Aufgaben zur Topologie

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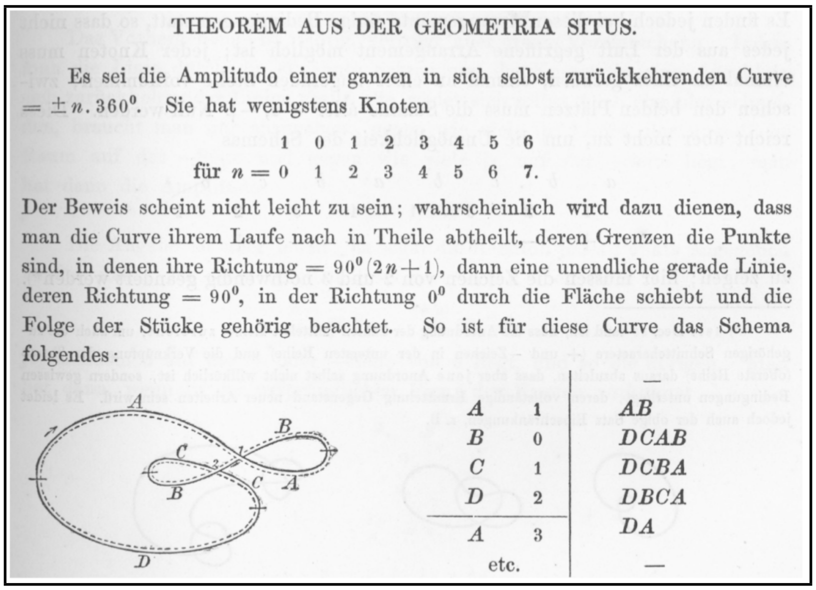
Week 2 — Umlauf, linking and winding numbers of curves to be done by: 02.11.2016



THEOREM AUS DER GEOMETRIA SITUS.

Es sei die Amplitudo einer ganzen in sich selbst zurückkehrenden Curve $= \pm n . 360^0$. Sie hat wenigstens Knoten

	1	0	1	2	3	4	5	6
für $n =$	0	1	2	3	4	5	6	7.

Der Beweis scheint nicht leicht zu sein; wahrscheinlich wird dazu dienen, dass man die Curve ihrem Laufe nach in Theile abtheilt, deren Grenzen die Punkte sind, in denen ihre Richtung $= 90^0 (2n+1)$, dann eine unendliche gerade Linie, deren Richtung $= 90^0$, in der Richtung 0^0 durch die Fläche schiebt und die Folge der Stücke gehörig beachtet. So ist für diese Curve das Schema folgendes:

		—
A	1	AB
B	0	$DCAB$
C	1	$DCBA$
D	2	$DBCA$
A	3	DA
etc.		—

Figure 1: From the notes of C.-F. Gauß, probably between 1823 and 1827.

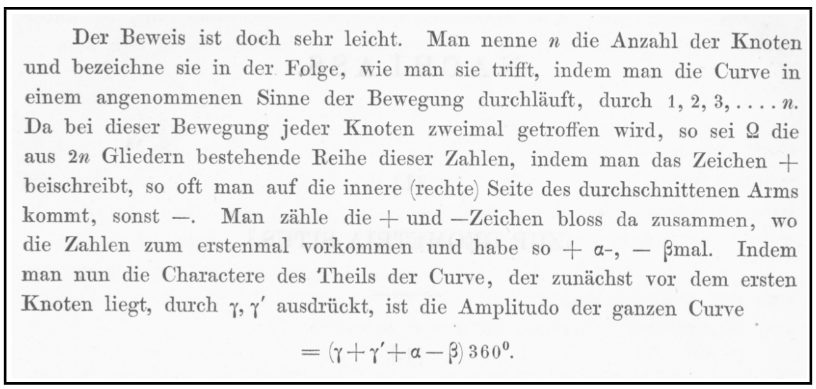
Exercise 1.1 ('Two-dimensional intermediate value theorem')
Let $f\colon \mathbb{D}^2 \to \mathbb{R}^2$ be a smooth function and $P \in \mathbb{R}^2$ a point such that $P \notin f(\mathbb{S}^1) = f(\partial\mathbb{D}^2)$. Recall the definition of the *umlauf number* $U(f,P)$ of the curve $f|_{\mathbb{S}^1}\colon \mathbb{S}^1 \to \mathbb{R}^2$ about the point P. Assuming that $U(f,P) \neq 0$, show that there exists a point $z \in \mathbb{D}^2$ such that $f(z) = P$.

Exercise 1.2 (Linking numbers of curves in $\mathbb{R}^3$)
(a) Let L be any one-dimensional affine subspace of $\mathbb{R}^3$ and let $f\colon \mathbb{S}^1 \to \mathbb{R}^3$ be a smooth curve which is disjoint from L. By considering a projection of f onto a plane orthogonal to L, define the *umlauf number* $U(f,L)$ of f about L.
(b) Show that if f and g are curves in the complement of L that are homotopic via a homotopy that never intersects L, then $U(f,L) = U(g,L)$.
(c) Fix a smooth curve $f\colon \mathbb{S}^1 \to \mathbb{R}^3$ and let $H\colon \mathbb{R} \times [0,1] \to \mathbb{R}^3$ be a continuous function that never intersects f and has the property that for each fixed t the function $H(-,t)\colon \mathbb{R} \to \mathbb{R}^3$ is an affine injection. Denote by L_t the one-dimensional affine subspace $H(\mathbb{R} \times \{t\})$ of $\mathbb{R}^3$. Show that $U(f,L_0) = U(f,L_1)$.
(d) We return to the situation where we have a fixed one-dimensional affine subspace L of $\mathbb{R}^3$ and a smooth curve $f\colon \mathbb{S}^1 \to \mathbb{R}^3$ in its complement. Consider a two-dimensional half-space with L as its boundary. (There are many possibilities – they are all related to each other by a rotation about L.) Define what it means for the half-space to *intersect f transversely*, and, if it does, the *sign* of the intersection. How can one calculate $U(f,L)$ using these signed intersections ?
(e)* Now let C be the unit circle in a two-dimensional affine subspace of $\mathbb{R}^3$. Find a continuous function $K\colon \mathbb{S}^3 \times [0,1] \to \mathbb{S}^3$ such that (i) each $K(-,t)$ is a diffeomorphism of $\mathbb{S}^3$, (ii) $K(-,0)$ is the identity on $\mathbb{S}^3$ and (iii) $K(C,1)$

is a one-dimensional affine subspace of $\mathbb{R}^3$ together with the point at infinity.
(f)* Using this, define the *umlauf number* $U(f, C)$ of any smooth curve $f\colon \mathbb{S}^1 \to \mathbb{R}^3$ which does not intersect C.
(g)* Consider a two-dimensional disc embedded into $\mathbb{R}^3$ in such a way that its boundary is C. Similarly to part (d), define a *transverse intersection* of f with the disc, the *sign* of such an intersection and show how one can compute $U(f, C)$ using these concepts.

Exercise 1.3 (Circumnavigating the world)
Let B be a sailing boat on the surface of the world $\mathbb{S}^2$, and assume that it never visits the north or the south pole. What does it mean for B to *circumnavigate the world* ? If you are given a map of its journey (which starts and ends at the basepoint Hamburg $\in \mathbb{S}^2$), how can you use lines of longitude to detect whether it has really circumnavigated the world ?

Exercise 1.4 (Local index of vector fields)
For each of the following sets of integers, give a vector field V on $\mathbb{S}^2$ with precisely k zeros $\zeta_1, \ldots, \zeta_k$ having the listed integers as their local indices $\mathrm{ind}(V, \zeta_i)$:
(a) $k = 2$ and indices $1, 1$,
(b) $k = 1$ and index 2,
(c) $k = 4$ and indices $1, 1, 1, -1$,
(d) $k = 2$ and indices $2, 0$,
(e) $k = 7$ and indices $1, 1, 1, 1, 1, -1, -2$.
(Do this by a drawing.)



Der Beweis ist doch sehr leicht. Man nenne n die Anzahl der Knoten und bezeichne sie in der Folge, wie man sie trifft, indem man die Curve in einem angenommenen Sinne der Bewegung durchläuft, durch $1, 2, 3, \ldots n$. Da bei dieser Bewegung jeder Knoten zweimal getroffen wird, so sei Ω die aus $2n$ Gliedern bestehende Reihe dieser Zahlen, indem man das Zeichen $+$ beischreibt, so oft man auf die innere (rechte) Seite des durchschnittenen Arms kommt, sonst $-$. Man zähle die $+$ und $-$Zeichen bloss da zusammen, wo die Zahlen zum erstenmal vorkommen und habe so $+\ \alpha$-, $-\ \beta$mal. Indem man nun die Charactere des Theils der Curve, der zunächst vor dem ersten Knoten liegt, durch γ, γ' ausdrückt, ist die Amplitudo der ganzen Curve

$$= (\gamma + \gamma' + \alpha - \beta)\, 360^0.$$

Figure 2: A little later in the notes of C.-F. Gauß.

Exercise 1.5 (Symmetric vector fields on spheres)
In this exercise we will take the point of view that a tangent vector field on $\mathbb{S}^2 \subset \mathbb{R}^3$ is a continuous function $V\colon \mathbb{S}^2 \to \mathbb{R}^3$ with the property that x and $V(x)$ are orthogonal vectors for each $x \in \mathbb{S}^2$.
(a) Suppose that $V(x) = V(-x)$ for each $x \in \mathbb{S}^2$. If x is a zero of V, what is the relationship between the local index of V at x and the local index of V at $-x$?
(b) Now fix an angle $\theta \in [0, 2\pi]$. Consider the following equation (where $x \in \mathbb{S}^2$):

$$V(-x) \;=\; \cos(\theta)\, V(x) + \sin(\theta)\, (x \times V(x)), \tag{1}$$

where $\times$ denotes the cross product in $\mathbb{R}^3$. Show that (1) holds for x if and only if it holds for $-x$.
(c) Now assume that (1) holds for all $x \in \mathbb{S}^2$. If x is a zero of V, what is the relationship between the local index of V at x and the local index of V at $-x$? How does this relationship depend on θ?
(d) Now let V be an arbitrary vector field (not necessarily satisfying (1)) and define another vector field V_θ for $\theta \in [0, 2\pi]$ by

$$V_\theta(x) \;=\; \cos(\theta)\, V(x) + \sin(\theta)\, (x \times V(x)).$$

What are the zeros of V_θ in comparison to the zeros of V? How is the local index of a zero of V_θ determined by the local indices of the zeros of V?

(e)* Now let $F \subset \mathbb{R}^3$ be a torus, symmetrically embedded into $\mathbb{R}^3$ such that $x \in F$ implies that $-x \in F$ and the tangent spaces T_xF and $T_{-x}F$ are parallel. Let us say that it is the z-axis that passes through the 'hole' in the torus (so F is disjoint from the z-axis).
We will think of vector fields on F as continuous functions $V\colon F \to \mathbb{R}^3$ with the property that $V(x)$ is parallel to T_xF for each $x \in F$. Let $p\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the projection onto the (x, y)-axis. Show that, when $x \in F$, the vector $W(x) = p(x) \times (0, 0, 1)$ always lies in the tangent plane T_xF, where $\times$ is the cross product in $\mathbb{R}^3$. This is therefore an example of a non-vanishing vector field on F. Now suppose that V is any vector field on F satisfying the following property: for each $x \in F$,

$$V(-x) \;=\; \text{the reflection of } V(x) \text{ about the axis } \{\lambda.W(x) \mid \lambda \in \mathbb{R}\} \text{ in the tangent plane } T_xF.$$

If x is a zero of V, what is the relationship between the local index of V at x and the local index of V at $-x$? Give an example of a vector field V that has zeros and satisfies this property.

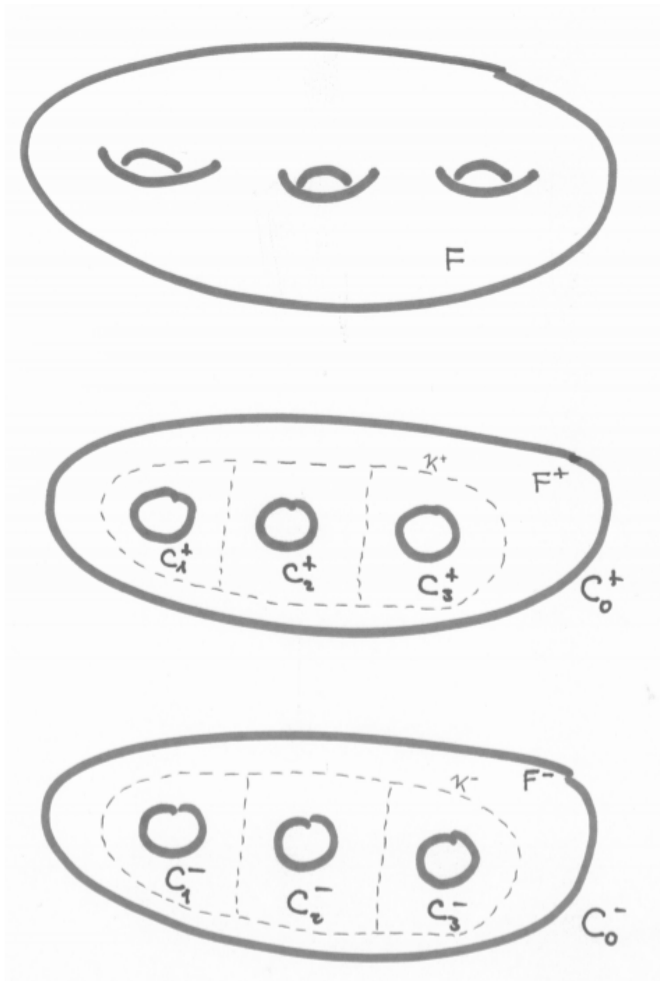


Figure 3: The surface F from Exercise 1.6.

Exercise 1.6 ('Satz vom Igel' for higher-genus surfaces)
The orientable surface F of genus g we imagine as smoothly embedded in $\mathbb{R}^3$ (see Figure 3 on page 3), symmetric with respect to the origin and to the (x, y)-plane. The intersection with this plane consists of an outer curve C_0 and g further curves $C_1, \dots, C_g$. We have an upper resp. lower half-surface F^+ and F^-, both a disc with an outer

boundary curve C_0^+ resp. C_0^- and g inner boundary curves $C_1^+, \ldots, C_g^+$ resp. $C_1^-, \ldots, C_g^-$.
(a) Show that $T(F)$ is trivial over both F^+ and F^-. (Move a point ζ towards the point on the dotted core curve K^+ resp. K^- closest to ζ and slide the tangent plane along. The curves K^+ and K_- lie in planes parallel to the (x, y)-plane, where all tangent planes have a canonical isomorphism to $\mathbb{R}^2$.) In the end we have for each curve a clutching function $\phi_i \colon C_i \to \mathrm{GL}_2(\mathbb{R})$, $i = 0, 1, \ldots, g$.
(b) In the case of a torus, $g = 1$, conclude that the tangent bundle is trivial over the whole surface.
(c) Now assume a vector field is given. This vector field is a function $V \colon F \to \mathbb{R}^3$ such that $V(\zeta)$ is tangent to F at ζ, i.e., $V(\zeta) \in T_\zeta(F)$. What can we conclude about the zeroes of V? If we assume that V has no zeroes; then — since F is orientable — in each tangent plane we get by left-rotation of $V(\zeta)$ with angle $90°$ a second and linearly independent vector and thus a basis in each tangent space. Restricting to the curves C_i we have functions $\psi_i \colon C_i \to \mathrm{GL}_2(\mathbb{R})$. But these functions must satisfy relations. What are these relations, and what do they imply for non-vanishing vector fields on F?