Aufgaben zur Topologie

Prof. Dr. C.-F. Bödigheimer

Wintersemester 2016/17

Week 13 — Products of spheres, linking number, cellular homology. Not due for handing-in. (Non-compulsory)

Exercise 13.1 (Homology of products of spheres.)
Recall from the lectures that, for spaces X and Y with "good" basepoints (this means that the basepoint is closed as a subset and also has an open neighbourhood that deformation retracts onto it), there are split short exact sequences

$$0 \to \widetilde{H}_i(X) \oplus \widetilde{H}_i(Y) \longrightarrow \widetilde{H}_i(X \times Y) \longrightarrow \widetilde{H}_i(X \wedge Y) \to 0$$

for $i \geqslant 0$. Note that the smash product $\mathbb{S}^n \wedge X$ is homeomorphic to the n-fold reduced suspension $\widetilde{\Sigma}^n X = \widetilde{\Sigma} \cdots \widetilde{\Sigma} X$.
(a) Use this fact, the above short exact sequences and the Suspension Theorem to show that

$$\widetilde{H}_i(\mathbb{S}^n \times X) \cong \begin{cases} \widetilde{H}_i(X) & 0 \leqslant i \leqslant n-1 \\ \mathbb{Z} \oplus \widetilde{H}_n(X) \oplus \widetilde{H}_0(X) & i = n \\ \widetilde{H}_i(X) \oplus \widetilde{H}_{i-n}(X) & i \geqslant n+1. \end{cases}$$

(b) Calculate the homology of a product of two spheres $\mathbb{S}^k \times \mathbb{S}^\ell$.
(c)* More generally, what is the homology of an iterated product of spheres $\mathbb{S}^{k_1} \times \cdots \times \mathbb{S}^{k_i}$?

Exercise 13.2 (Homology of knot complements.)
Let $f\colon \mathbb{S}^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{R}^3$ be a framed knot in Euclidean space, i.e., an embedding of $\mathbb{S}^1 \times \mathbb{D}^2$ into $\mathbb{R}^3$. The complement of its image, $M = \mathbb{R}^3 \smallsetminus f(\mathbb{S}^1 \times \mathbb{D}^2)$, is then a non-compact 3-manifold.
(a) Describe an open covering $\{U, V\}$ of $\mathbb{R}^3$ such that $U \simeq M$, $V \simeq \mathbb{S}^1$ and $U \cap V \simeq \mathbb{S}^1 \times \mathbb{S}^1$.
(b) Using the Mayer-Vietoris sequence for this covering, calculate the homology of the knot-complement M, in particular concluding that $H_1(M) \cong \mathbb{Z}$.
(c) Draw a 1-cycle μ representing a generator of $H_1(M)$.

Exercise 13.3 (Linking number.)
As in the previous exercise, let $f\colon \mathbb{S}^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{R}^3$ be a framed knot, write $K = f(\mathbb{S}^1 \times \mathbb{D}^2)$ and $M = \mathbb{R}^3 \smallsetminus K$. Fix a generator $[\mu]$ of $H_1(M) \cong \mathbb{Z}$ as in part (c) of the previous exercise. For any curve $c\colon \mathbb{S}^1 \to M$ we may define its *linking number with* K, denoted $L(c, K)$ or just $L(c)$, to be the unique integer such that

$$c_*([\omega_1]) = L(c, K).[\mu],$$

where $[\omega_1] \in H_1(\mathbb{S}^1)$ is a generator. Note that $L(c, K)$ depends on the choices of μ and ω_1. See the figure on the next page for an example.

Show:
(a) $L(c_1) = L(c_2)$, if $c_1 \simeq c_2 \colon \mathbb{S}^1 \to M$.
(b) $L(c) = 0$, if the image of c and K may be separated by a plane in $\mathbb{R}^3$.
(c) Suppose that $\Phi\colon \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ is an ambient isotopy, i.e., each $\Phi_t = \Phi(-, t)$ is a self-homeomorphism of $\mathbb{R}^3$ and Φ_0 is the identity. Then $L(c, K) = L(c', K')$, where $c' = \Phi_1 \circ c$ and $K' = \Phi_1(K)$, and we use the generator $[\mu'] = (\Phi_1)_*([\mu])$ of $H_1(\mathbb{R}^3 \smallsetminus K')$.
(d)* Now let $f_1, f_2 \colon \mathbb{S}^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{R}^3$ be two non-intersecting framed knots in $\mathbb{R}^3$. Let $K_i = f_i(\mathbb{S}^1 \times \mathbb{D}^2)$ and $c_i = f_i \circ c$, where $c\colon \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{D}^2$ is defined by $c(t) = (t, 0)$ (so c_i is the "core" of the framed knot f_i). Then we may define a difference map

$$D\colon \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{S}^2$$

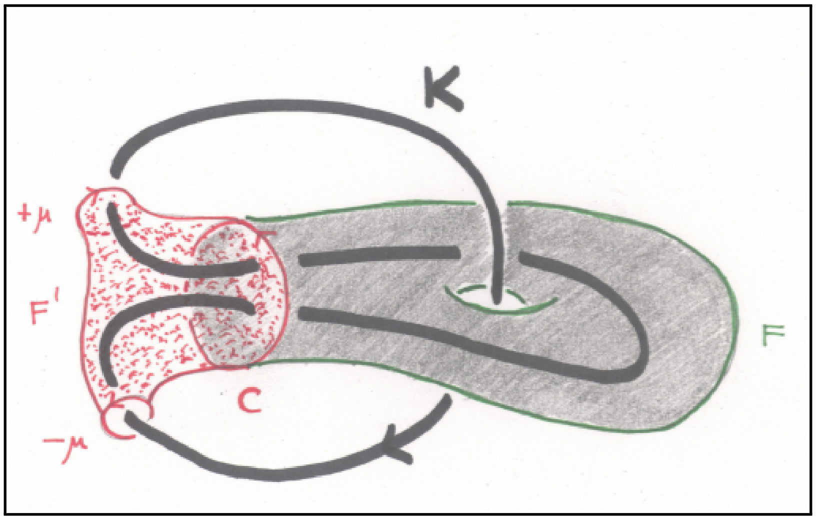


Figure for Exercise 13.3: The union of the curves K and C is the Whitehead link. Note that $L(C, K) = 0$. The surface F shows that C is nullhomologous in $\mathbb{R}^3 \setminus K$, and F' shows that C is homologous to $\mu + (-\mu) = 0$. (But still, K and C cannot be isotoped to curves separated by a plane.)

by the formula

$$D(t_1, t_2) \;=\; \frac{c_1(t_1) - c_2(t_2)}{\|c_1(t_1) - c_2(t_2)\|}.$$

Consider the induced homomorphism $D_*\colon H_2(\mathbb{S}^1 \times \mathbb{S}^1) \to H_2(\mathbb{S}^2)$. Prove that $D_* = 0$ if $L(c_1, K_2) = L(c_2, K_1) = 0$. See the figure above for an example.

Exercise 13.4 (Mapping degree for tori.)
We know that for the torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ we have $H_2(\mathbb{T}) \cong \mathbb{Z}$. So let us choose a generator $[\tau]$ of $H_2(\mathbb{T})$ and define the *mapping degree* of a self-map f of the torus to be the unique integer $\deg(f)$ such that

$$f_*([\tau]) = \deg(f).[\tau].$$

(a) This definition is independent of whether we choose $[\tau]$ or $-[\tau]$ as our generator.
(b) If f and g are homotopic, then $\deg(f) = \deg(g)$.
(c) If $f_1, f_2\colon \mathbb{S}^1 \to \mathbb{S}^1$ are two self-maps of the circle, and $f = f_1 \times f_2\colon \mathbb{T} \to \mathbb{T}$ is their product – a self-map of the torus – then we have:

$$\deg(f) \;=\; \deg(f_1) \cdot \deg(f_2).$$

Show this using the following steps (or via another argument if you prefer):
(i) We may assume without loss of generality that f_1 is the map $z \mapsto z^m$ and f_2 is $z \mapsto z^n$ for some $m, n \in \mathbb{Z}$.
(ii) Recall the comultiplication $\nabla\colon \mathbb{S}^2 \to \mathbb{S}^2 \vee \mathbb{S}^2$ and the fold map $F\colon \mathbb{S}^2 \vee \mathbb{S}^2 \to \mathbb{S}^2$ from Exercise 12.1. These may be iterated, leading to maps $\nabla_k\colon \mathbb{S}^2 \to \bigvee^k \mathbb{S}^2$ and $F_k\colon \bigvee^k \mathbb{S}^2 \to \mathbb{S}^2$. On second homology groups, we have

$$(\nabla_k)_*(1) = (1, \ldots, 1) \in \mathbb{Z}^k$$
$$(F_k)_*(0, \ldots, 0, 1, 0, \ldots, 0) = 1 \in \mathbb{Z}.$$

(iii) Let $A = \mathbb{S}^1 \vee \mathbb{S}^1 \subset \mathbb{S}^1 \times \mathbb{S}^1$. Then the quotient map $q\colon \mathbb{S}^1 \times \mathbb{S}^1 \to (\mathbb{S}^1 \times \mathbb{S}^1)/A \cong \mathbb{S}^2$ induces an isomorphism on $H_2(-)$.
(iv) Let $B \subset \mathbb{S}^1 \times \mathbb{S}^1$ be an $(m \times n)$ rectangular grid in the usual picture of the torus as a square with edge identifications. Then $(\mathbb{S}^1 \times \mathbb{S}^1)/B$ is homeomorphic to a wedge sum of mn copies of $\mathbb{S}^2$. Under this identification, the quotient map $\mathbb{S}^1 \times \mathbb{S}^1 \to (\mathbb{S}^1 \times \mathbb{S}^1)/B \cong \bigvee^{mn} \mathbb{S}^2$ is homotopic to $\nabla_{mn} \circ q$.
(v) The following diagram is commutative up to homotopy, and so the result follows.

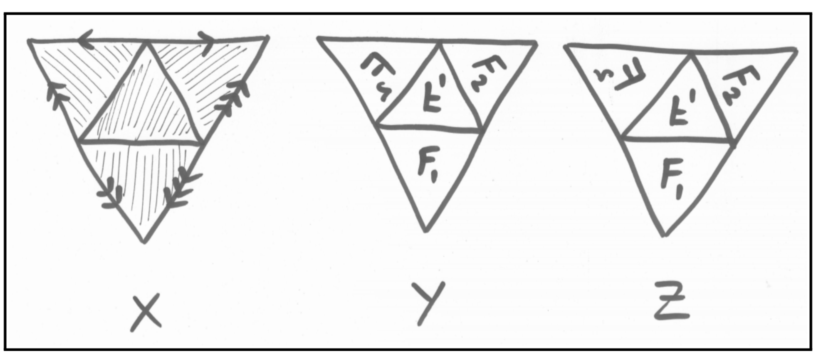
$$\begin{array}{ccccc} \mathbb{S}^1\times\mathbb{S}^1 & & \xrightarrow{\quad f = f_1\times f_2 \quad} & & \mathbb{S}^1\times\mathbb{S}^1 \\ \Big\downarrow q & & & & \Big\downarrow q \\ \mathbb{S}^2 & \xrightarrow{\nabla_{mn}} & \bigvee^{mn}\mathbb{S}^2 & \xrightarrow{F_{mn}} & \mathbb{S}^2 \end{array}$$

Exercise 13.5 (Cellular homology of quotients of the 3-simplex.)
Let X be the 3-simplex, the 2-skeleton of which is depicted on the left-hand side in the figure below, and identify its four faces in two pairs, as indicated in the middle part of the figure, to obtain a quotient space Y.
(a) Describe the natural cell complex structure on X and the induced structure on Y, with two 0-cells $\{P, Q\}$, three 1-cells $\{a, b, c\}$, two 2-cells $\{F_1, F_2\}$ and one 3-cell ω.
(b) Compute the differentials in the cellular chain complex

$$0 \leftarrow \mathbb{Z}\langle P, Q\rangle \leftarrow \mathbb{Z}\langle a, b, c\rangle \leftarrow \mathbb{Z}\langle F_1, F_2\rangle \leftarrow \mathbb{Z}\langle \omega\rangle \leftarrow 0 \leftarrow 0 \leftarrow \cdots$$

of Y, and thus compute its homology.
(c) Now identify the faces of the 3-simplex as indicated on the right-hand side of the figure, to obtain a quotient space Z. Describe the induced cell structure on Z, with one 0-cell, two 1-cells, two 2-cells and one 3-cell.
(d) Compute the cellular homology of Z.

Exercise 13.6* (Cellular homology of a quotient of the dodecahedron.)
Let X be the dodecahedron, which has a cell structure with 20 zero-cells, 30 one-cells, 12 two-cells and one three-cell. For each face, imagine pushing it through the interior of the dodecahedron until it lies in the same plane as the opposite face, and then rotating it by $\frac{\pi}{5}$ radians. This gives a homeomorphism between each pair of opposite faces. Let $\sim$ be the equivalence relation generated by $x \sim \phi(x)$, where ϕ is one of these homeomorphisms. Describe the induced cell structure on the quotient space $X/{\sim}$ and its cellular chain complex. Prove that the (cellular) homology of $X/{\sim}$ is the same as the homology of $\mathbb{S}^3$. This is the famous *Poincaré homology sphere.*



Figures for Exercise 13.5: The left-hand figure is the 2-skeleton (the union of all cells of dimension at most 2) of the 3-simplex X. The middle figure describes how to identify two pairs of faces of X to obtain the quotient space Y. Similarly, the right-hand figure shows how to identify the same two pairs of faces – in a *different* way – to obtain the quotient space Z.

Dodekaederraum.

Von Poincaréschen Räumen mit endlicher Fundamentalgruppe sind uns zwei bekannt. Ihre Fundamentalgruppen stimmen mit denen des Dodekaederraumes überein. Wir wissen nicht, ob die beiden Räume untereinander und mit dem Dodekaederraum homöomorph sind [31]). Dagegen gibt es außer der Fundamentalgruppe noch eine andere Eigenschaft, die Poincarés Poincaréscher Raum mit dem Dodekaederraum gemein hat, nämlich die Zerlegbarkeit in zwei Doppelringe.

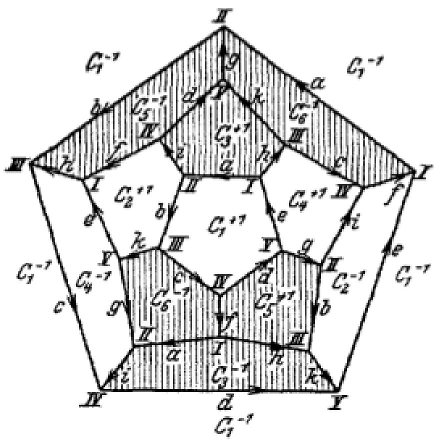


Fig. 11.

Wir benennen die Kanten und Flächenstücke des Diskontinuitätsbereiches, der uns den Dodekaederraum liefert, wie in der Fig. 11 (schematisches Netz der Dodekaederfläche) angegeben ist, und schreiben die wesentlichen Relationen an, die man aus den Kantenumläufen gewinnt. Wir sind dann nach § 10 sicher, daß die von den $\alpha_2 = 6$ Erzeugenden $C_1, \ldots, C_6$ und den $\alpha_1 = 10$ (den Kanten a), …, k) entsprechenden) wesentlichen Relationen definierte Gruppe mit der binären Ikosaedergruppe (§ 6, S. 26)

From W. Threlfall, H. Seifert, *Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes*, Math. Ann. (1931).