Aufgaben zur Topologie

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Wintersemester 2016/17

Week 12 — Suspensions, coefficient rings, (co)invariants, surgery. Due: 1. February 2017

Exercise 12.1 (Sums of maps.)
Let $\tilde{\Sigma}X$ denote the reduced suspension $\Sigma X/\Sigma x_0$ of a based space X with x_0 the basepoint; and denote by

$$\nabla\colon \tilde{\Sigma}X \longrightarrow \tilde{\Sigma}X \vee \tilde{\Sigma}X$$

the so-called *co-multiplication* defined by

$$\nabla([x,t]) = \begin{cases} [x,2t] \text{ in the left summand}, & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ [x,2t-1] \text{ in the right summand}, & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Here we denote the points of $\tilde{\Sigma}X = X \times [0,1]/A$ with $A = (X \times \{0,1\}) \cup (\{x_0\} \times [0,1])$ by $[x,t]$, using their X-coordinate and their height t in the double cone. We denote by $p_i\colon \tilde{\Sigma}X \vee \tilde{\Sigma}X \to \tilde{\Sigma}X$ for $i = 1,2$ the projection onto the left resp. right summand, which collapses the other summand to a point. And by $\iota_i\colon \tilde{\Sigma}X \to \tilde{\Sigma}X \vee \tilde{\Sigma}X$ we denote the inclusions of the left resp. right summand.

(1) Show that $p_i \circ \nabla \simeq \mathrm{id}_{\tilde{\Sigma}X}$ for $i = 1,2$.

(2) Show that $(a,b) \mapsto \iota_{1*}(a) + \iota_{2*}(b)$ is an isomorphism $\Phi\colon H_n(\tilde{\Sigma}X) \oplus H_n(\tilde{\Sigma}X) \to H_n(\tilde{\Sigma}X \vee \tilde{\Sigma}X)$, for $n > 0$.

(3) Conclude that the homomorphism $\Phi^{-1} \circ \nabla_*\colon H_n(\tilde{\Sigma}X) \to H_n(\tilde{\Sigma}X) \oplus H_n(\tilde{\Sigma}X)$ is the diagonal.

Now we write the n-sphere $\mathbb{S}^n = \Sigma\mathbb{S}^{n-1}$ (for $n \geq 1$) as a suspension of $\mathbb{S}^{n-1}$. For two based maps $f,g\colon \mathbb{S}^n \to \mathbb{S}^n$ we declare their sum $f+g\colon \mathbb{S}^n \to \mathbb{S}^n$ by $f+g := \mathrm{F} \circ (f \vee g) \circ \nabla$, where $\mathrm{F}\colon \tilde{\Sigma}X \vee \tilde{\Sigma}X \to \tilde{\Sigma}X$ is the folding map.

(4) Prove the formula:

$$\deg(f+g) = \deg(f) + \deg(g).$$

(5)* More generally, prove that $(f+g)_* = f_* + g_*\colon H_n(\tilde{\Sigma}X) \to H_n(\tilde{\Sigma}X)$.

Exercise 12.2 (An application of mapping degree: fixed and antipodal points of self-maps of spheres.)
Let $n \geqslant 1$ and let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be a self-map of the n-sphere.
(a) If n is even, show that f must have either a fixed point or an antipodal point ($x \in \mathbb{S}^n$ such that $f(x) = -x$).
(b) More generally, if n is even, any two self-maps f,g of $\mathbb{S}^n$ must have either an incidence point ($x \in \mathbb{S}^n$ such that $f(x) = g(x)$) or an opposite point ($x \in \mathbb{S}^n$ such that $f(x) = -g(x)$), unless they both have degree 0.
(c) For n odd, give an example of a self-map $f\colon \mathbb{S}^n \to \mathbb{S}^n$ with no fixed point and no antipodal point.
(d) More generally, given a self-map $f\colon \mathbb{S}^n \to \mathbb{S}^n$, construct (when n is odd) another self-map $g\colon \mathbb{S}^n \to \mathbb{S}^n$ such that f and g have no incidence points and no opposite points.

Exercise 12.3 (Coefficient rings.)
Let Z be the space $\mathbb{D}^2 \times \{0,1\}$, i.e., the disjoint union of two closed 2-discs, and let $A \subset Z$ be its boundary $\mathbb{S}^1 \times \{0,1\}$. Let $Y = \mathbb{S}^1$ and consider a map $f\colon A \to Y$ that sends $\mathbb{S}^1 \times \{0\}$ to Y by a map of degree m and sends $\mathbb{S}^1 \times \{1\}$ to Y by a map of degree n. See the figure on the next page.
(a) Using the Mayer-Vietoris sequence for an appropriate open covering of $X = Z \cup_f Y$, show that there is an exact sequence

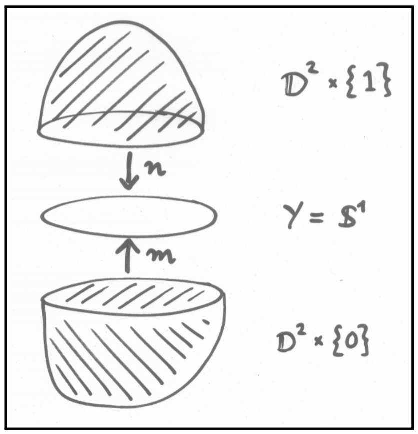
$$0 \to H_2(X) \to \mathbb{Z}^2 \xrightarrow{\phi} \mathbb{Z} \to H_1(X) \to 0,$$

where ϕ is given by the matrix $(m\ n)$, and hence that $H_2(X) \cong \mathbb{Z}$ (unless $m = n = 0$, in which case $H_2(X) \cong \mathbb{Z}^2$) and $H_1(X) \cong \mathbb{Z}/h\mathbb{Z}$, where $h = \gcd(m,n)$ is the greatest common divisor of m and n if they are both non-zero,

and is $\max(|m|,|n|)$ otherwise.
(b) What happens when we compute homology not with $\mathbb{Z}$ coefficients, but rather with R coefficients, where
(i) $R = \mathbb{Q}$,
(ii) $R = \mathbb{F}_p$, for a prime p,
(iii) $R = \mathbb{Z}[\frac{1}{p}]$, for a prime p, where $\mathbb{Z}[\frac{1}{p}] = \{\frac{a}{b} \in \mathbb{Q} \mid a \text{ and } b \text{ are coprime and } b = p^c \text{ for some integer } c \geqslant 0\}$?
(c)* Now take $Y' = \mathbb{S}^1 \vee \mathbb{S}^1$ and consider a map $g\colon A \to Y'$ that sends $\mathbb{S}^1 \times \{0\}$ to Y' as a loop that winds 3 times around the left-hand circle of the "figure-of-eight" and twice around the right-hand circle, and sends $\mathbb{S}^1 \times \{1\}$ to Y' as a loop that winds 5 times around the left-hand circle and 7 times around the right-hand circle. Let $X' = Z \cup_g Y'$. Similarly to part (a), show that there is an exact sequence

$$0 \to H_2(X') \to \mathbb{Z}^2 \xrightarrow{\psi} \mathbb{Z}^2 \to H_1(X') \to 0,$$

where ψ is given by the matrix $\left(\begin{smallmatrix} 3 & 5 \\ 2 & 7 \end{smallmatrix}\right)$, and hence that $H_2(X') = 0$ and $H_1(X') \cong \mathbb{Z}/11\mathbb{Z}$.
(d)* What happens if we change the ring of coefficients as in part (b)?



The attaching map f for the space $X = Z \cup_f Y$ in Exercise 12.3(a).

Exercise 12.4 (Invariants and coinvariants.)
Let X be a space with an action of a group G. We write X^G for the subspace of X consisting of all fixed points under the action (the *invariants*) and X/G for the quotient space $\{x.G \mid x \in X\}$ (the *orbit space*).
Fix a commutative ring R with unit. If M is an R-module with a G-action by R-linear automorphisms, we define the *invariants* M^G, as above, to be the submodule of all elements that are fixed under the action. The module of *coinvariants* M_G is the quotient of M by the submodule generated by the set $\{m - m.g \mid m \in M, g \in G\}$.
(a) If X is a space with a G-action, then $M = H_n(X;R)$ is an R-module with a G-action. There are inclusion and quotient maps $X^G \hookrightarrow X \twoheadrightarrow X/G$ and also $M^G \hookrightarrow M \twoheadrightarrow M_G$. Complete the following commutative diagram by defining the dotted arrows:

$$\begin{tikzcd}
H_i(X^G) \arrow[dr] \arrow[dd, dotted, "f_i"'] & & H_i(X)_G \arrow[dd, dotted, "g_i"] \\
& H_i(X) \arrow[ur] \arrow[dr] & \\
H_i(X)^G \arrow[ur] & & H_i(X/G)
\end{tikzcd}$$

(b) Take $R = \mathbb{Z}$ and let $X = \mathbb{S}^n$ $(n \geqslant 2)$ with $G = \mathbb{Z}/2\mathbb{Z}$ acting by a reflection. Show that
(i) g_i is an isomorphism for $i < n$, but g_n is not injective;
(ii) f_{n-1} is also not injective.
(c) Now consider the same set-up, except that $G = \mathbb{Z}/2\mathbb{Z}$ acts by the antipodal map instead. Show that
(i) f_0 is not surjective;

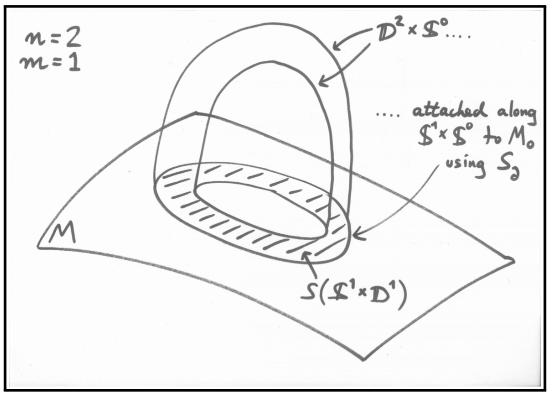
(ii) f_n is an isomorphism if and only if n is even; whereas g_n is injective but not surjective for n odd and is surjective but not injective for n even;
(iii) in degrees $0 < i < n$ we have: g_i is an isomorphism if and only if i is even.

(d) In part (c), replace $R = \mathbb{Z}$ with $R = \mathbb{Q}$ or $R = \mathbb{F}_p$ for an odd prime p. Now g_i is an isomorphism for all i.

Exercise 12.5 (Surgery on a manifold.)
Recall that an n-dimensional topological manifold is a Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$. A *framed embedded sphere* S in M of dimension m is an embedding $S\colon \mathbb{S}^m \times \mathbb{D}^{n-m} \hookrightarrow M$. Write S_∂ to denote the restriction of S to $\mathbb{S}^m \times \partial\mathbb{D}^{n-m} = \mathbb{S}^m \times \mathbb{S}^{n-m-1} = \partial\mathbb{D}^{m+1} \times \mathbb{S}^{n-m-1}$. We then define

$$M(S) \;=\; \left(\mathbb{D}^{m+1} \times \mathbb{S}^{n-m-1}\right) \cup_{S_\partial} M_\circ, \qquad \text{where} \qquad M_\circ \;=\; M - S(\mathbb{S}^m \times \mathring{\mathbb{D}}^{n-m}),$$

and call this *the result of surgery on M along S*. For example, the result of surgery along a framed embedded 1-sphere in a surface look like the following:



(a) Draw a sketch to show why this is again a manifold.
(b) Explain why we have a diagram of the form

$$\begin{array}{ccccccc}
 & & H_{i+1}(M(S), M_\circ) & & & & \\
 & & \downarrow & & & & \\
H_{i+1}(M, M_\circ) & \longrightarrow & H_i(M_\circ) & \longrightarrow & H_i(M) & \longrightarrow & H_i(M, M_\circ) \\
 & & \downarrow & & & & \\
 & & H_i(M(S)) & & & & \\
 & & \downarrow & & & & \\
 & & H_i(M(S), M_\circ) & & & &
\end{array} \tag{1}$$

with one exact row and one exact column. To relate $H_i(M(S))$ to $H_i(M)$, it is important to understand the relative homology groups appearing in (1).
(c) Using Excision, show that

$$H_i(M, M_\circ) \cong H_i(\mathbb{S}^m \times \mathbb{D}^{n-m}, \mathbb{S}^m \times \mathbb{S}^{n-m-1})$$
$$H_i(M(S), M_\circ) \cong H_i(\mathbb{D}^{m+1} \times \mathbb{S}^{n-m-1}, \mathbb{S}^m \times \mathbb{S}^{n-m-1}).$$

(d) Explain why the inclusion map $\mathbb{S}^a \times \mathbb{S}^b \hookrightarrow \mathbb{S}^a \times \mathbb{D}^{b+1}$ induces surjections on homology in every degree.
(*Hint*: Apart from degree 0, it is enough to show that a certain homology class in $H_*(\mathbb{S}^a \times \mathbb{D}^{b+1})$ is in the image.)
(e) You may from now on assume the following fact:

$$\widetilde{H}_i(\mathbb{S}^a \times \mathbb{S}^b) \;\cong\; \mathbb{Z}^{\delta_{i,a}} \oplus \mathbb{Z}^{\delta_{i,b}} \oplus \mathbb{Z}^{\delta_{i,(a+b)}},$$

where $\delta_{i,j}$ is the Kronecker delta function: $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ for $i \neq j$.
(If you like, try to prove this inductively using the Mayer-Vietoris sequence. Find an open cover $\{U, V\}$ of $\mathbb{S}^a \times \mathbb{S}^b$

such that $U \simeq V \simeq \mathbb{S}^a$ and $U \cap V \simeq \mathbb{S}^a \times \mathbb{S}^{b-1}$. For the base case, note that $\mathbb{S}^a \times \mathbb{S}^0 = \mathbb{S}^a \sqcup \mathbb{S}^a$.)
(f) Using parts (c)–(e), compute:

$$H_i(M, M_\circ) \cong \begin{cases} \mathbb{Z}^2 & i = n \text{ and } m = 0 \\ \mathbb{Z} & (i = n \text{ or } i = n - m) \text{ and } m \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H_i(M(S), M_\circ) \cong \begin{cases} \mathbb{Z}^2 & i = n \text{ and } m = n - 1 \\ \mathbb{Z} & (i = n \text{ or } i = m + 1) \text{ and } m \neq n - 1 \\ 0 & \text{otherwise} \end{cases}$$

(g) Assume that $n \geqslant 1$ and $m \leqslant \frac{n}{2}$. Use these calculations and (1) to show that in degrees $i \leqslant m - 2$,

$$H_i(M(S)) \;\cong\; H_i(M).$$

(h)* Now we consider a more specific example. Let M be a 7-manifold and let $S\colon \mathbb{S}^4 \times \mathbb{D}^3 \hookrightarrow M$ be a framed embedded 4-sphere. Show that

$$H_4(M(S)) \;\cong\; H_4(M)/\langle [c] \rangle,$$

where $[c]$ is the image under S_* of a generator of $H_4(\mathbb{S}^4 \times \mathbb{D}^3) \cong \mathbb{Z}$.

Exercise 12.6* (H-spaces and co-H-spaces.)
A based space C is called a *co-H-space*, if there is a map $\nabla\colon C \to C \vee C$ such that

$$p_i \circ \nabla \simeq \mathrm{id}_C \quad \text{for} \quad i = 1, 2$$

where p_1 and p_2 are the projections onto the first resp. second summand, which collapse the other summand to a point. One calls C *co-associative*, if

$$(\nabla \vee \mathrm{id}_C) \circ \nabla \simeq (\mathrm{id}_C \vee \nabla) \circ \nabla.$$

Example: the reduced suspension $C = \tilde{\Sigma} X$ of a based space X is a co-associative co-H-space.

By $\iota_i\colon C \to C \vee C$ for $i = 1, 2$ we will denote the inclusions of the left resp. right summand. With the same proof as in Exercise 12.1 we see that $(a, b) \mapsto \iota_{1*}(a) + \iota_{2*}(b)$ is an isomorphism $\Phi\colon H_n(C) \oplus H_n(C) \to H_n(C \vee C)$, for $n > 0$.

(1) Show that $\Phi^{-1} \circ \nabla_*\colon H_n(C) \to H_n(C) \oplus H_n(C)$ is the diagonal map.

For any two based maps $f, g\colon C \to C$ we can define their sum by $f + g := F \circ (f \vee g) \circ \nabla$, where $F\colon C \vee C \to C$ is the folding map.

(2) We have $(f + g)_* = f_* + g_*\colon H_n(C) \to H_n(C)$.

A based space M is called an *H-space*, if there is a map $\mu\colon M \times M \to M$, such that

$$\mu \circ \iota_i \simeq \mathrm{id}_M \quad \text{for} \quad i = 1, 2,$$

where $\iota_1\colon M \to M \times M$ sends m to (m, m_0), where m_0 is the basepoint of M, and similarly ι_2 sends m to (m_0, m). One calls M *associative*, if

$$\mu \circ (\mathrm{id}_M \times \mu) \simeq \mu \circ (\mu \times \mathrm{id}_M).$$

Example: A topological group, in particular a Lie group, is an H-space.

For a co-H-space C and a based space Y, we set $M := \mathrm{maps}_0(C, Y)$, the space of all based maps $f\colon C \to Y$. These are important spaces when C is a sphere.

(3) Show that M is an H-space, and it is associative if C is co-associative.