Uniform twisted homological stability II

(after Miller - Patzt - Petersen - Randal-Williams)

IMAR

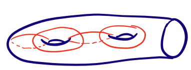
31 July 2024

The proof of high-connectivity

Aim: Uniform homological stability for B_{2g+1} with coefficients in V_λ.

Precisely:

integral reduced Burau repr.



$$\begin{tikzcd}
B_{2g+1} \arrow[r] \arrow[d] \arrow[rr, bend left, "\text{integral reduced Burau repr.}"] & \mathrm{Mod}(\Sigma_{g,1}) \arrow[r] & Sp_{2g}(\mathbb{Z}) \arrow[d] \curvearrowright V_\lambda \\
B_{2g+3} \arrow[rr] & & Sp_{2g+2}(\mathbb{Z}) \curvearrowright V_\lambda
\end{tikzcd}$$

$\}$ any irreducible repr. of $Sp_{2g}(\mathbb{Z})$

$$H_i(B_{2g+1}; V_\lambda) \longrightarrow H_i(B_{2g+3}; V_\lambda) \quad \text{is an isomorphism}$$

for all $i \leq f(g)$

- diverging function of g
- independent of λ

"Filling in the gaps": odd symplectic groups

For $n \geq 0$ let $M_n = \mathbb{Z}^n = \mathbb{Z}\{e_1, \dots, e_n\}$ equipped with

$$\langle -,- \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z} \qquad \langle e_i, e_j \rangle = \begin{cases} 1 & i<j \\ -1 & i>j \\ 0 & i=j \end{cases}$$

$$\phi : \mathbb{Z}^n \longrightarrow \mathbb{Z} \qquad \text{given by addition.}$$

Obs $\langle -,- \rangle$ is bilinear and alternating.

If $n = 2g$ it is non-degenerate and hence symplectic.

If $n = 2g+1$ it is degenerate: $\left\langle \sum_{i=1}^{2g+1} (-1)^i e_i , - \right\rangle \equiv 0$

but its restriction to $\ker(\phi)$ is non-degenerate, thus symplectic.

Def $T_n = \mathrm{Aut}(M_n)$

group automorphisms preserving $\langle -,- \rangle$ and ϕ

Lemma For $n = 2g+1$, $T_{2g+1} \cong Sp_{2g}(\mathbb{Z})$

Proof: M_n is spanned by $e_i - e_{i+1}$ $\quad i = 1, \dots, 2g \quad \Big\}\ \ker(\phi)$

and $\sum_{i=1}^{2g+1} (-1)^i e_i = v$

In this basis, T_{2g+1} acts by $\left(\begin{array}{c|c} Sp_{2g}(\mathbb{Z}) & 0 \\ \hline 0 & 1 \end{array}\right)$

Def [Gelfand-Zelevinsky] For all $n \in \mathbb{N}$, $Sp_n(\mathbb{Z}) := T_{n+1}$

(Equivalent def^n of $Sp_{2g-1}(\mathbb{Z})$: the subgroup of $Sp_{2g}(\mathbb{Z})$ stabilising a unimodular element of $\mathbb{Z}^{2g}$.)

The image of the integral Burau representation

Replacing $\mathbb{Z}$ with $\mathbb{Z}/2$ we similarly define $Sp_n(\mathbb{Z}/2)$ and

$$\text{mod } 2 : Sp_n(\mathbb{Z}) \twoheadrightarrow Sp_n(\mathbb{Z}/2).$$

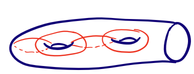
Obs There is an embedding $\mathfrak{S}_{n+1} \hookrightarrow Sp_n(\mathbb{Z}/2)$

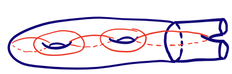
permutation matrices in $GL_{n+1}(\mathbb{Z}/2)$

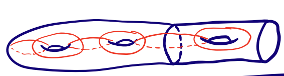
Def $Q_{n+1} := (\text{mod } 2)^{-1}(\mathfrak{S}_{n+1})$

The integral Burau representation extends as follows:

action on $H_1(-;\mathbb{Z})$







$$\begin{tikzcd}
B_{2g+1} \arrow[r] \arrow[d, hook] & \mathrm{Mod}(\Sigma_{g,1}) \arrow[r] \arrow[d, hook] & Sp_{2g}(\mathbb{Z}) \arrow[r, phantom, "="] \arrow[d, hook] & T_{2g+1} \arrow[d, hook] \\
B_{2g+2} \arrow[r] \arrow[d, hook] & \mathrm{Mod}(\Sigma_{g,2}) \arrow[r] \arrow[d, hook] & Sp_{2g+1}(\mathbb{Z}) \arrow[r, phantom, "="] \arrow[d, hook] & T_{2g+2} \arrow[d, hook] \\
B_{2g+3} \arrow[r] & \mathrm{Mod}(\Sigma_{g+1,1}) \arrow[r] & Sp_{2g+2}(\mathbb{Z}) \arrow[r, phantom, "="] & T_{2g+3}
\end{tikzcd}$$

(To justify this, we need to check that the action of $\mathrm{Mod}(\Sigma_{g,2}) \subset \mathrm{Mod}(\Sigma_{g+1,1})$ on $H_1(\Sigma_{g+1,1};\mathbb{Z}) \cong \mathbb{Z}^{2g+2}$ stabilises a unimodular element. It stabilises the generator .)

Thm [A'Campo '79 + Bloomquist-Patzt-Scherich '22]

The image of Burau : $B_n \to T_n$ is Q_n.

Pulling back uniform homological stability

(from last time)

Suppose we have:

groups G_n $(n \in \mathbb{N})$ (with G_0 trivial)

injections $G_m \times G_n \xrightarrow{\oplus} G_{m+n}$ (assoc., unital)

automorphisms $G_{m+n} \xrightarrow{b_{m,n}} G_{m+n}$

$$b_{m,n+p} = (1_n \oplus b_{m,p}) \circ (b_{m,n} \oplus 1_p)$$
$$b_{m+n,p} = (b_{m,p} \oplus 1_n) \circ (1_m \oplus b_{n,p})$$

This determines a sequence of "destabilisation complexes" $W_n(G)$.

Now suppose that we have:

- $\{G_n\}$ and $\{Q_n\}$ as above
- surjections $p_n : G_n \twoheadrightarrow Q_n$ compatible with all structure
- (!!) $W_n(G)$ and $W_n(Q)$ are highly-connected of slope $\alpha > 0$ & offset $\beta \geq 0$

  $(\pi_i(-) = 0$ for $i < \alpha n - \beta)$

Thm [MPPRW '24]

If Q_n is UHS of slope $\gamma > 0$ (uniformly homologically stable)
for a class of coeff systems (closed under taking shifts)

$(H_i(Q_n; \mathcal{V}) \longrightarrow H_i(Q_{n+1}; \mathcal{V})$ isom. for $i < \gamma n -$ constant$)$

then G_n is UHS of slope $\min(\frac{\alpha}{\beta+1}, \gamma)$ for the pullbacks of these coeff systems.

Reduction to high-connectivity

Now set $G_n = B_n$

$Q_n = Q_n \subset T_n = Sp_{n-1}(\mathbb{Z})$

$p_n = \mathrm{Bur} : B_n \twoheadrightarrow Q_n$

These admit all of the above structure, so we have destabilisation complexes

$W_n(B)$

$W_n(Q)$

Calculations of Borel

$\rightsquigarrow$

arithmetic subgroup

UHS of slope 1 for $Q_{2g+1} \subset Sp_{2g}(\mathbb{Z})$ with coeffs in V_λ

$\rightsquigarrow$ Lemma expressing $H_*(Q_{2g}; V_\lambda)$ in terms of $H_*(Q_{2g-1}; V_\lambda)$ in a range.

UHS of slope $\frac{1}{2}$ for $Q_n \subset Sp_{n-1}(\mathbb{Z})$ with coeffs in V_λ

and hence $\bigoplus_{i=1}^{k} V_{\lambda_i}$

closed under taking shifts due to branching rule

$\Sigma V_\lambda \cong \bigoplus_\mu V_\mu$

[MPPRW]

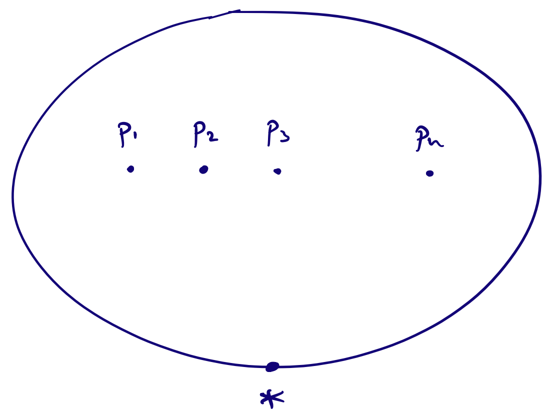
\+ high-connectivity of $W_n(B)$ of slope ∞ / $W_n(Q)$ $\frac{1}{4}$ and offset $\frac{11}{4}$

$\downarrow$

UHS of slope $\frac{1}{15}$ for B_n with coeffs in V_λ pulled back along

$\mathrm{Bur} : B_n \twoheadrightarrow Q_n \subset T_n = Sp_{n-1}(\mathbb{Z})$

$W_n(B)$ is contractible [Damiolini '13, Hatcher-Vogtmann '17]



$W_n(B)$ has vertices $= \{$ isotopy classes of arcs from $*$ to $p_i \}$

and $\{\gamma_0, \dots, \gamma_p\}$ span a p-simplex $\iff$

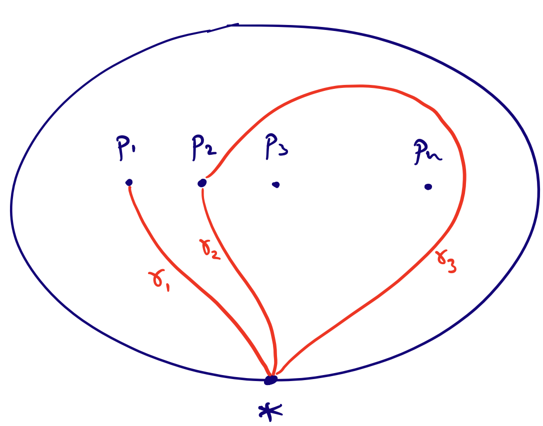
$\exists$ representatives that are disjoint except at $*$.

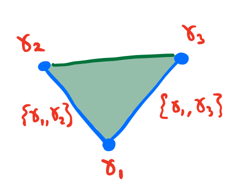
$\widetilde{W_n(B)}$ has the same vertices

and $\{\gamma_0, \dots, \gamma_p\}$ span a p-simplex $\iff$

$\exists$ representatives that are disjoint except at $*$ or $p_1, \dots, p_n$.

Examples:





Remarks :

- ∞ many vertices
- $\dim W_n(B) = n-1$
- $\dim \widetilde{W_n(B)} = \infty$

Proposition : Let δ be any vertex of $\widetilde{W_n(B)}$. Then $\widetilde{W_n(B)}$ deformation retracts onto $\{\delta\}$.

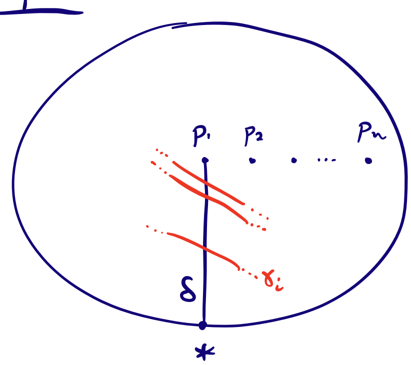
Proof :

Point $p \in |\widetilde{W_n(B)}| \longleftrightarrow$ collection of arcs $\gamma_0, \dots, \gamma_p$ (that form a simplex)

& weights $t_0, \dots, t_p \in [0,1]$ summing to 1

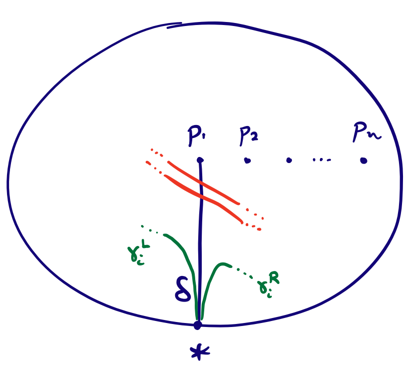
(barycentric coordinates)

Step 1



$\gamma_0, \dots, \gamma_p$ in general position w.r.t. δ & pairwise disjoint

Define γ_i^L and γ_i^R as in the figure.



Exactly one of γ_i^L and γ_i^R has an endpoint at $p_1, \dots, p_n$ and is thus a vertex of $\widetilde{W_n(B)}$.

Call this γ_i^N.

During a time interval proportional to t_i, gradually decrease the weighting (initially t_i) of γ_i and increase the weighting (by t_i) of γ_i^N.

Repeat finitely many times $\longrightarrow$ $\gamma_0', \dots, \gamma_q'$ $\quad (q \leq p)$
$t_0', \dots, t_q'$

with $\gamma_i' \cap \delta = \{*\}$, $i = 0, \dots, q$.

<u>Step 2</u>. Consider this as $\left.\begin{array}{l} \gamma_0', \dots, \gamma_q', \delta \\ t_0', \dots, t_q', 0 \end{array}\right\}$ This forms a valid $(q+1)$-simplex!

Now gradually decrease the t_i' to 0 (at speed t_i')
and increase the weight of δ to 1 (at speed 1). $\square$

(<u>Remark</u> The same proof also works for the generalisation of $\widetilde{W_n}(B)$ where there are $m \geq 1$ basepoints $*_1, \dots, *_m$ on the boundary. Call this $\widetilde{W_{n,m}}(B)$.

<u>Proposition</u> For any $n, m \geq 1$, $\widetilde{W_{n,m}}(B)$ is contractible.)

<u>Theorem</u>: The inclusion $W_n(B) \hookrightarrow \widetilde{W_n}(B)$ is a (weak) homotopy equivalence, and hence $W_n(B)$ is contractible.

The proof of this uses the "<u>bad simplex argument</u>" ...

Bad simplex argument

$Y \subset X$ simplicial complexes

B = collection of simplices of $X - Y$ ("bad simplices")

so that (1) if no face of $\sigma \subset X$ is bad, then $\sigma \subset Y$

(2) if $\sigma_1, \sigma_2 \subset X$ are bad

and $\underbrace{\sigma_1 * \sigma_2}$ is also a simplex of X

"join" = the simplex spanned by the union of the vertices of σ_1 and σ_2

then $\sigma_1 * \sigma_2$ is also bad.

For $\sigma \in B$ set G_σ = subcomplex of X of simplices τ such that

- $\tau * \sigma$ is a simplex of X (τ is in the link of σ)
- all bad faces of $\tau * \sigma$ lie in σ

("good link of σ")

Proposition Suppose $\exists d \geq 0$: $\forall \sigma \in B$, G_σ is $(d - \dim(\sigma) - 1)$-connected.
Then $Y \hookrightarrow X$ induces isom's on π_i $(i \leq d-1)$ and a surjection on π_d.

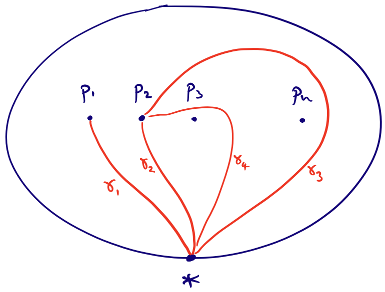
In our case $X = \widetilde{W_n(B)}$

$Y = W_n(B)$

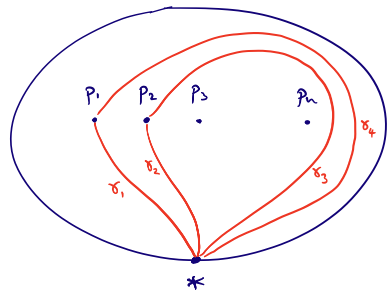
Define:

$$\mathcal{B} := \left\{ \sigma : \forall i = 1, \dots, n, \text{ if an arc of } \sigma \text{ ends at } p_i \text{ then } \geq 2 \text{ arcs of } \sigma \text{ end at } p_i \right\}$$

Eg



Not bad.



Bad.

Proof of Theorem

Induction on n. ($n=1$ is trivial)

$\forall \sigma \in B$,

cut along $\sigma \longrightarrow$ disjoint union of discs, each with $n_j < n$ punctures and $m_j \geq 1$ boundary basepoints

$G_\sigma \cong$ join of $W_{n_j, m_j}(B)$ over all j

Key observation: one of the m_j is equal to 1 (say $m_1 = 1$)

(i.e. for at least one of the complementary regions of σ, the basepoint $*$ does not get split into several copies when cutting along σ)

Hence $G_\sigma \cong \underbrace{W_{n_1}(B)}_{\text{contractible by induction}} * (\text{the rest})$

So G_σ is contractible, hence the Proposition above applies with $d=\infty$. $\square$

$W_n(Q)$ is $\left(\frac{n-12}{4}\right)$-connected [MPPRW '24]

Recall that $Q_n \subset T_n \subset GL_n(\mathbb{Z})$
$$T_n \cong Sp_{n-1}(\mathbb{Z})$$

Def A Q-basis of $\mathbb{Z}^n$ is the image of the standard basis $(e_1, \dots, e_n)$ under some matrix in Q_n.

A partial Q-basis is a subset of a Q-basis.

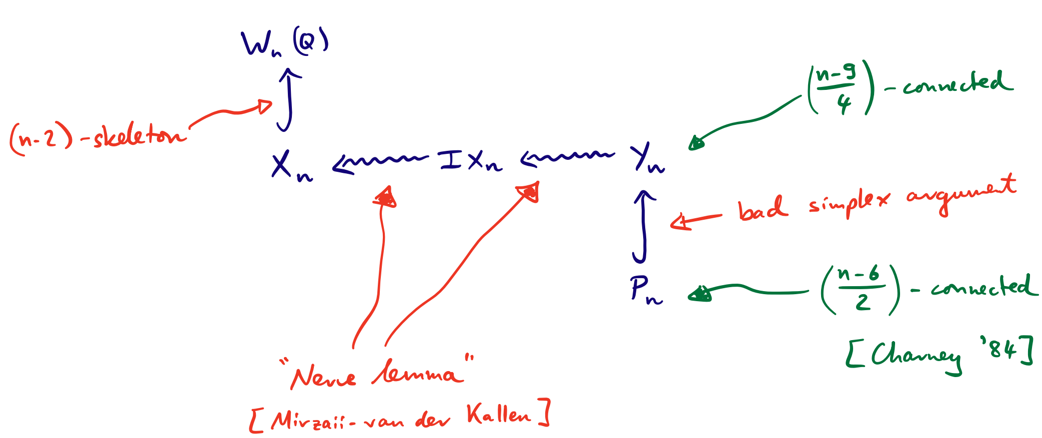
Fact $W_n(Q)$ is the simplicial complex whose simplices are partial Q-bases of $\mathbb{Z}^n$.

(This is what comes directly out of the "homological stability machine".)

Obs $\dim(W_n(Q)) = n-1$

Theorem [MPPRW '24] $W_n(Q)$ is $\left(\frac{n-12}{4}\right)$-connected.

Outline of proof (slightly simplified)



$X \rightsquigarrow Y$ means $\{\text{simplices of } X\} \xrightarrow{f} \{\text{subcomplexes of } Y\}$

$$\sigma \subseteq \tau \implies f(\sigma) \supseteq f(\tau)$$

We'll leave the nerve lemma & bad simplex arguments a black boxes, and instead just describe the general context of Charney's result.

P_n = "complex of partial bases of $\mathbb{Z}^{n-1}$ that reduce mod 2 to a subset of a fixed partial basis of $(\mathbb{Z}/2)^{n-1}$"

Complexes of lifts of partial bases

R ring (with unit)

$A \subseteq R$ ideal (2-sided)

Def $\mathcal{U}_A(R^{k,\ell})$ has p-simplices $\{x_0,\dots,x_p\} \subseteq R^{k+\ell}$ such that

- $\{x_0,\dots,x_p\}$ is unimodular (basis of direct summand of $R^{k+\ell}$)

  [Note: In general this is weaker than assuming it is a partial basis. But when R is a P.I.D. (eg. $R=\mathbb{Z}$) it is equivalent.]

- each x_i is congruent mod A to one of $e_1,\dots,e_k$.

$P_n = \mathcal{U}_{2\mathbb{Z}}(\mathbb{Z}^{k,\ell})$ with $k = \lfloor \frac{n}{2} \rfloor$, $k+\ell = n-1$ (... when n is odd. When n is even it is slightly different.)

Obs $\dim\left(\mathcal{U}_A(R^{k,\ell})\right) = k-1$

Assume that R satisfies Bass' stable range condition of sdim $= d$.

↳ E.g. [Bass] if R is commutative Noetherian of Krull dimension d, then sdim $= d$

↳ E.g. if $R =$ field sdim $= 0$

if $R = \mathbb{Z}$ sdim $= 1$ (P.I.D. not a field)

<u>Theorem</u> [Charney '84]

$\mathcal{U}_A(R^{k,\ell})$ is $(k-d-2)$- connected.

In our case $k = \lfloor \frac{n}{2} \rfloor$

$d = 1$ (since $R = \mathbb{Z}$)

connectivity $\geq \lfloor \frac{n}{2} \rfloor - 3 = \lfloor \frac{n-6}{2} \rfloor$