Aufgaben zur Topologie

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Week 10 — Excision and mapping tori Due: 18. January 2017

Exercise 10.1 (Reduction Lemma.)
Let $A_\bullet$ be a subcomplex of a chain complex $B_\bullet$ and assume the following two hypotheses:
(R1) Each cycle $b \in B_\bullet$ is homologous (in $B_\bullet$) to a cycle in $A_\bullet$.
(R2) If two cycles a, a' in $A_\bullet$ are homologous in $B_\bullet$, then they are homologous in $A_\bullet$.
Prove that the inclusion $\iota_\bullet\colon A_\bullet \to B_\bullet$ induces an isomorphism $\iota_*\colon H_n(A_\bullet) \to H_n(B_\bullet)$.

In addition show that (R1) and (R2) follow from:
(R3) Each chain in $B_\bullet$ is homologous (in $B_\bullet$) to a chain in $A_\bullet$.

Exercise 10.2 (Transformators.)
A *transformator* is a natural self-transformation $\tau_\bullet$ of the functor $S_\bullet$ with $\tau_0 = \mathrm{id}$. Spelled out in detail this is, for each space X and each $n \geq 0$, a homomorphism $\tau_n^X : S_n(X) \to S_n(X)$ such that
(1) $\tau_0^X = \mathrm{id}_{S_0(X)}$,
(2) $\partial \circ \tau_n^X = \tau_{n-1}^X \circ \partial$,
(3) $S_n(f) \circ \tau_n^X = \tau_n^Y \circ S_n(f)$ for any map $f\colon X \to Y$.
Examples:
(i) $\tau_n^X = \mathrm{id}_{S_n(X)}$ is a trivial example.
(ii) The composition of two transformators is a transformator.
(iii) The barycentric subdivision $\tau = B$ (as defined in the lectures) is the fundamental example.
(iv) If $\omega_n\colon \Delta^n \to \Delta^n$ is the affine homeomorphism permuting the vertices $e_0, e_1, \ldots, e_n$ of Δ^n like $\omega_n(e_i) = e_{n-i}$, then $\tau_n^X(a) := \epsilon(n).a \circ \omega_n$ for a simplex $a\colon \Delta^n \to X$, where the sign $\epsilon(n)$ is -1 when $n \equiv 1, 2 \bmod 4$ and is $+1$ when $n \equiv 0, 3 \bmod 4$, defines a transformator.

Prove the following:
(a) If $\tau_\bullet$ is a transformator, then its image $S_\bullet^\tau(X) := \mathrm{im}(\tau_\bullet^X : S_\bullet(X) \to S_\bullet(X))$ defines a subcomplex of $S_\bullet(X)$, which satisfies the two hypotheses (R1) and (R2) of the Reduction Lemma (Exercise 10.1).
(b) $\tau_\bullet^X$ induces the identity in homology, since there is a natural chain homotopy between $\tau_\bullet^X$ and the identity.
(c)* (A generalisation of part (a).) If $\Lambda_n\colon \mathfrak{B}_n(X) \to \mathbb{N}$ is a family of functions from the basis $\mathfrak{B}_n(X)$ of $S_n(X)$, denote by $S_n^\Lambda(X)$ the subgroup of $S_n(X)$ generated by the elements $\tau^k(a)$ for all $a \in \mathfrak{B}_n(X)$ and all $k \geq \Lambda_n(a)$. Assume that the functions satisfy the following property: whenever $a \in \mathfrak{B}_n(X)$ and b is a basis element in $\partial(a)$, we have $\Lambda_n(a) \geq \Lambda_{n-1}(b)$. Then $S_\bullet^\Lambda(X)$ is a subcomplex of $S_\bullet(X)$ satisfying (R1) and (R2) of the Reduction Lemma. (Cf. the preparation for the Excision Theorem.)

Exercise 10.3 (Local homology.)
For a space X and a point $x \in X$, the *local homology groups* of X at x are by definition the relative homology groups $H_n(X, X - \{x\})$, in other words, the homology groups of the quotient chain complex $S_\bullet(X)/S_\bullet(X - \{x\})$. The name *local* comes from the following property of these groups:
(1) Let U be an open neighbourhood of x in X. Assume that the one-point subspace $\{x\}$ is closed in X (for example, this is always true if X is Hausdorff). Using the Excision Theorem, show that there are isomorphisms

$$H_n(X, X - \{x\}) \cong H_n(U, U - \{x\}).$$

Thus the local homology of a space X at a point depends only on its topology arbitrarily close to that point.
(2) If F is a surface, show that for any point $x \in F$ the relative homology $H_n(F, F - \{x\})$ is trivial for $n \neq 2$ and isomorphic to $\mathbb{Z}$ for $n = 2$. There are three steps:

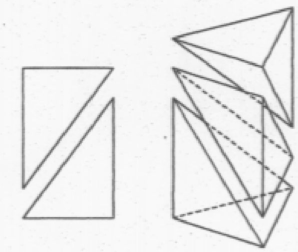


Fig. 59 Triangulations of cylinders over simplices

From: A. Fomenko, D. Fuchs, *Homotopical Topology.*

(i) $H_n(F, F - \{x\}) \cong H_n(\mathbb{D}^2, \mathbb{D}^2 - \{0\})$
(ii) $H_n(\mathbb{D}^2, \mathbb{D}^2 - \{0\}) \cong H_n(\mathbb{D}^2, \partial\mathbb{D}^2)$
(iii) $H_n(\mathbb{D}^2, \partial\mathbb{D}^2)$ is isomorphic to $H_{n-1}(\mathbb{S}^1)$ for $n \geq 2$ and is zero for $n = 0, 1$. (cf. Exercise 9.3)
(You may assume the fact that the homology $H_n(\mathbb{S}^1)$ of $\mathbb{S}^1$ is isomorphic to $\mathbb{Z}$ for $n = 0, 1$ and is trivial for $n \geq 2$.)
(3)* If F is a surface with boundary and x is a point on the boundary ∂F, show that the relative homology groups $H_n(F, F - \{x\})$ are all trivial, including for $n = 2$.

Exercise 10.4 (Quotient Theorem : Relative homology and collapsing a subspace to a point.)
Let X be a space and let $C \subseteq U \subseteq X$ be subspaces, where C is closed in X and U is open in X.
(1) Using inclusions and (restrictions of) the quotient map $X \to X/C$ that collapses C to a point, construct a commutative diagram of pairs of spaces as follows:

$$\begin{tikzcd} (X - C, U - C) \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & (X, U) \arrow[d, "\delta"] \\ (X/C - C/C, U/C - C/C) \arrow[r, "\beta"] & (X/C, U/C) \end{tikzcd} \tag{$\odot$}$$

(2) Show that, for any subspace $A \subseteq X - C$, the restriction to A of the quotient map $X \to X/C$ is a homeomorphism onto its image.
(3) The quotient map $X \to X/C$ induces a map of long exact sequences from the long exact sequence associated to the pair $(X - C, U - C)$ to the long exact sequence associated to the pair $(X/C - C/C, U/C - C/C)$.
(4) Using the five-lemma (cf. Exercise 8.1), show that the map γ in ($\odot$) induces isomorphisms on homology.
(5) Applying excision to the maps α and β, show that $H_*(X, U)$ is isomorphic to $H_*(X/C, U/C)$ via δ_*.
(6)* Now assume that U deformation retracts onto C. Show that U/C deformation retracts onto the point C/C.
(7)* Under the same assumption, use the long exact sequences associated to the pairs (X, U) and $(X/C, U/C)$ to show that

$$H_*(X/C, C/C) \cong H_*(X/C, U/C) \cong H_*(X, U) \cong H_*(X, C).$$

So the relative homology of the pair (X, C) is isomorphic to the homology of the quotient space X/C relative to a point.

Exercise 10.5 (Homology of mapping tori.)
Given a continuous map $f \colon X \to X$, the *mapping torus* $T(f)$ of f is defined to be the quotient space $X \times [0,1]/\sim$, where $\sim$ is the equivalence relation generated by the relations $(x, 0) \sim (f(x), 1)$ for all $x \in X$.
(a) Using Exercise 8.3, explain why $f \simeq f'$ implies that $T(f) \simeq T(f')$.
In Exercise 10.6* you will (optionally) construct the associated long exact sequence, which is of the form

$$\cdots H_{i+1}(T(f)) \to H_i(X) \to H_i(X) \to H_i(T(f)) \to H_{i-1}(X) \to \cdots, \tag{$\star$}$$

where the map $H_i(X) \to H_i(X)$ is defined by $H_i(f) - \mathrm{id}$. Explain why this is the zero map for $i = 0$ when X is path-connected.

(b) Now take $X = \mathbb{S}^1$ and let f be a self-map of degree $d \neq 1$. Using the long exact sequence ($\star$) and the techniques from Exercise 9.5, together with the fact (which you may assume) that $H_i(\mathbb{S}^1) \cong \mathbb{Z}$ for $i = 0, 1$ and $H_i(\mathbb{S}^1) = 0$ for $i \neq 0, 1$, compute the homology of the mapping torus $T(f)$.
Also show that $T(f)$ is homotopy equivalent to the Klein bottle when $d = -1$ and to $\mathbb{S}^1$ when $d = 0$. (In fact, for any space X, if f is nullhomotopic, then $T(f)$ is homotopy equivalent to $\mathbb{S}^1$.)
(c) Now let X be any space and take f to be the identity $X \to X$. Show that the homology of $T(f)$ is related to that of X via short exact sequences $0 \to H_i(X) \to H_i(T(f)) \to H_{i-1}(X) \to 0$. Therefore, if the homology of X is free abelian in each degree, we have isomorphisms $H_i(T(f)) \cong H_i(X) \oplus H_{i-1}(X)$. As a special case, we obtain the homology of the torus $\mathbb{S}^1 \times \mathbb{S}^1$.
(d)* (The homology of some 3-manifolds.) If X is a (smooth) surface and $f \colon X \to X$ is a homeomorphism (resp. diffeomorphism) then $T(f)$ is a (smooth) 3-manifold. In this part we take X to be the genus-2 surface F_2.
(i) First suppose that f is a Dehn twist. Recall (cf. Exercise 6.2) that, if one chooses the generators of $H_1(F_2) \cong \mathbb{Z}^4$ in an appropriate way, the induced homomorphism $H_1(f)$ is the elementary matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

You may assume the following facts: $H_2(F_2) \cong \mathbb{Z}$ and $H_i(F_2) = 0$ for $i \geqslant 3$. Moreover, any self-homeomorphism of F_2 which is the identity on some open subset induces the identity homomorphism on $H_2(F_2)$. Compute the homology of the mapping torus $T(f)$.
(ii) Now suppose that F_2 is embedded into $\mathbb{R}^3$ in such a way that it is symmetric with respect to reflection in a plane P that cuts it into two punctured tori, and let $f \colon F_2 \to F_2$ be the self-homeomorphism induced by reflection in P. Show (by a picture) that one may choose generators for $H_1(F_2)$ in such a way that $H_1(f)$ is given by the permutation matrix

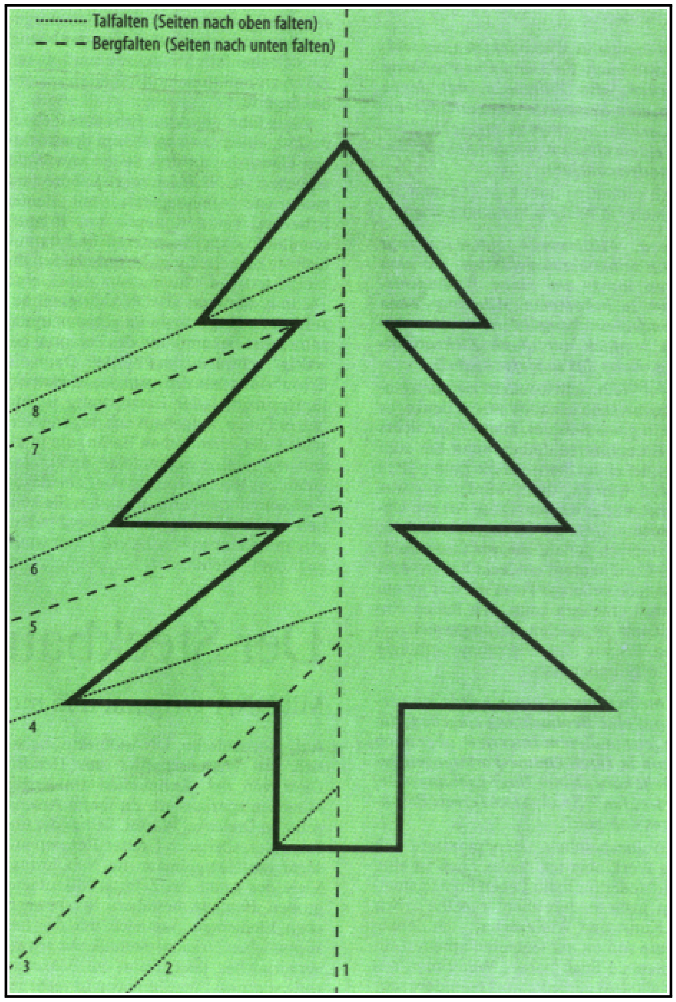
$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

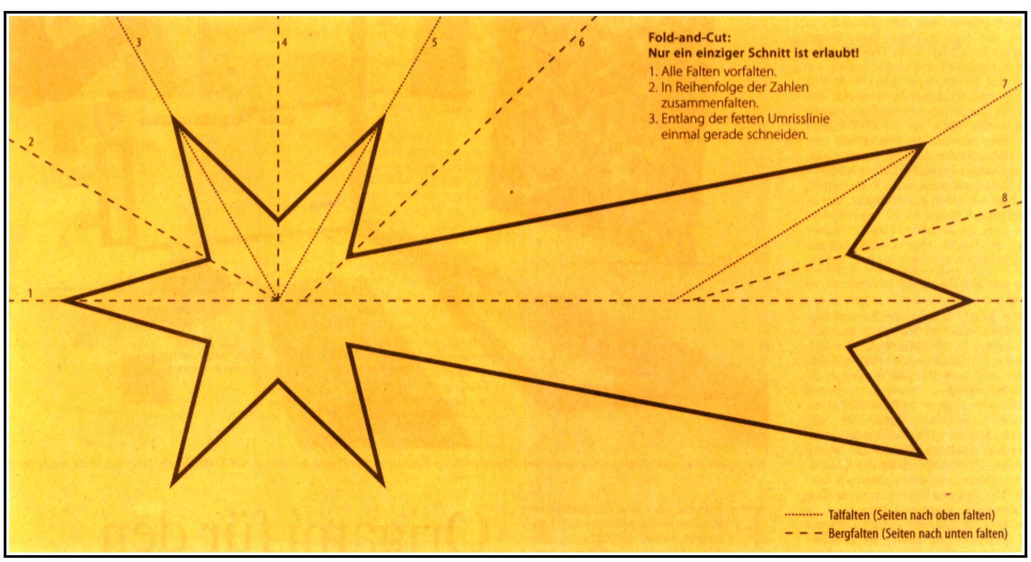
with respect to these generators. Calculate the homology of $T(f)$ in this case. (You may use the fact that, in this case, f acts on $H_2(F_2)$ by multiplication by -1.)

Exercise 10.6* (The long exact sequence associated to a mapping torus.)
Recall the definition of the *mapping torus* $T(f)$ of a map $f \colon X \to X$ from Exercise 10.5, and let $q \colon X \times [0,1] \to T(f)$ denote the quotient map.
(1) Show that the image of $X \times \{0,1\}$ under q is a subspace of $T(f)$ homeomorphic to X. Moreover, if we identify $X \times \{0,1\}$ with $X \sqcup X$ and $q(X \times \{0,1\})$ with X, then the restriction of q to $X \times \{0,1\}$ is $f \sqcup \mathrm{id}$.
(2) We therefore have a map of pairs of topological spaces of the form $(X \times [0,1], X \times \{0,1\}) \to (T(f), X)$. Each pair of spaces has an associated long exact sequence of homology groups; draw a diagram of the *map* of long exact sequences induced by the map of pairs.
(3) The desired long exact sequence ($\star$) is very similar to the long exact sequence associated to the pair $(T(f), X)$. To derive ($\star$) from this sequence, you just need to explain why the terms $H_i(T(f), X) = H_i(C_\bullet(T(f))/C_\bullet(X))$ that appear in the long exact sequence for the pair $(T(f), X)$ can be replaced by $H_{i-1}(X)$, and why the map $H_i(X) \to H_i(X)$ after doing this replacement is given by $H_i(f) - \mathrm{id}$. Do this using the following steps:
(a) The map $H_i(X \times \{0,1\}) \to H_i(X \times [0,1])$ in your diagram is surjective (for every i).
(b) The map $H_{i+1}(X \times [0,1], X \times \{0,1\}) \to H_i(X \times \{0,1\})$ is an isomorphism onto a subgroup which is isomorphic to $H_i(X)$.
(c) The map $H_i(X \times [0,1], X \times \{0,1\}) \to H_i(T(f), X)$ is an isomorphism. (Hint: each pair of spaces has the property that the subspace has a neighbourhood in the larger space that deformation retracts onto it.)
(d) Thus there is an isomorphism $H_i(X) \cong H_{i+1}(T(f), X)$. Using commutativity of your diagram, and part (1), it follows that the map $H_i(X) \cong H_{i+1}(T(f), X) \to H_i(X)$ is given by $H_i(f) - H_i(\mathrm{id}_X) = H_i(f) - \mathrm{id}_{H_i(X)}$.

Der Meister der Origami-Technik

Erik Demaine wurde 1981 in Halifax (Kanada) geboren. Seit 2001 ist er Professor am Massachusetts Institute of Technology und interessiert sich für fast alle Bereiche der Mathematik und Informatik, die mit Algorithmen zu tun haben, insbesondere für Faltungsprobleme der diskreten und rechnerbasierten Geometrie. Sein Vater, der Künstler Martin Demaine, nahm ihn mit sieben Jahren aus der Schule, um mit ihm vier Jahre lang durch die Vereinigten Staaten zu reisen. In dieser Zeit entdeckte Demaine seine Begeisterung für das Programmieren, die ihn mit zwölf Jahren ein Studium der Informatik beginnen ließ. Am MIT arbeiten Vater und Sohn nach wie vor eng zusammen. Im Jahr 1999 bewiesen beide das „Fold-and-Cut-Theorem“, das besagt, dass man jede polygone Form mit nur einem einzigen geraden Schnitt aus einem gefalteten Papier schneiden kann. Als mathematisches Rätsel wurde dieses Problem in Japan bereits 1721 beschrieben. Einer Anekdote zufolge war die einfache Erzeugung fünfzackiger Sterne per Fold-and-Cut-Technik auch der Grund dafür, dass die amerikanische Flagge diese anstatt sechszackiger Sterne verwendet. Wir haben für Sie hier zwei Faltanleitungen abgedruckt, mit deren Hilfe Sie diese Technik zur Produktion von Weihnachtsmotiven nutzen können. (sian)





For those who missed Christmas completely: an origami from FAZ 21.12.2016.