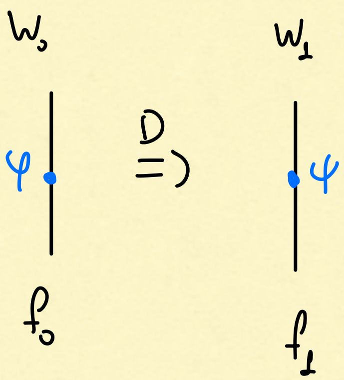
$\mathcal{F}^0(W) \rightsquigarrow$ space of excellent Morse functions

$\mathcal{F}^1_\alpha(W) \rightsquigarrow$ space of functions with a single cusp crit. point

$\mathcal{F}^1_\beta(W) \rightsquigarrow$ space of functions that have exactly two Morse critical points at the same critical level

Cerf paths

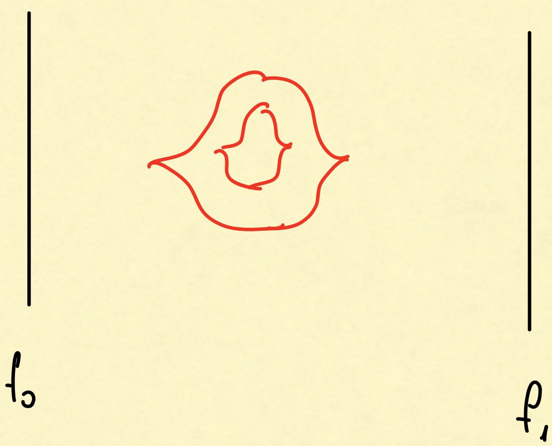




$f_1 : W_1 \longrightarrow \mathbb{R}$

$W_0 : f_0, \quad D^* f_1$

Can I find a path of Morse functions without any crit pts connecting f_0 and f_1 ?



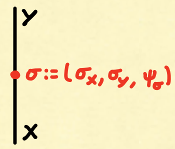
$\mathrm{Sur}_{(d-1;d;\sim)}$

- **0-cells** Closed, oriented, $(d-1)$-dim manifolds

- **Generating 1-cells**

– Diffeomorphism type gen. 1-cells

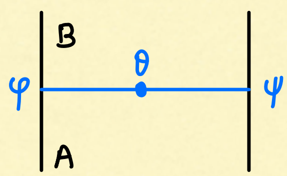
– Surgery type gen. 1-cells



- **Generating 2-cells**

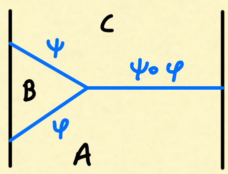
1. Diffeomorphism type generating 2-cells:

Isotopy 2-cells



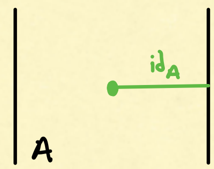
Compositors

(y)



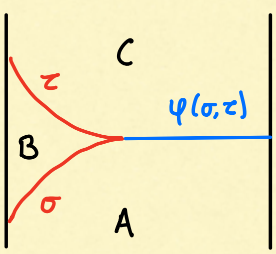
Unitor

(y)

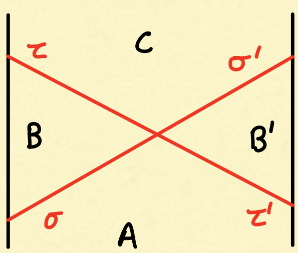


2. Surgery type generating 2-cells:

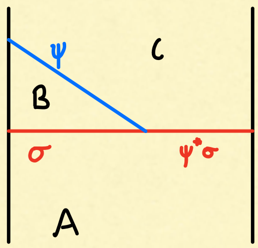
Cusp

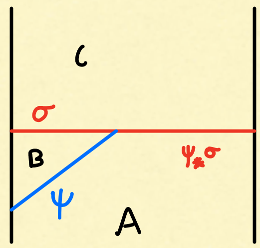


Crossing

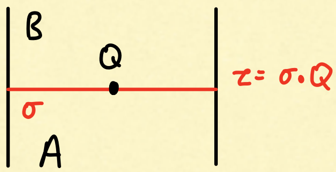


3. Naturality type generating 2-cells :





4. Frame change type generating 2-cells :



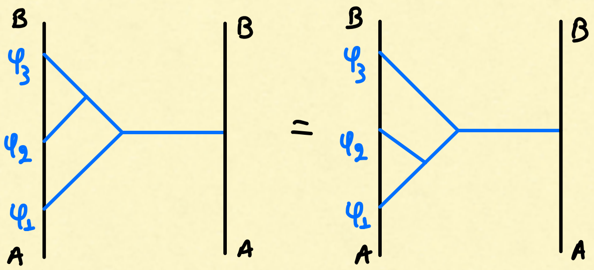
where

- $(\sigma\cdot Q)_A = \sigma_A \circ Q|_{S^{k-1}\times D^{n-k}}$
- $(\sigma\cdot Q)_B = \sigma_B \circ Q|_{D^k \times S^{n-k-1}}$
- $\varphi_{(\sigma\cdot Q)} = \varphi_\sigma$

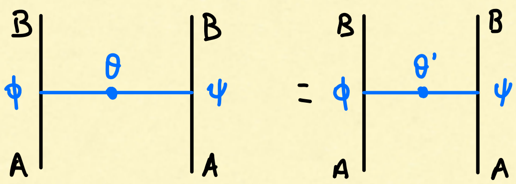
Relations (partial list)

1. Diffeomorphism type relations

Associativity relations



Isotopy invariance



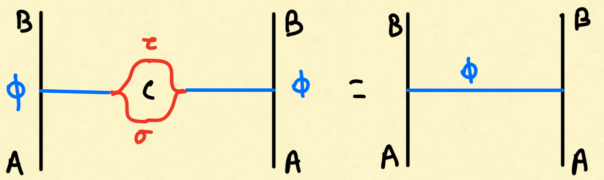
if θ and θ' are isotopic rel ∂

2. Surgery type relations :

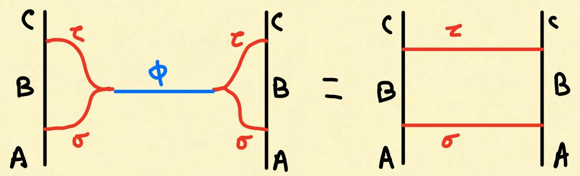
A. Coming from single jet singularities

Cusp invertibility relations

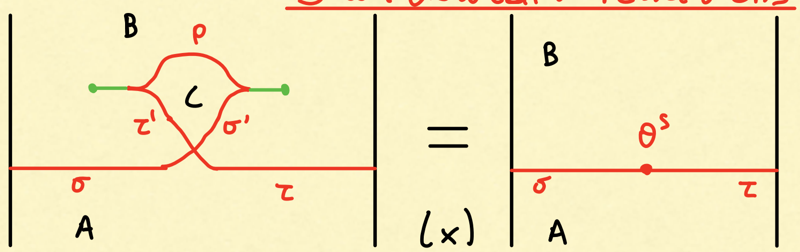
Eye birth death



Merge - unmerge relations

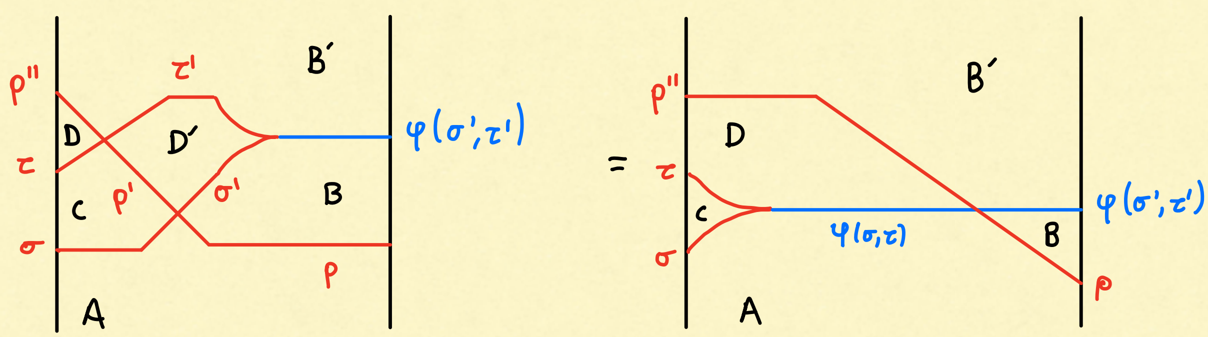


Swallowtail relations

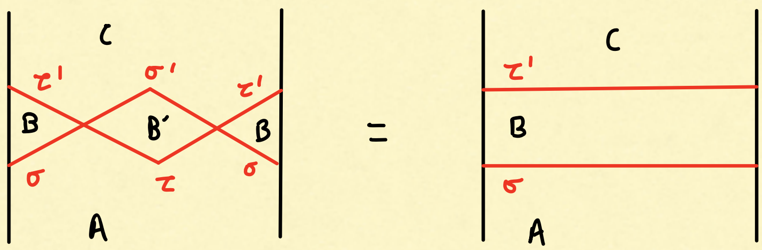


B. Coming from multijet singularities

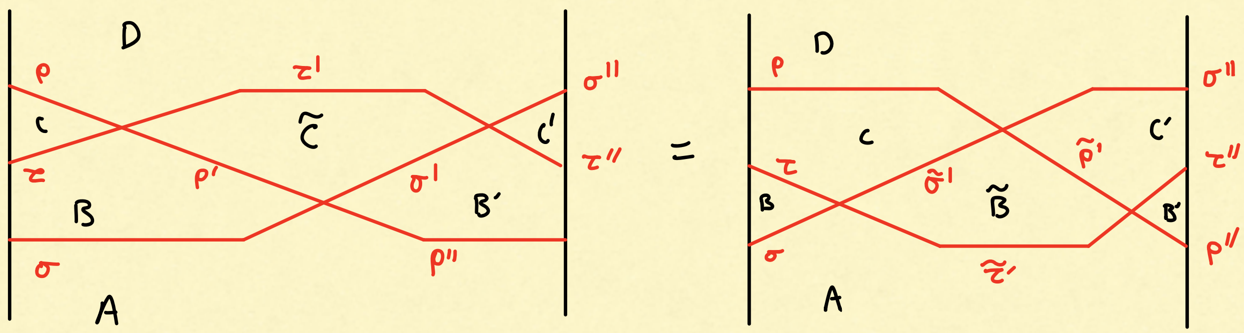
Beak



Reidemeister II

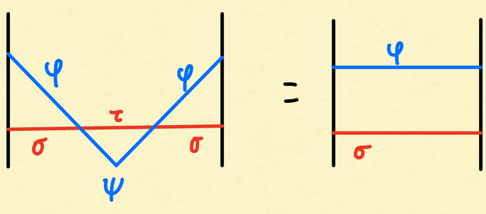


Reidemeister III

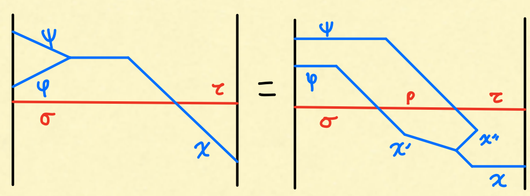


3. Naturality type relations

Naturality – Reidemeister II

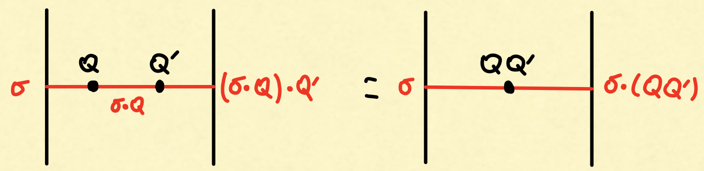


Surgery-compositor relations

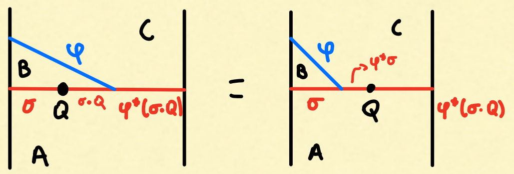


4. Frame change type relations

Composition of frame change 2-cells

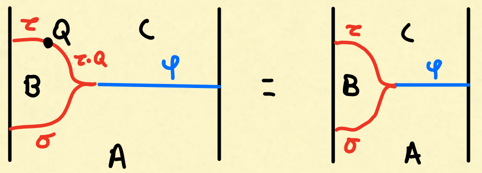


Frame change – naturality relations



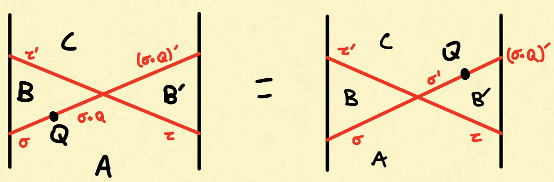
(x) (y)

Frame change – cusp relations



(x) (y)

Frame change – crossing relations



(y)

Theorem (B. Bartlett, C. Douglas, F.S)

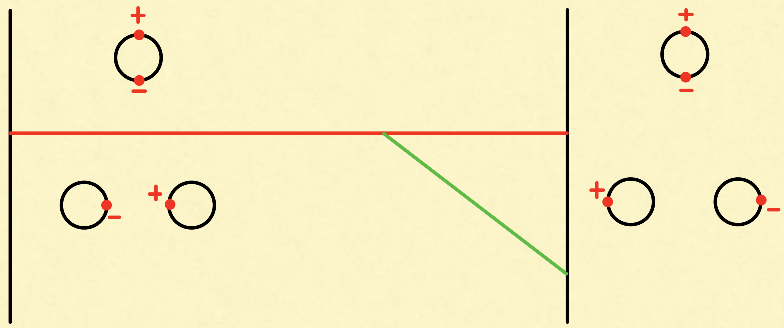
$\mathcal{F}(Sur_{(d-1; d; \sim)})$ is equivalent to $Bord_{(d-1; d; \sim)}$.

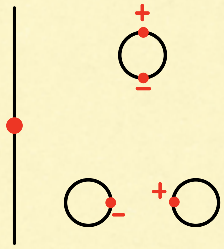
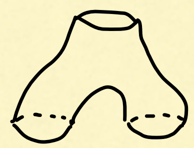
$$Sur_{(d-1; d; \sim)} \rightleftarrows Bord^{\text{cert data}}_{(d-1; d; \sim)} \longrightarrow Bord^{\text{cert}}_{(d-1; d; \sim)} \longrightarrow Bord_{(d-1; d; \sim)}$$

Skeletalization

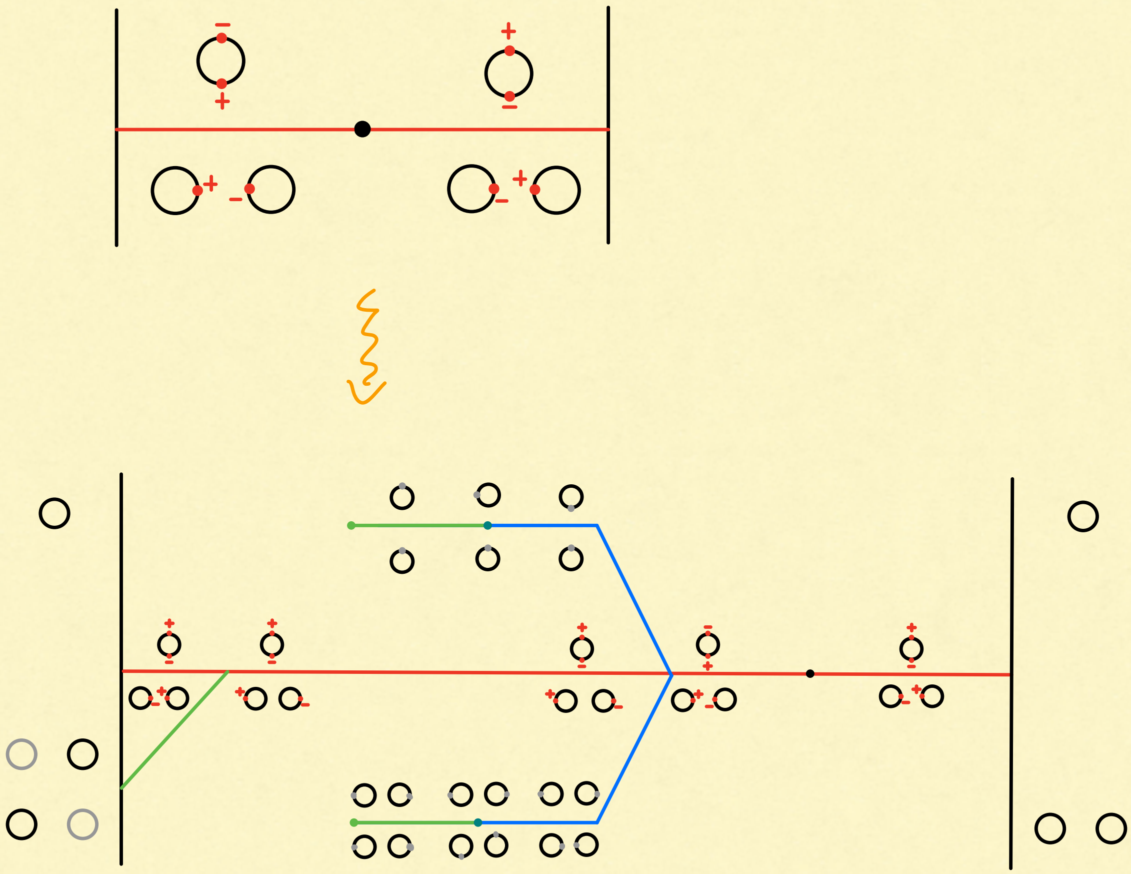
Issue: $Sur_{12\sim}$ is still big $\rightsquigarrow$ Skeletalize

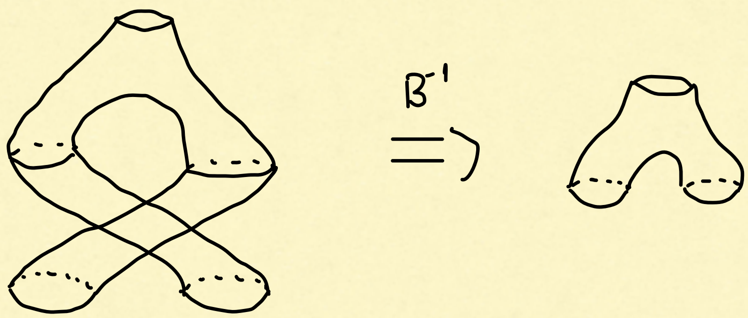
should not be both included since

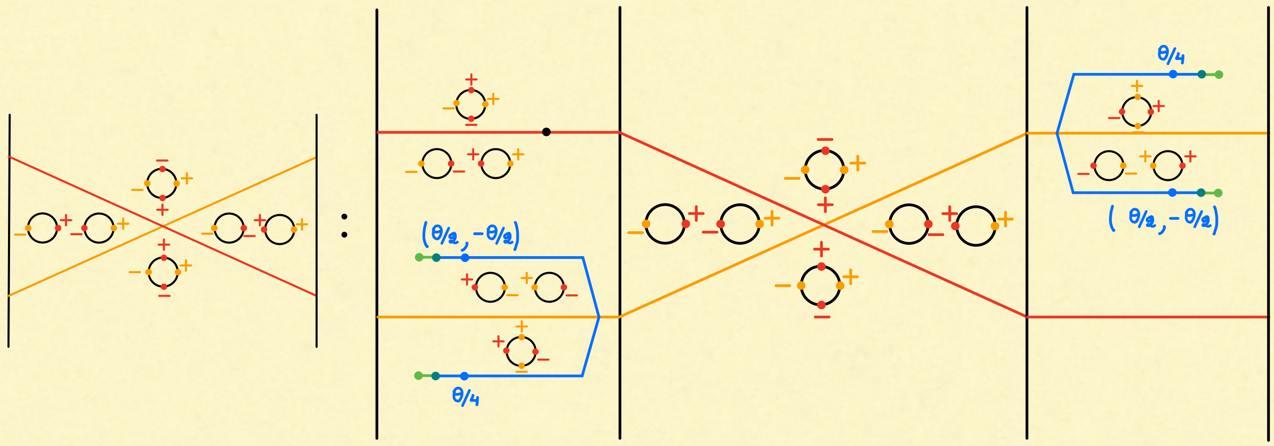


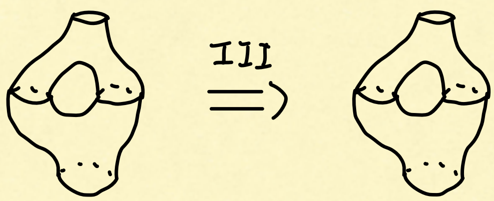
I choose to keep  and call it 

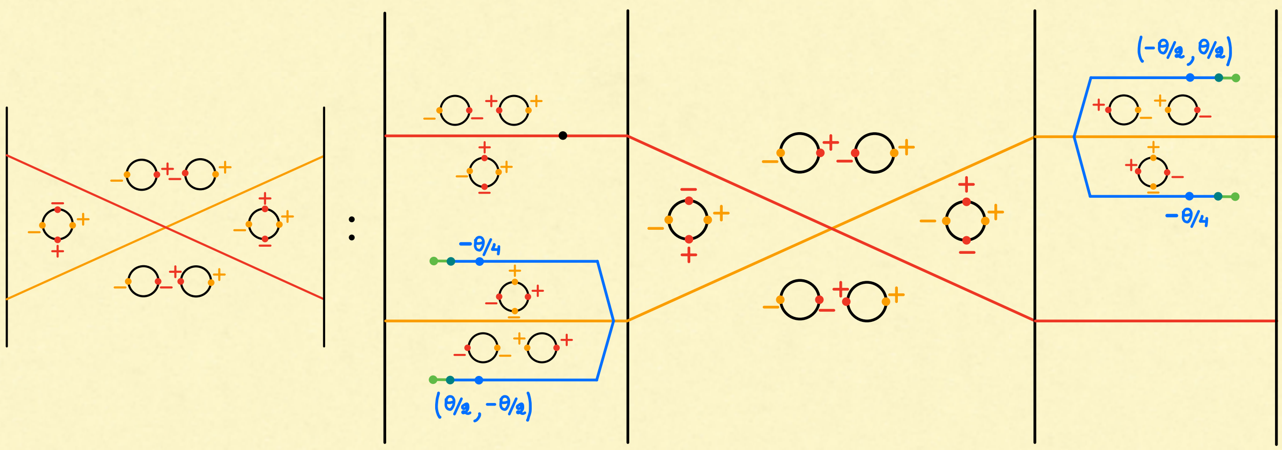
But then, what do I do with :

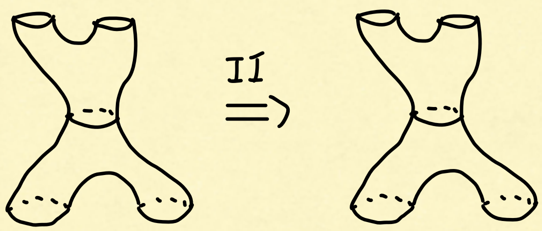





θ/4
(θ/2, -θ/2)
θ/4
(θ/2, -θ/2)

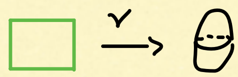

III


(-θ/2, θ/2)
-θ/4
-θ/4
(θ/2, -θ/2)


II'

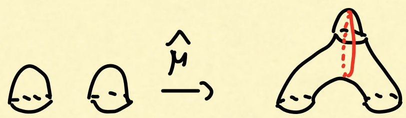
Extending to 3d (speculation, in progress following work of B. Haioun)

Level 0

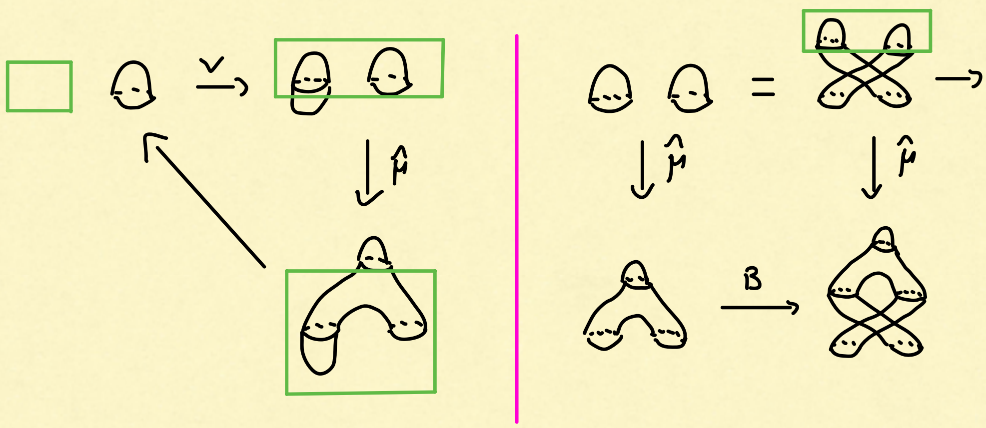
Extra gen. 2-cells: 

Extra relations: None

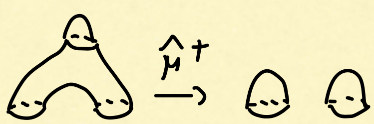
Level 1

Extra gen. 2-cells: 

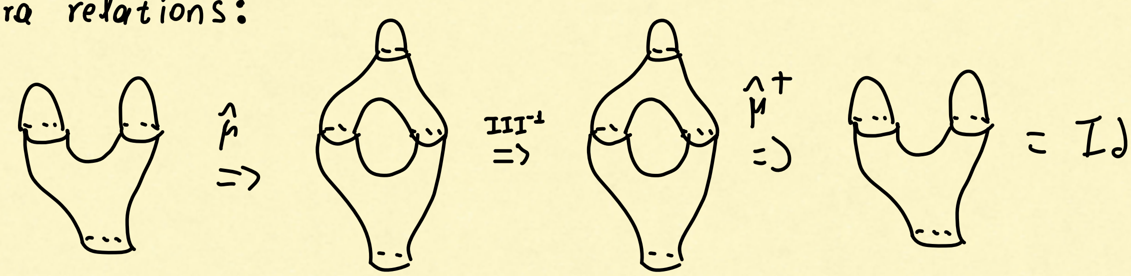
Extra relations:

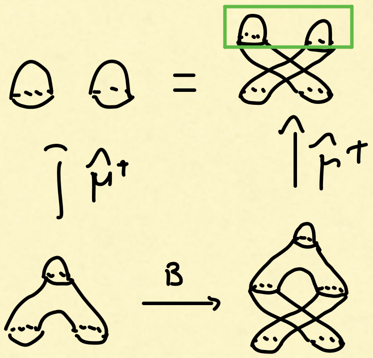


Level 2

Extra gen. 2-cells: 

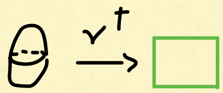
Extra relations:



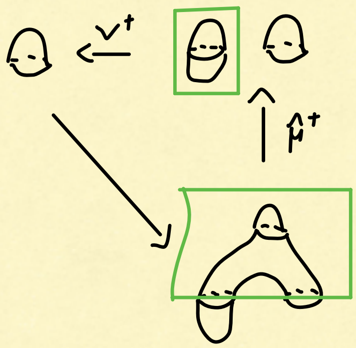


Level 3

Extra gen. 2-cells:



Extra relations:



D1: Classifications of partially defined extended 3d TQFTs?

D2: Are there "twisted S^1-compactifications" $\mathrm{Bord}^{or}_{12\sim} \longrightarrow \mathrm{Bord}^{or}_{23\sim}$?