Uniform twisted homological stability III

(after Miller - Patzt - Petersen - Randal-Williams)

IMAR

The proof of high-connectivity (continued!) 14 August 2024

Reminder:

Aim

w.r.t. λ

Uniform homological stability for B_{2g+1} with coefficients given by the representation $B_{2g+1} \longrightarrow Sp_{2g}(\mathbb{Z}) \longrightarrow \mathrm{Aut}_{\mathbb{Q}}(V_\lambda)$

integral reduced Burau

irrep.

$\Uparrow$ [MPPRW '24] as long as $W_n(\mathbb{Q})$ and $W_n(B)$ are highly-connected

image of integral reduced Burau

Uniform homological stability for Q_{2g+1} with coefficients given by the representation $Q_{2g+1} \subset Sp_{2g}(\mathbb{Z}) \longrightarrow \mathrm{Aut}_{\mathbb{Q}}(V_\lambda)$

Theorem (Borel)

Last time: $W_n(B)$ is contractible (i.e. d-connected for all d)

Today: $W_n(Q)$ is $\left(\frac{n-12}{4}\right)$-connected

$W_n(Q)$ is $\left(\frac{n-12}{4}\right)$-connected [MPPRW '24]

Recall that $Q_n \subset T_n \subset GL_n(\mathbb{Z})$
$$T_n \cong Sp_{n-1}(\mathbb{Z})$$

Def A *Q-basis* of $\mathbb{Z}^n$ is the image of the standard basis $(e_1, \dots, e_n)$ under some matrix in Q_n.

A *partial Q-basis* is a subset of a Q-basis.

Fact $W_n(Q)$ is the simplicial complex whose simplices are partial Q-bases of $\mathbb{Z}^n$.

(This is what comes directly out of the "homological stability machine".)

Obs $\dim(W_n(Q)) = n-1$

Theorem [MPPRW '24] $W_n(Q)$ is $\left(\frac{n-12}{4}\right)$-connected.

Outline of proof (slightly simplified)

$$\begin{tikzcd}
W_n(Q) & & \\
X_n \arrow[u, hook, "(n-2)\text{-skeleton}"'] & IX_n \arrow[l, rightsquigarrow] & Y_n \arrow[l, rightsquigarrow] \\
& & P_n \arrow[u, hook, "\text{bad simplex argument}"']
\end{tikzcd}$$

Y_n: $\left(\frac{n-9}{4}\right)$-connected

P_n: $\left(\frac{n-6}{2}\right)$-connected [Charney '84]

"Nerve lemma" [Mirzaii-van der Kallen]

$X \rightsquigarrow Y$ means $\{\text{simplices of } X\} \xrightarrow{f} \{\text{subcomplexes of } Y\}$
$$\sigma \subseteq \tau \implies f(\sigma) \supseteq f(\tau)$$

Plan

1. Define the various simplicial complexes
2. Results of Charney
3. A bad simplex argument
4. Nerve lemma arguments

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1 Definitions

Recall that we have equipped $\mathbb{Z}^n$ with

- $\langle -,- \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$ $\qquad \langle e_i, e_j \rangle = \begin{cases} +1 & i<j \\ -1 & i>j \\ 0 & i=j \end{cases}$
- $\phi : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ $\qquad \phi(e_i) = 1$
- $v_n \in \mathbb{Z}^n$ $\qquad v_n := \sum_{i=1}^{n} (-1)^{i+1} e_i$

We also define

- $\rho : \mathbb{Z}^n \longrightarrow P(\{1,\dots,n\})$ $\qquad \rho\left(\sum_{i=1}^n a_i e_i\right) = \{1 \le i \le n \;:\; a_i \text{ is odd}\}$

Def: Z_n = simplicial complex whose p-simplices are $u_0, \dots, u_p \in \mathbb{Z}^n$ such that
- $\{u_0, \dots, u_p, v_n\}$ is a partial $\mathbb{Z}$-basis of $\mathbb{Z}^n$
- $\rho(u_0), \dots, \rho(u_p)$ are pairwise disjoint

Def: Y_n = subcomplex where we also have $\phi(u_i) = 0$, $|\rho(u_i)| = 2$

IX_n = subcomplex where we also have $\phi(u_i) = 0$, $|\rho(u_i)| = 2$, $\langle u_i, u_j \rangle = 0$

X_n = subcomplex where we also have $\phi(u_i) = 1$, $|\rho(u_i)| = 1$, $\langle u_i, u_j \rangle = 1$ for $i<j$, for some ordering of $u_0, \dots, u_p$ (necessarily unique)

Obs: $Q_n \subset GL_n(\mathbb{Z})$ acts simplicially on Z_n and each of these subcomplexes.

Obs: $\dim(X_n) = n-2$ (e.g. $e_1, \dots, e_{n-1}$ is a top-dim. simplex)

Lemma: $\{\text{simplices of } X_n\} = \{\text{partial } \mathbb{Q}\text{-bases of size} \le n-1\}$ (Proof uses a result of Vaserstein.)

Cor: $X_n = (n-2)$-skeleton of $W_n(\mathbb{Q})$.

② Complexes of lifts of partial bases (results of Charney)

R ring (with unit)

$A \subseteq R$ ideal (2-sided)

Def $\mathcal{U}_A(R^{k,\ell})$ has p-simplices $\{x_0, \dots, x_p\} \subseteq R^{k+\ell}$ such that

- $\{x_0, \dots, x_p\}$ is unimodular (basis of direct summand of $R^{k+\ell}$)

  [Note: In general this is weaker than assuming it is a partial basis. But when R is a P.I.D. (eg. $R = \mathbb{Z}$) it is equivalent.]

- each x_i is congruent mod A to one of $e_1, \dots, e_k$.

Obs $\dim(\mathcal{U}_A(R^{k,\ell})) = k-1$

Assume that R satisfies Bass' stable range condition of sdim $= d$.

$\hookrightarrow$ E.g. [Bass] if R is commutative Noetherian of Krull dimension d, then sdim $= d$

$\hookrightarrow$ E.g. if $R =$ field sdim $= 0$
if $R = \mathbb{Z}$ sdim $= 1$ (P.I.D. not a field)

Theorem [Charney '84]

$\mathcal{U}_A(R^{k,\ell})$ is $(k-d-2)$-connected.

Def: P_n = simplicial complex whose p-simplices are $u_0, \dots, u_p \in \ker(\phi) \subset \mathbb{Z}^n$ such that
- $\{u_0, \dots, u_p\}$ is a partial $\mathbb{Z}$-basis of $\ker(\phi)$
- $\mathfrak{s}(u_0), \dots, \mathfrak{s}(u_p)$ are pairwise disjoint
- each $\mathfrak{s}(u_i) = \{2j-1, 2j\}$ for some $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ $(*)$

Obs: (1) When n is odd, P_n = the subcomplex of Y_n of simplices satisfying the additional condition $(*)$. [Note: $\mathbb{Z}^n \cong \ker(\phi) \oplus \langle v_n \rangle$ for n odd.]

(This is also true when n is even, but with a slightly different definition of P_n in that case.)

(2) $P_n \cong$ complex of partial $\mathbb{Z}$-bases of $\underset{\substack{\cong \\ \ker(\phi)}}{\mathbb{Z}^{n-1}}$ that reduce mod 2 to a subset of a $\underbrace{\text{fixed partial } \mathbb{Z}/2\text{-basis}}_{\text{namely, the partial } \mathbb{Z}/2\text{-basis } \{e_{2j-1}+e_{2j} \;:\; 1 \leq j \leq \lfloor \frac{n}{2} \rfloor\}}$ of $\underset{\substack{\cong \\ \ker(\phi) \otimes \mathbb{Z}/2}}{(\mathbb{Z}/2)^{n-1}}$.

Hence, by changing basis, we have $P_n \cong \mathcal{U}_{2\mathbb{Z}}(\mathbb{Z}^{k,l})$

with $k = \lfloor \frac{n}{2} \rfloor$
$k + l = n-1$

Applying Charney's theorem with $k = \lfloor \frac{n}{2} \rfloor$ and $d = 1$ we obtain:

since $R = \mathbb{Z}$, which is a P.I.D.

Coro: P_n is $\lfloor \frac{n-6}{2} \rfloor$ - connected.

More generally: **Def**: For $S \subseteq \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$

$P_n(S)$ = subcomplex of P_n where $j \in S$ in $(*)$

Obs: $P_n(S) \cong \mathcal{U}_{2\mathbb{Z}}(\mathbb{Z}^{k,l})$ with $k = |S|$
$k + l = n-1$

Coro: $P_n(S)$ is $(|S|-3)$-connected.

③ From P_n to Y_n via bad simplices

(Take n odd; the case n even is slightly different.)

Recall:

$$P_n \hookrightarrow Y_n$$

$g(u_i) \in \mathcal{A}$ $\quad$ $\{u_0, \dots, u_p\}$ partial $\mathbb{Z}$-basis of $\ker(\phi)$

$g(u_i)$ pairwise disjoint and of size 2

$$\mathcal{A} = \{ \{2j-1, 2j\} \; : \; 1 \leq j \leq \lfloor \tfrac{n}{2} \rfloor \}$$

To prove that the inclusion is d-connected, we need a family of "bad simplices" so that, for each bad p-simplex σ, G_σ is $(d-p-1)$-connected.

Reminder of the bad simplex argument from last time:

$Y \subset X$ simplicial complexes

B = collection of simplices of $X - Y$ ("bad simplices")

so that (1) if no face of $\sigma \subset X$ is bad, then $\sigma \subset Y$

(2) if $\sigma_1, \sigma_2 \subset X$ are bad and $\underbrace{\sigma_1 * \sigma_2}$ is also a simplex of X

"join" = the simplex spanned by the union of the vertices of σ_1 and σ_2

then $\sigma_1 * \sigma_2$ is also bad.

For $\sigma \in B$ set G_σ = subcomplex of X of simplices τ such that

- $\tau * \sigma$ is a simplex of X (τ is in the link of σ)
- all bad faces of $\tau * \sigma$ lie in σ

("good link of σ")

Proposition Suppose $\exists d \geq 0$: $\forall \sigma \in B$, G_σ is $(d - \dim(\sigma) - 1)$-connected.
Then $Y \hookrightarrow X$ induces isom's on π_i $(i \leq d-1)$ and a surjection on π_d.

Say a simplex σ of Y_n is <u>bad</u> if none of its vertices lie in P_n.

This clearly satisfies (1) and (2).

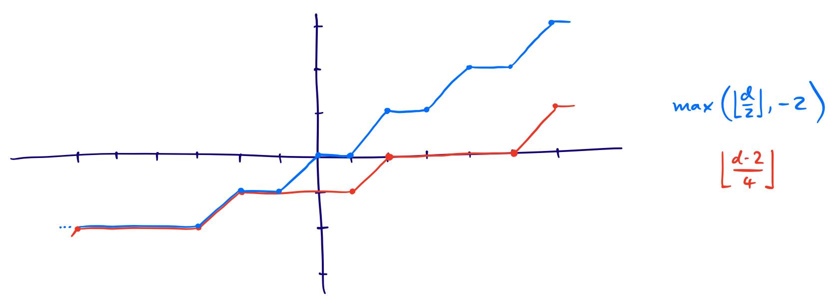
For a bad simplex $\underset{\{u_0,\dots,u_p\}}{\sigma}$, we have $G_\sigma = P_n(S)$, where $S \subseteq \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ is the set of indices j such that $\{2j-1, 2j\}$ is disjoint from every $g(u_0), \dots, g(u_p)$.

Each $|g(u_i)| = 2$, so $|S| \geq \lfloor \frac{n}{2} \rfloor - 2(p+1)$,

so $\text{connectivity}(G_\sigma) \geq \underbrace{\lfloor \tfrac{n}{2} \rfloor - 2p - 5}$

<u>not</u> of the required form, due to the factor of 2 on the p term!

But if X is $\frac{d}{2}$-connected then it is $\frac{d-2}{4}$-connected — since $\lfloor \frac{d-2}{4} \rfloor \leq \max\left(\lfloor \frac{d}{2} \rfloor, -2\right)$ and every space is (-2)-connected.



So $\text{connectivity}(G_\sigma) \geq \left\lfloor \dfrac{n-4p-10}{2} \right\rfloor$

$\Rightarrow \text{connectivity}(G_\sigma) \geq \left\lfloor \dfrac{n-4p-12}{4} \right\rfloor = \left\lfloor \dfrac{n}{4} \right\rfloor - p - 3$

Bad simplex argument $\Longrightarrow$ the inclusion $P_n \hookrightarrow Y_n$ is $\left(\lfloor \frac{n}{4} \rfloor - 2\right)$-connected.

P_n is $\left\lfloor \frac{n-6}{2} \right\rfloor$-connected $\Longrightarrow$ it is $\left\lfloor \frac{n-8}{4} \right\rfloor$-connected.

$\Rightarrow$ <u>Proposition</u>: Y_n is $\left\lfloor \frac{n-8}{4} \right\rfloor$-connected.

(Note: this is valid for n odd. When n is even the details are slightly different and we conclude that Y_n is $\lfloor \frac{n-9}{4} \rfloor$-connected.

4 Nerve lemma arguments

Finally, we have to prove

$$\underset{\frac{n-9}{4}}{(\text{high-conn. of } Y_n)} \overset{①}{\Longrightarrow} \underset{\frac{n-11}{4}}{(\text{high-conn. of } IX_n)} \overset{②}{\Longrightarrow} \underset{\frac{n-12}{4}}{(\text{high-conn. of } X_n)}$$

Reminder: simplices are $u_0, \dots, u_p \in \mathbb{Z}^n$:

- $\{u_0, \dots, u_p, v_n\}$ partial $\mathbb{Z}$-basis
- $\varsigma(u_0), \dots, \varsigma(u_p)$ pairwise disjoint

$\Big\}$ (Z_n)

- and (Y_n) $\phi(u_i) = 0$, $|\varsigma(u_i)| = 2$

  (IX_n) $\phi(u_i) = 0$, $|\varsigma(u_i)| = 2$, $\langle u_i, u_j \rangle = 0$

  (X_n) $\phi(u_i) = 1$, $|\varsigma(u_i)| = 1$, $\langle u_i, u_j \rangle = 1$ for $i<j$, for some ordering

Arguments ① and ② are similar — focus on ②.

Notation $A \overset{f}{\rightsquigarrow} B$ means $\{\text{simplices of } A\} \xrightarrow{f} \{\text{subcomplexes of } B\}$ (non-empty!)

$$\sigma \subseteq \tau \implies f(\sigma) \supseteq f(\tau)$$

Nerve lemma (Mirzaii–van der Kallen)

Let $A \overset{f}{\rightsquigarrow} B$ and suppose A is d-connected. Then B is also d-connected if:

Ⓐ The $(d+1)$-skeleton of B is contained in $\bigcup_{\sigma \in A} f(\sigma)$.

Ⓑ For every p-simplex σ of A,

$(p \geq 1)$: $f(\sigma)$ is $(d-p)$-connected

$(p=0)$: $f(\sigma)$ is $(d-1)$-connected and $f(\sigma) \hookrightarrow B$ factors through a d-connected subcomplex

Ⓒ For every p-simplex τ of B,

$A_\tau = \{\sigma \in A : \tau \in f(\sigma)\}$ is $(d-p)$-connected.

"preimage" of τ

Applying the Nerve lemma to ②

Aim: construct $IX_n \overset{f}{\rightsquigarrow} X_n$ and check Ⓐ, Ⓑ, Ⓒ with $d = \frac{n-12}{4}$.

Work by induction on n.

$n \leq 7$ vacuous so assume that $n \geq 8$ $\quad (\Rightarrow n \geq d+9)$

Recall:

$$\begin{tikzcd} & Z_n & \\ IX_n \arrow[ur, hook] & & X_n \arrow[ul, hook'] \end{tikzcd}$$

Construction:

$$\sigma \in IX_n \longrightarrow \underbrace{\mathrm{Link}_{Z_n}(\sigma)}_{\{\tau \in Z_n \;:\; \tau * \sigma \in Z_n\}} \cap X_n =: f(\sigma)$$

check: order-reversing ✓

Ⓐ τ p-simplex of X_n, $p \leq n-4$

First: $\tau = \{e_{n-p}, \ldots, e_n\}$

This lies in $f(\{e_1 - e_2\})$ ↖ vertex of IX_n

Recall: Q_n acts transitively on p-simplices of X_n

$f : IX_n \rightsquigarrow X_n$ is Q_n-equivariant

Hence $\bigcup_{\sigma \in IX_n} f(\sigma)$ contains the $(n-4)$-skeleton of X_n.

$$\begin{array}{c} (n-4) \\ \rotatebox{90}{$\leq$} \\ d+5 \\ \rotatebox{90}{$\leq$} \\ d+1 \end{array}$$

(C) **Lemma:** $\forall$ p-simplex τ of X_n, $(IX_n)_\tau \cong IX_{n-p-1}$

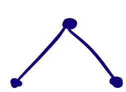
$$\text{connectivity} \geq \frac{n-p-12}{4} = d - \frac{p}{4} \geq d - p .$$

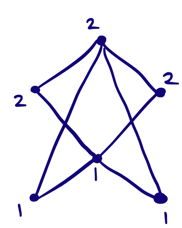
(B) **Lemma:** $\forall$ p-simplex σ of IX_n,

$$f(\sigma) = \operatorname{Link}_{Z_n}(\sigma) \cap X_n \cong X_{n-2p-2} \langle (2\mathbb{Z})^{p+1} \rangle$$

$\sigma = \{u_0, \dots, u_p\}$
$|\rho(u_i)| = 2$

Def: A (directed) simplicial complex with vertices A_0
S non-$\emptyset$ set
$A\langle S \rangle$ = (directed) simplicial complex with vertices $A_0 \times S$
and $(a_0, s_0), \dots, (a_p, s_p)$ forms a p-simplex
iff $a_0, \dots, a_p$ forms a p-simplex in A.

Eg: $A =$  $S = \{1, 2\}$

$A\langle S \rangle =$ 

There are simplicial maps $A \hookrightarrow A\langle S \rangle \twoheadrightarrow A$

label with $s_0 \in S$ forget labels

Def: $\sigma \in W$ directed simplicial complex
$$\operatorname{LLink}_W(\sigma) = \{\tau \in W : \underbrace{\tau * \sigma \in W}_{\text{directed join}}\}$$ ("left link")

Lemma: $\forall$ q-simplex τ of X_n, $\operatorname{LLink}_{X_n}(\tau) \cong X_{n-q-1}$

Prop (Charney)

W directed simplicial complex

S non-$\emptyset$ set

Suppose
- $\exists d : \forall q$-simplex τ of W, $\mathrm{LLink}_W(\tau)$ is $(d-q-1)$-connected. $(*)$
- W is $\min(1, d-1)$-connected

Then the inclusion $W \longrightarrow W\langle S\rangle$ is d-connected.

In particular, if W is d-connected and $(*)$ then $W\langle S\rangle$ is d-connected.

In our case:

$$X_{n-2p-2} \xrightarrow{\text{inc.}} X_{n-2p-2}\langle (2\mathbb{Z})^{p+1} \rangle \cong f(\sigma)$$

$$\underset{\tau}{\cup\!\!\!\shortmid} \quad q\text{-simplex}$$

$$\mathrm{LLink}_{X_{n-2p-2}}(\tau) \cong X_{n-2p-q-3}$$

connectivity $\geq \frac{n-2p-q-15}{4} \geq \frac{n-2p-11}{4} - q - 1$

by inductive hypothesis

Prop (Charney) $\Rightarrow$ inc. is $\left(\frac{n-2p-11}{4}\right)$-connected

X_{n-2p-2} is $\left(\frac{n-2p-14}{4}\right)$-connected by inductive hypothesis

Hence: connectivity$(f(\sigma)) \geq \frac{n-2p-14}{4}$
$\begin{cases} \geq \frac{n-4p-12}{4} = d-p & p \geq 1 \\ = d - \frac{1}{2} \geq d-1 & p = 0 \end{cases}$

Final step

When $p=0$ (so $\sigma = v$ is a vertex) we need to show that $f(v) \hookrightarrow X_n$ factors through a d-connected subcomplex.

(i) **Lemma**: The Q_n-action on vertices of IX_n is transitive when $n \geq 3$. (Recall we are assuming $n \geq 8$.)

Hence we may assume that $v = e_{n-1} - e_n$.

(ii) **Obs**: For this choice of v, the composition

$$X_{n-2} \longrightarrow X_{n-2}\langle 2\mathbb{Z}\rangle \cong f(v) \hookrightarrow X_n$$

is the canonical inclusion $X_{n-2} \hookrightarrow X_n$.

(iii) **Obs**: $\forall \tau \in X_{n-2}$, $\tau * e_n \in X_n$.

Hence the canonical inclusion factors as

$$X_{n-2} \hookrightarrow \underbrace{X_{n-2} * \{e_n\}}_{\text{This is a cone, hence contractible!}} \longrightarrow X_n$$

(iv) Consider the pushout:

$$\begin{tikzcd}
 & f(v) \arrow[d] \arrow[dr, bend left] & \\
X_{n-2} \arrow[ur] \arrow[dr] & f(v) \cup_{X_{n-2}} C \arrow[r, hook] & X_n \\
 & \underbrace{X_{n-2} * \{e_n\}}_{\substack{\| \\ C}} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

contractible

By above:

$$\operatorname{conn}(X_{n-2} \hookrightarrow f(v)) \geq \frac{n-11}{4}$$

By general properties of (homotopy) pushouts:

$$\operatorname{conn}\Big(f(v) \cup_{X_{n-2}} C\Big) \geq \underset{\substack{\| \\ d + \frac{1}{4} \geq d}}{\frac{n-11}{4}}$$

$\square$