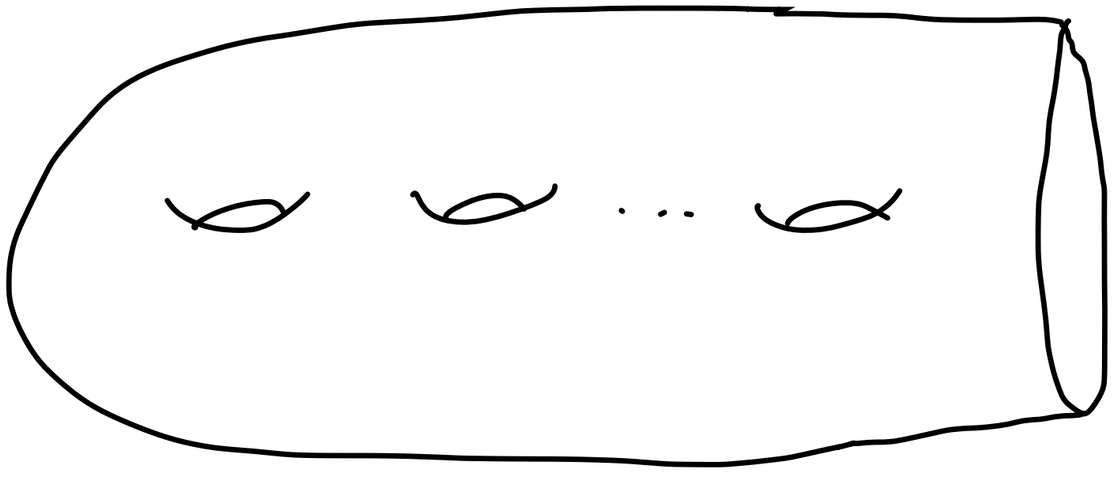
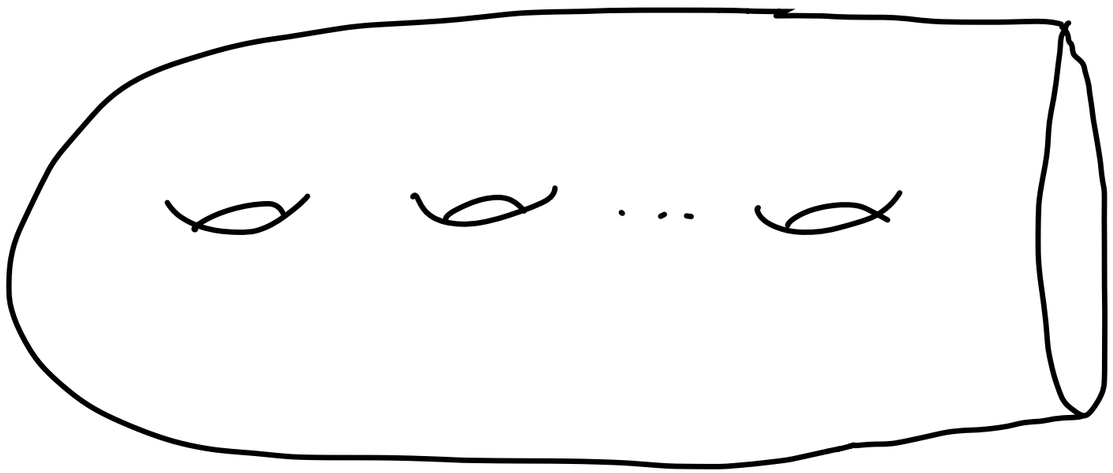
Cohomology of Handlebody Torelli Groups

Annie Holden

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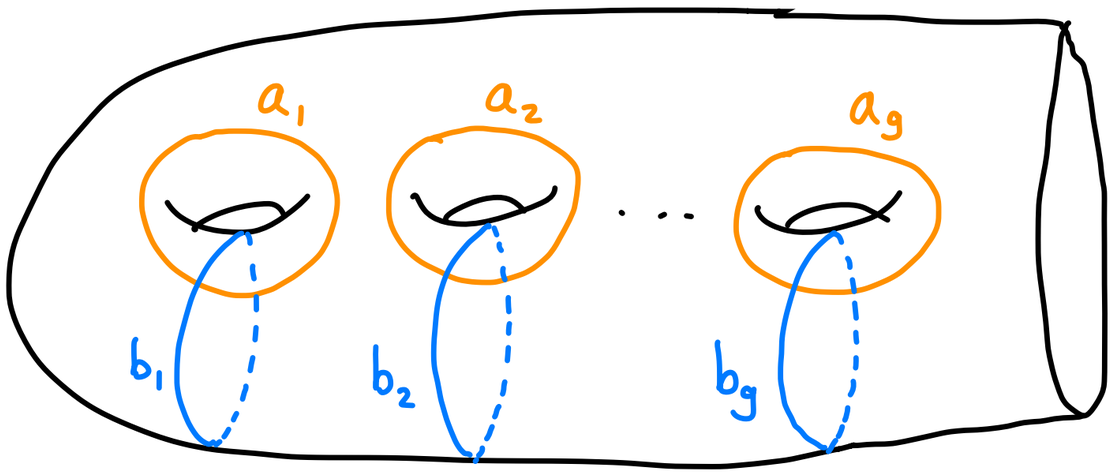


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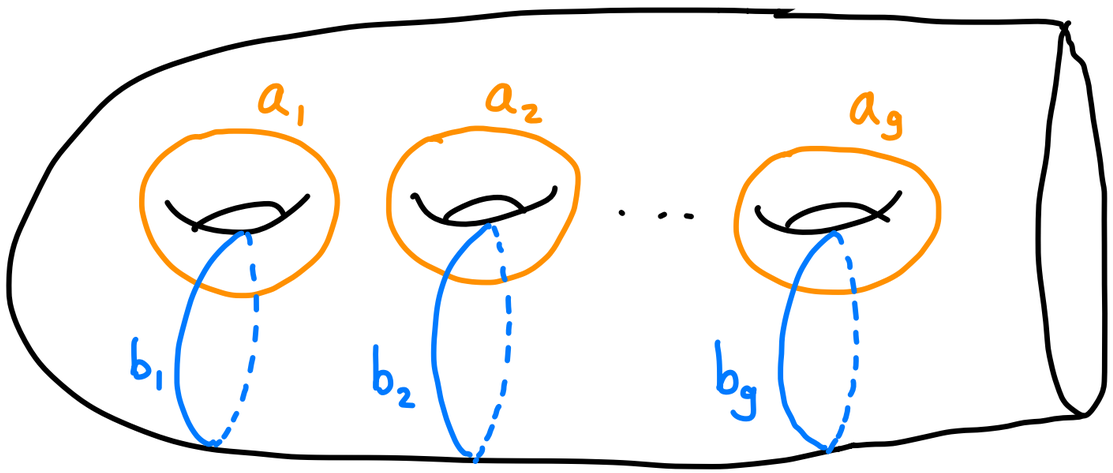


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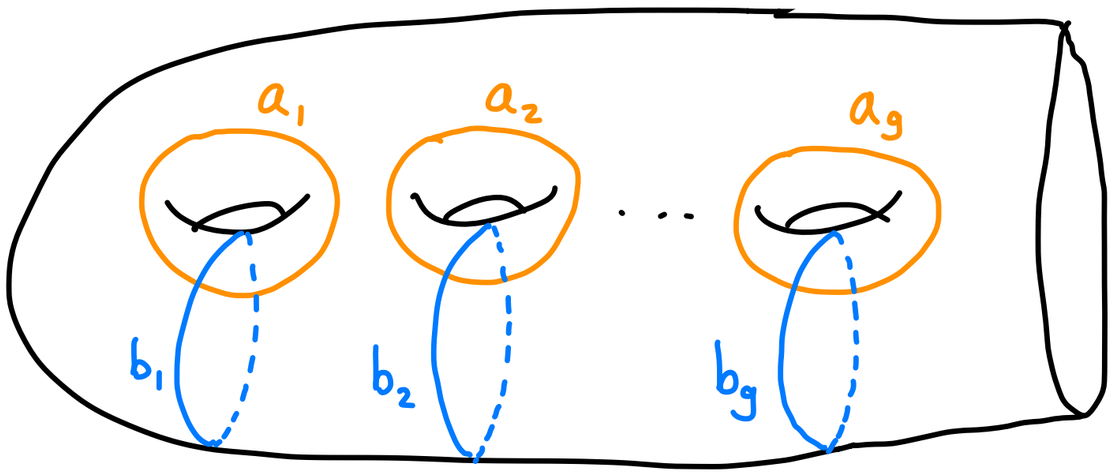


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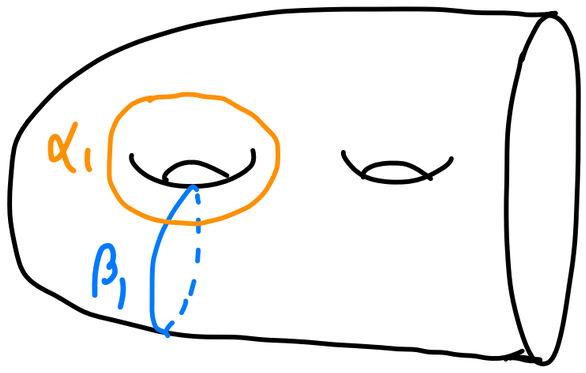
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Torelli group $\mathcal{I}_g^1$ = the subgroup of Mod_g^1 that acts trivially on H

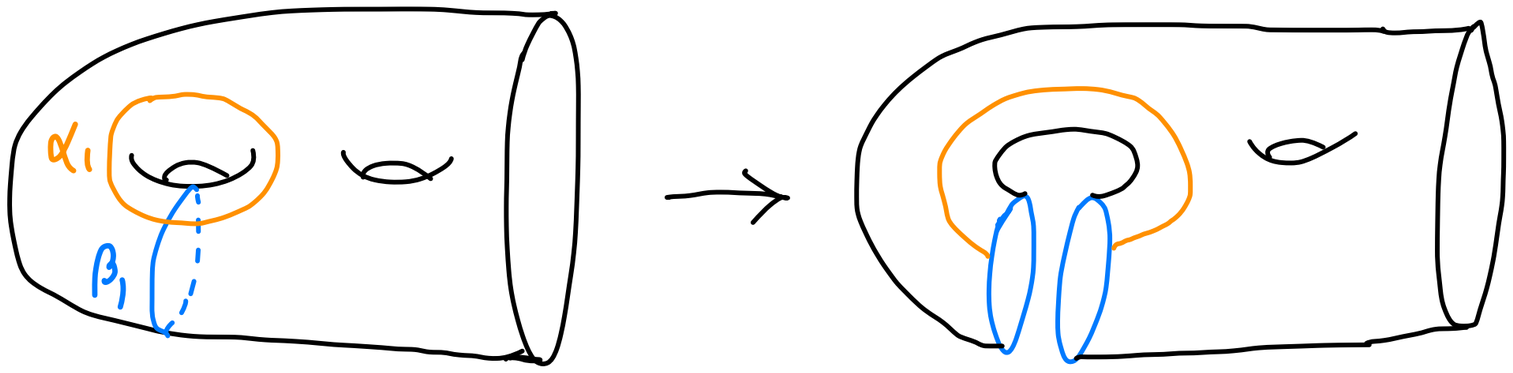
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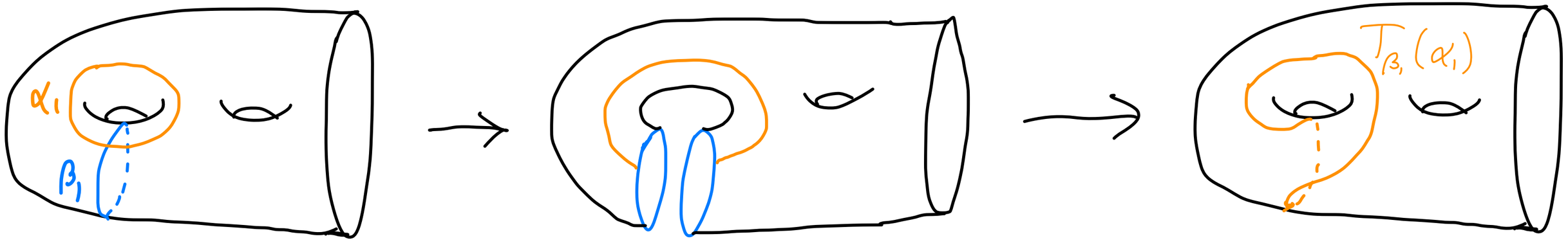
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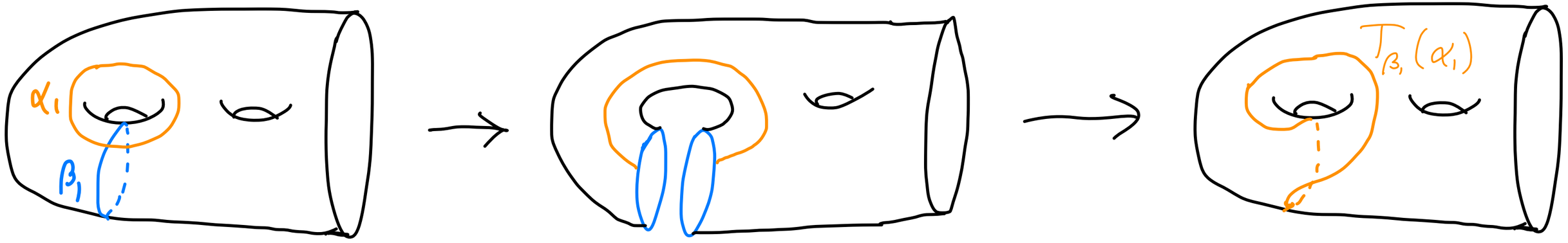
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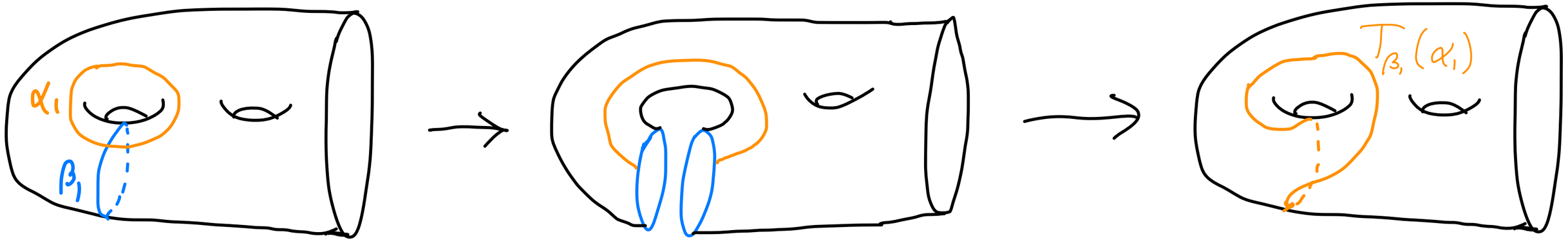


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T_{β_1} acts on H by sending $a_1 \to a_1 + b_1$

fixing all other generators

Examples that act trivially on H

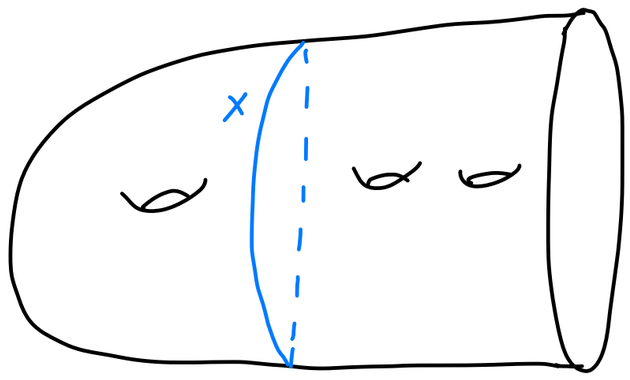
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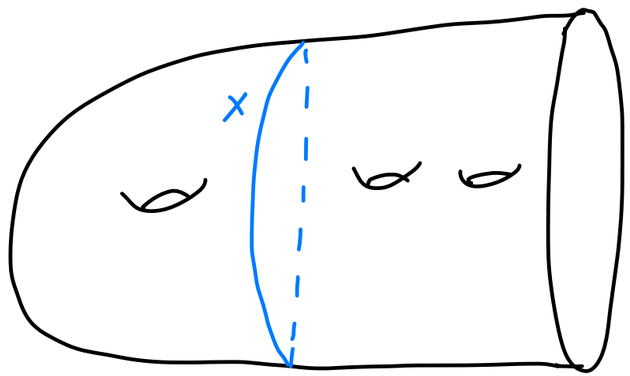
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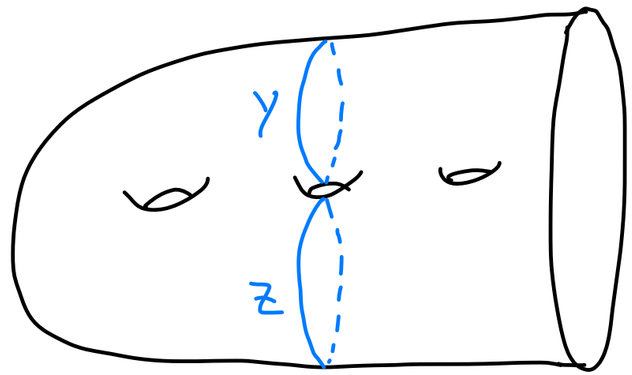
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$[x] = 0$ in H

$T_x \in \mathcal{I}_g^1$

ex) bounding pair maps



$[y] = [z]$

$\Updownarrow$

$y \cup z$ bounds a subsurface

$T_y T_z^{-1} \in \mathcal{I}_g^1$

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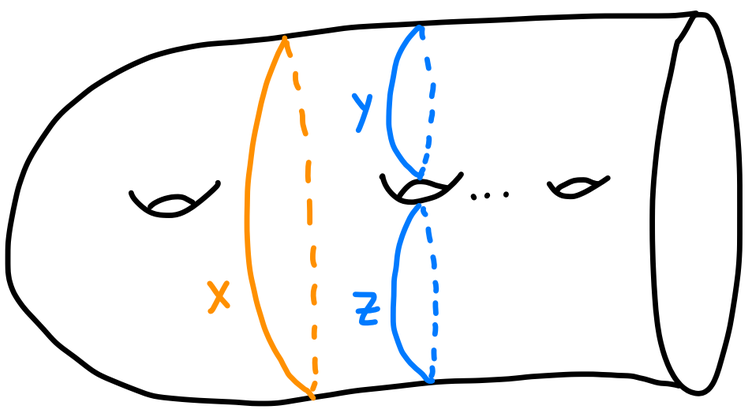
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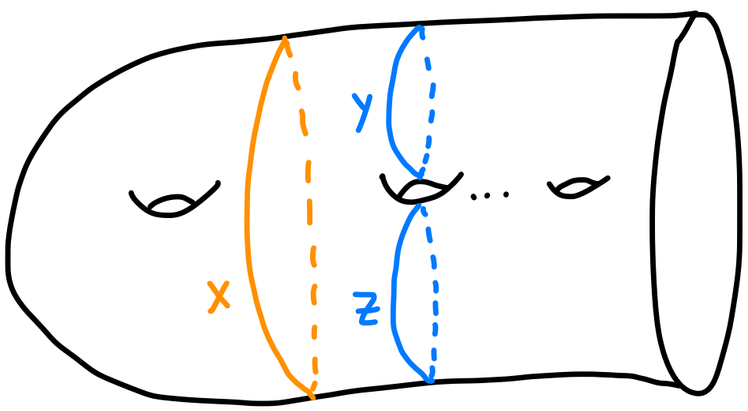
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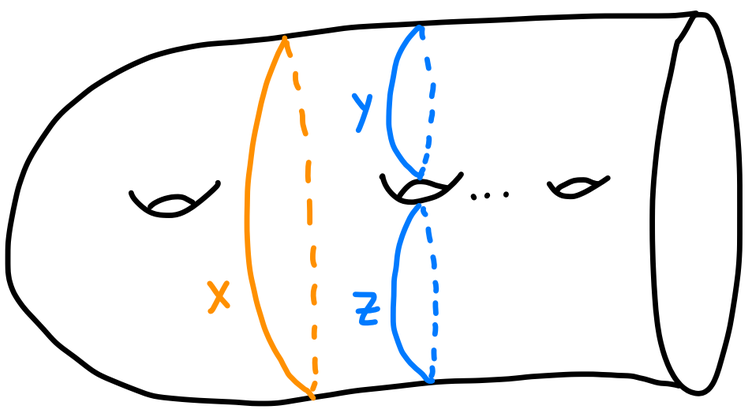
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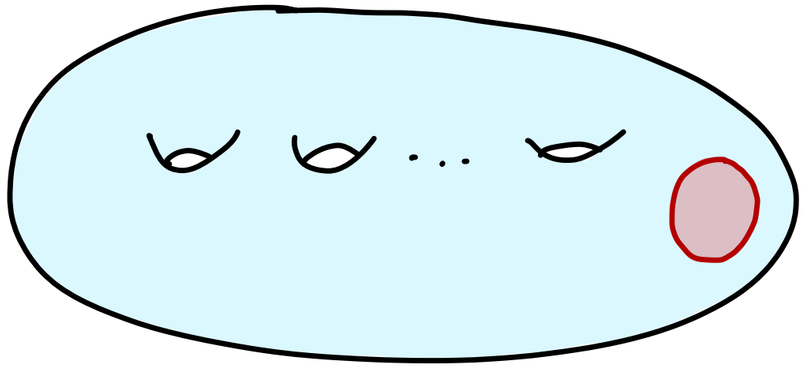
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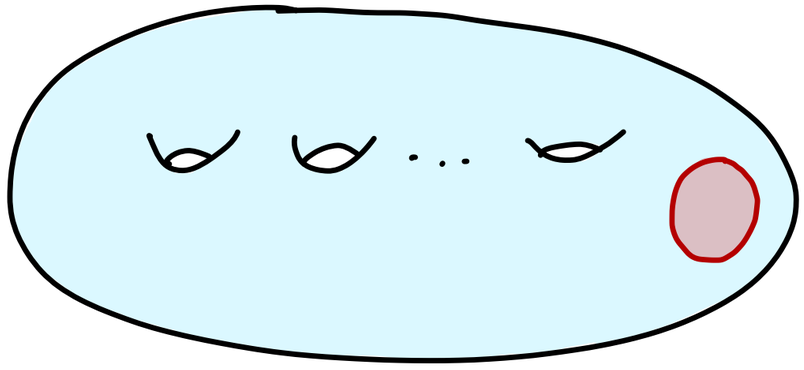
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Handlebody Groups



Fix a genus g handlebody V_g' with $\partial V_g' = \Sigma_g'$

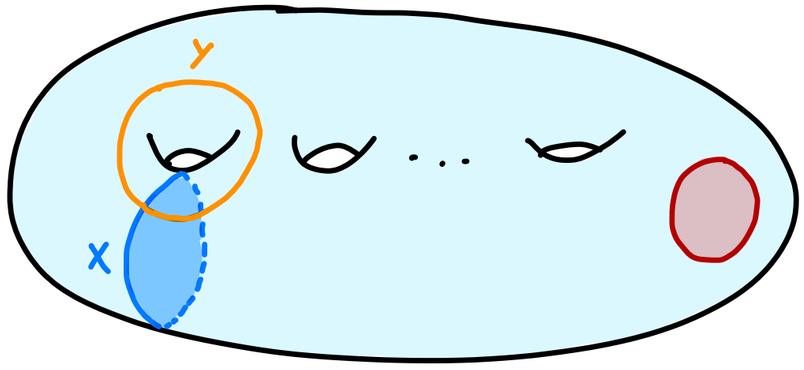
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Handlebody group $H_g' =$ subgroup of Mod_g' that extends to V_g'

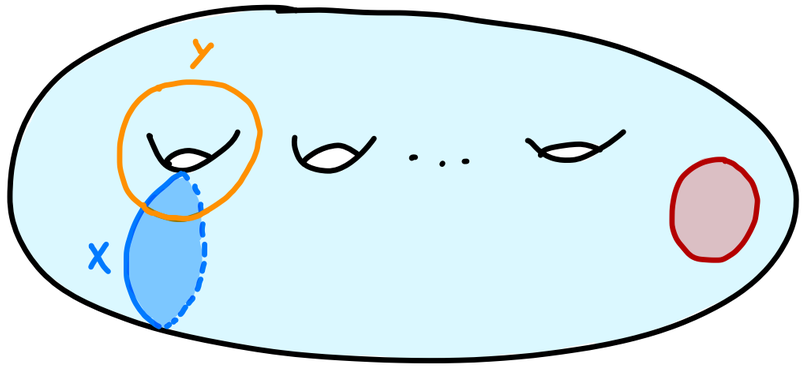
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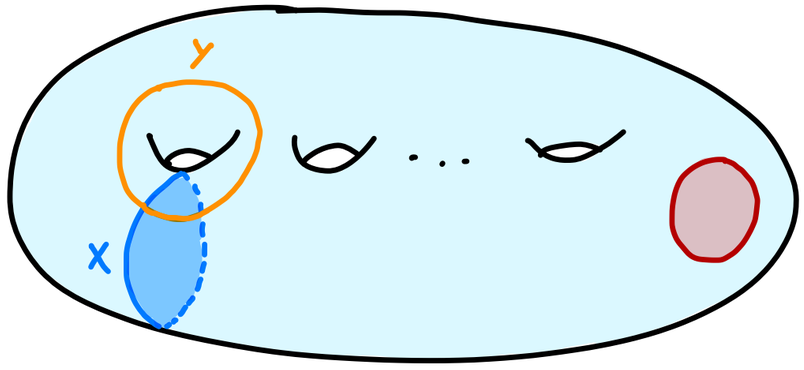


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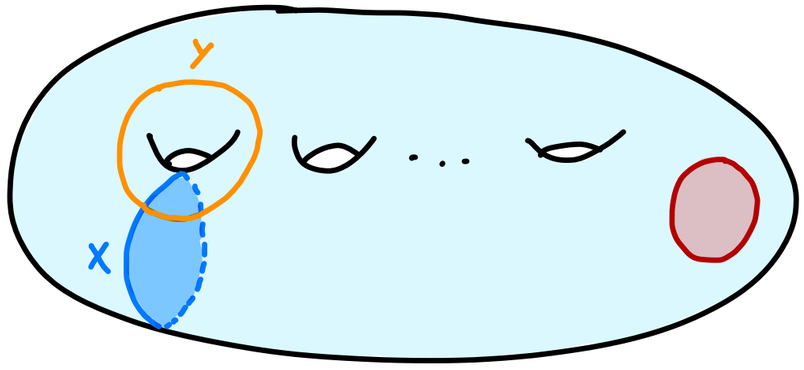
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Bridges two important groups in GGT:

$$\begin{array}{ccc} \mathcal{H}_g' & \hookrightarrow & Mod_g' \\ \downarrow\downarrow & & \\ Aut(F_g) & & \end{array}$$

via action on $\pi_1(V_g') \cong F_g$

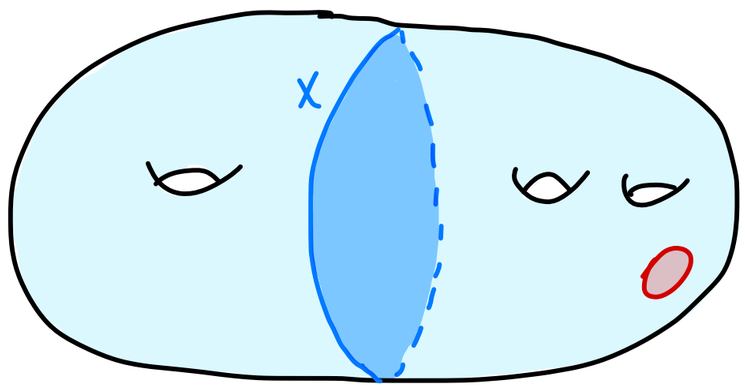
A handlebody Torelli group

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ex) Separating disk twists



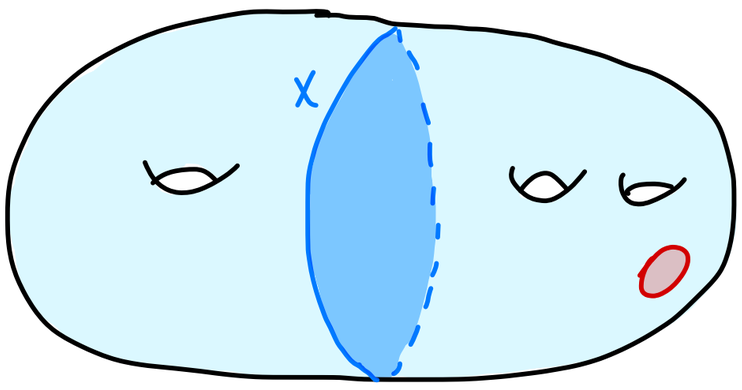
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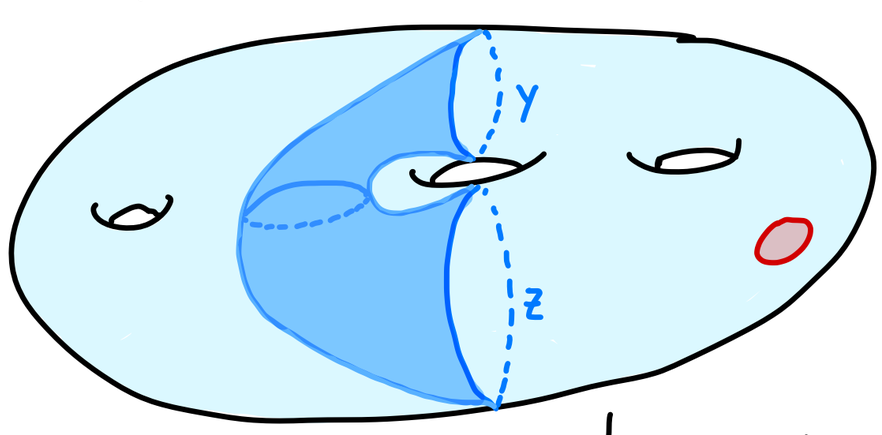
ex) Separating disk twists



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$$T_x \in \mathcal{HI}_g'$$

ex) annulus bounding pair maps



$y \cup z$ separates Σ_g' and bounds an annulus in V_g'

$$T_y T_z^{-1} \in \mathcal{HI}_g'$$

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Solution: $SL_g(\mathbb{Q})$-equivariant

$$H_2(\Lambda^3 H) = 2\mathbb{Q} \oplus 3\Gamma_{0,1} \oplus \Gamma_{0,2} \oplus \Gamma_{1,0,1} \oplus 2\Gamma_{0,0,0,1} \oplus \Gamma_{0,1,0,1} \oplus \Gamma_{0,0,0,0,0,1}$$

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highest weight vector: $[a_1 \wedge a_2 \wedge a_3] \wedge [a_1 \wedge a_2 \wedge a_4]$

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highest weight vector:

$$\sum_{1 \le i \ne j \le g-2} ([b_{g-1} \wedge b_g \wedge b_i] \wedge [a_i \wedge a_j \wedge b_j - \frac{1}{g-1} a_i \wedge \omega] + [b_{g-1} \wedge b_g \wedge a_i] \wedge [b_i \wedge a_j \wedge b_j - \frac{1}{g-1} b_i \wedge \omega])$$

$$+ \sum_{i=1}^{g-2} ([b_i \wedge b_{g-1} \wedge b_g] \wedge [(a_{g-1} \wedge b_{g-1} + a_g \wedge b_g) \wedge a_i - \frac{2}{g-1} a_i \wedge \omega] + [a_i \wedge b_{g-1} \wedge b_g] \wedge [(a_{g-1} \wedge b_{g-1} + a_g \wedge b_g) \wedge b_i - \frac{2}{g-1} b_i \wedge \omega]$$

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Thm (H) Explicitly calculated $\ker(\tau^*)$ and $\mathrm{im}(\tau^*)$

- $\ker(\tau^*)$ = direct sum of || $SL_9(\mathbb{Q})$ reps
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