Uniform twisted homological stability

(after Miller - Patzt - Petersen - Randal-Williams)

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Recall from last time:

Our aim is to prove that, for all fixed $g \gg r$:

$$\frac{1}{q^{2g+1}} \sum_{d} \underbrace{L(\tfrac{1}{2}, \chi_d)^r} \quad \sim \quad Q_r(2g+1) \qquad \text{as } g \to \infty$$

d: $\in \mathbb{F}_q[t]$ monic, sq-free, degree $= 2g+1$

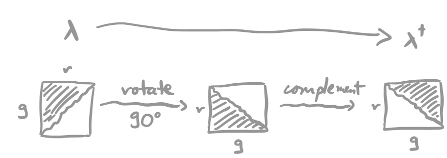
$L(\tfrac12,\chi_d) = \sum_{\substack{f \in \mathbb{F}_q[t] \\ \text{monic}}} \chi_d(f)\, |f|^{-1/2}$

χ_d: unique real primitive Dirichlet character mod d

$|f| = q^{\deg(f)}$

Q_r: explicit poly of degree $\frac{1}{2} r(r+1)$

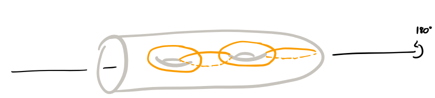
By the Grothendieck-Lefschetz trace formula,
+ Artin comparison theorem



$$\text{LHS} = \sum_{\substack{\lambda \\ \cap \\ g \times r \text{ box} \\ \| \\ (r^g)}} \overbrace{\sum_{k=0}^{\infty} (-1)^k \operatorname{trace}\left(\mathrm{Frob}_q \circlearrowright H_k(\mathrm{Hyp}_g^1 ; V_\lambda)\right)}^{tr_\lambda(q)} \cdot \dim(V_{\lambda^t})$$

Hyp_g^1: hyperelliptic mapping class group in genus g with one ∂-component

V_λ: irred. repr. of $Sp_{2g}(\mathbb{C})$ assoc. to $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq 0)$



$$\mathrm{Hyp}_g^1 \hookrightarrow \mathrm{Mod}_g^1 \twoheadrightarrow Sp_{2g}(\mathbb{Z}) \subset Sp_{2g}(\mathbb{C})$$
$$\mathrm{Hyp}_g^1 \cong B_{2g+1}$$

Denote this function of g — equivalently, of $2g+1$ — by $M_r(2g+1)$.

Define

$$Q_r(2g+1) := \sum_{\lambda \subseteq \infty \times r} \underbrace{\sum_{k=0}^{\infty} (-1)^k \operatorname{trace}\left(\mathrm{Frob}_q \circlearrowright H_k(\mathrm{Hyp}'_\infty ; V_\lambda)\right)}_{c_\lambda} \cdot \underbrace{\dim(V_{\lambda^+})}_{P_\lambda(g)}$$

i.e. $\lambda_1 \leq r$

c_λ independent of g

$P_\lambda(g)$ polynomial in $2g+1$ of degree $\frac{1}{2}r(r+1)$

This turns out to be the desired explicit polynomial — so in particular the notation is consistent.

Strategy: Prove that $M_r(2g+1) - Q_r(2g+1) \sim 0$ as $g \to \infty$.

1. Prove uniform twisted hom. stab^{r} for $\mathrm{Hyp}'_g \cong B_{2g+1}$ with V_λ coeffs. (today)

   uniform: stable range independent of λ

   $\longrightarrow$ terms in $M_r(2g+1) - Q_r(2g+1)$ with $k \leq$ stable range cancel.

2. Counting cells in a cell complex classifying space for B_{2g+1}. (easy)

   (+ Deligne bounds on Frobenius eigenvalues)

   $\longrightarrow$ bound on terms in $M_r(2g+1)$ with $k >$ stable range.

3. Calculation of stable homology $H_k(\mathrm{Hyp}'_\infty ; V_\lambda)$ (later talk)

   (+ Deligne bounds on Frobenius eigenvalues)

   $\longrightarrow$ bound on terms in $Q_r(2g+1)$ with $k >$ stable range.

$$\begin{tikzcd}
B_{2g+1} \arrow[r] \arrow[d, hook] & B_{2g+3} \arrow[d, hook] \\
\mathrm{Mod}'_g \arrow[r] \arrow[d] & \mathrm{Mod}'_{g+1} \arrow[d] \\
Sp_{2g}(\mathbb{C}) \arrow[r] & Sp_{2g+2}(\mathbb{C}) \\
V_\lambda(g) \arrow[r] \arrow[u, phantom, "\circlearrowleft"] & V_\lambda(g+1) \arrow[u, phantom, "\circlearrowleft"]
\end{tikzcd}$$

$$H_k(B_{2g+1}; V_\lambda(g)) \xrightarrow{(*)} H_k(B_{2g+3}; V_\lambda(g+1))$$

$$V_\lambda(g) = \begin{cases} \text{irred. rep. of highest weight } \lambda & \text{if } \ell(\lambda) \leq g \\ 0 & \text{if } \ell(\lambda) > g \end{cases}$$

Theorem (Miller-Patzt-Petersen-Randal-Williams '24)

(*) is an isomorphism for $k < \frac{n-13}{12}$

Rmk It was previously known by [Randal-Williams-Wahl '17] that (*) is an isomorphism for $k < \frac{n-2-|\lambda|}{2}$.

How to prove homological stability

(A) – with constant coefficients

Idea (Quillen '70s) (this axiomatisation: Hatcher-Wahl '10)

- $G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow \cdots$
- $G_n \circlearrowright X_n$ simplicial complex
- action is transitive on {p-simplices} $\forall p$
- $\mathrm{stab}(p\text{-simplex}) \sim G_{n-p-1}$
- X_n is $\left(\frac{n-2}{2}\right)$-connected (+ small technical cond$^{\text{n}}$s)

$\rightsquigarrow$ hom. stab$^{\text{y}}$ for $\{G_n\}$ with constant $\mathbb{Z}$ coeffs.

Example Symmetric groups $\Sigma_n \circlearrowright \Delta^{n-1}$

Idea of proof

Spectral sequence

$$E^1_{pq} = H_q(\mathrm{stab}(\sigma_p)) \quad \Rightarrow \quad 0 \quad \text{for} \quad p+q \leq \frac{n-1}{2}$$

d^1 - differentials are built out of stabilisation maps :

$$\begin{array}{ccc} H_q(\mathrm{stab}(\sigma_0)) & \longrightarrow & H_q(\mathrm{stab}(\sigma_{-1})) \\ \| & & \| \\ H_q(G_n) & & H_q(G_{n+1}) \end{array}$$

This allows a proof by induction on q ....

(B) — with twisted coefficients

[Randal-Williams–Wahl '17]

Inputs:

(1) G braided monoidal groupoid with objects $= \mathbb{N}$

↳ groups G_n

homom. $G_n \times G_m \xrightarrow{\oplus} G_{n+m}$ (assoc., unital)

isom. $G_{n+m} \xrightarrow{\cong} G_{m+n}$ (compatible with $\oplus$)

(E.g. braid groups
juxtaposition
conjugation by )

Assume that $G_0 = \{id\}$
and $G_n \longrightarrow G_{n+m}$ is injective.

(2) Coeff. system: $V(n)$ G_n-representation over R

$V(m) \to V(n+m)$ $(G_n \times G_m)$-equivariant

(E.g. $V(n) = R^{\oplus n}$ permutation representation)

Assume that V has finite polynomial degree d.

↳ $\Sigma V(n) = V(n+1)$ shift of V

$V \to \Sigma V$

$\deg(V) \leq d$ if $\ker(V \to \Sigma V) = 0$

$\operatorname{coker}(V \to \Sigma V)$ has degree $\leq d-1$

$\deg(V) = -1$ if $V = 0$

(E.g. $d = 0 \longleftrightarrow$ constant

$V(n) = R^{\oplus n}$ has $d = 1$)

Theorem (RWW'17)

There is a sequence of simplicial complexes (semi-simplicial sets) $X_n(G)$ naturally associated to G.

If $X_n(G)$ is $\left(\frac{n-2}{2}\right)$-connected $\longrightarrow$ hom. stab^{y} for $\{G_n\}$ with $V(n)$ coeffs in the range $k \leq \frac{n}{2} - d - 1$

"destabilisation complex" for G

degree of V

Example

$G_n = B_n$ braid groups

$V(n) = V_\lambda(n)$

$$B_n \longrightarrow Sp_{n-1}(\mathbb{Z}) \circlearrowright V_\lambda(n)$$

"odd symplectic groups" when n is even

This has degree $d = |\lambda|$.

Proof: Exercise 6.12 in [Fulton-Harris]

$$\Rightarrow \Sigma V_\lambda(n) = V_\lambda(n+1) \cong \bigoplus_{\mu} V_\mu(n)$$

$\mu \leftarrow \lambda$ by removing any # of boxes from its Young diagram, but no two in the same column

$$\Rightarrow \operatorname{coker}\left(V_\lambda \to \Sigma V_\lambda\right) \cong \bigoplus_{\substack{\mu \\ \mu \neq \lambda}} V_\mu$$

has degree $\max_{\mu} |\mu| = |\lambda| - 1$ by induction. //

Ⓒ — with twisted coefficients, II

[Miller-Patzt-Petersen-Randal-Williams '24]

Inputs:

① $\left.\begin{array}{l} G \\ Q \end{array}\right\}$ braided monoidal groupoids with objects $= \mathbb{N}$

$G \longrightarrow Q$ (id on objects)

Assume that $G_0 = \{id\}$

$G_n \twoheadrightarrow Q_n$ is surjective

$\left.\begin{array}{l} G_n \times G_m \longrightarrow G_{n+m} \\ Q_n \times Q_m \longrightarrow Q_{n+m} \end{array}\right\}$ injective.

② A class $\mathcal{D}$ of coeff. systems for Q that is closed under $\Sigma(-)$.

Do not assume that they are polynomial!

Theorem (MPPRW '24)

Suppose that • $X_n(G)$ and $X_n(Q)$ are both (αn)-connected.

• $\{Q_n\}$ is hom. stable in the range $k \leq \beta n$ w.r.t each coeff system in $\mathcal{D}$.

Then $\{G_n\}$ is hom. stable in the range $k \leq \min(\alpha, \beta) n$ w.r.t each coeff system in $\mathcal{D}$.

Rmk May replace $X_n(G)$ and $X_n(Q)$ —— destabilisation complexes with $S_n(G)$ and $S_n(Q)$ —— splitting complexes

[Galatius-Kupers-Randal-Williams '18]

[Randal-Williams '22]: high-conn for $X_n(G)$ ⟵- - -⟶ high-conn. for $S_n(G)$

Example ① $G_n = B_n$ braid groups

$B_n \longrightarrow Sp_{n-1}(\mathbb{Z})$

not surjective!

Def $Q_n :=$ image of $B_n \longrightarrow Sp_{n-1}(\mathbb{Z})$

(**Thm** (A'Campo '79)
$$\begin{array}{ccc} Sp_{n-1}(\mathbb{Z}) & \twoheadrightarrow & Sp_{n-1}(\mathbb{Z}/2) \\ \cup & \urcorner & \cup \\ Q_n & \twoheadrightarrow & \Sigma_n \end{array}$$
)

② $\mathcal{V}$ = all finite direct sums of V_λ for all λ

closed under $\Sigma(-)$ because $\Sigma V_\lambda \cong \bigoplus_\mu V_\mu$

To obtain uniform hom. stab^y for B_n with coeffs in V_λ, we need:

- High.-conn. of $X_n(B)$ — destabilisation complex asso. to B_n.

  ↳ **Thm** [Damiolini '13] $X_n(B)$ is contractible!

- High.-conn. of $X_n(Q)$ — destabilisation complex asso. to the level-2 symmetric congruence subgroups of the even/odd symplectic groups.

  ↳ **Thm** [MPPRW '24] $X_n(Q)$ is $\left(\frac{n-12}{4}\right)$-connected.

- Uniform hom. stab^y for $\{Q_n\}$ with coeffs in each V_λ.

  ↳ Follows from calculations of Borel....

[Borel]

$\Gamma \subset Sp_{2g}$ arithmetic subgroup

V irreducible real representation of Sp_{2g}

$$H^*(\Gamma; V) \cong \begin{cases} \mathbb{R}[x_2, x_6, x_{10}, \ldots] & V \equiv \mathbb{R} \\ 0 & V \not\equiv \mathbb{R} \end{cases}$$

$* < g$

independent of g

Apply this to $\Gamma = Q_{2g+1}$

$\rightsquigarrow$ uniform hom. stab^y for $\{Q_{2g+1}\}$ with coeffs in each V_λ

$\cap$
$Sp_{2g}(\mathbb{Z})$

$\rightsquigarrow$ uniform hom. stab^y for $\{Q_n\}$ with coeffs in each V_λ

extra "trick"

$\cap$
$Sp_{n-1}(\mathbb{Z})$

$$\begin{tikzcd} H_k(Q_{2n-1}; V_\lambda(2n-1)) \arrow[d] \arrow[dd, bend left, "\cong \text{ for } k < n-1"] \\ H_k(Q_{2n}; V_\lambda(2n)) \arrow[d] \\ H_k(Q_{2n+1}; V_\lambda(2n+1)) \end{tikzcd}$$

Calc: These have the same dim. when $k < n-1$.

Still to explain:

- Proof of high-conn. of $X_n(Q)$ $\big($& contractibility of $X_n(B)\big)$.
- Proof of "pulling back hom. stab$^{\text{y}}$" from Q to G.
- Calculation of stable homology $H_*(B_\infty; V_\lambda)$.