Aufgaben zur Topologie

Prof. Dr. C.-F. Bödigheimer

Wintersemester 2016/17

Week 1 — Exercises and Revision to be done by: 25.10.2016

Exercise 0.1 (Homotopy)
Recall the definition of homotopy and relative homotopy. Homotopy classes.

Exercise 0.2 (Free and based homotopy of maps $\mathbb{S}^1 \to \mathbb{S}^1$)
Show that, for self-maps of $\mathbb{S}^1$, free and based homotopy are the same.

Exercise 0.3 (Group structure of the fundamental group)
Recall the definition and verify the group axioms of $\pi_1(X, x_0)$.

Exercise 0.4 (Dependence on the fundamental group)
(a) Formulate correctly the phrase: $\pi_1(X, x_0)$ depends only on the path component of x_0.
(b) Formulate correctly the phrase: If X is path-connected, then $\pi_1(X, x_0)$ is independent of x_0.

Exercise 0.5 (Degree)
Recall the definition and the basic properties of the function $\mathbf{grad} : \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}$.

Exercise 0.6 (Applications)
Recall the statements and proofs of
(a) the fundamental theorem of algebra,
(b) the Brouwer fixed point theorem (in dimension 2),
(c) the Borsuk-Ulam theorem,
(d) the hairy ball theorem.

Aufgaben zur Topologie

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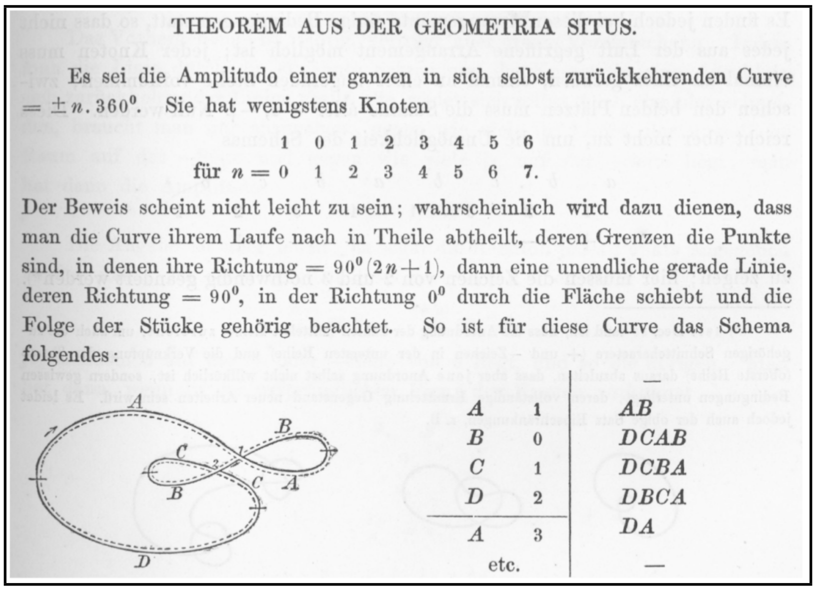
Wintersemester 2016/17

Week 2 — Umlauf, linking and winding numbers of curves to be done by: 02.11.2016



THEOREM AUS DER GEOMETRIA SITUS.

Es sei die Amplitudo einer ganzen in sich selbst zurückkehrenden Curve $= \pm n . 360^0$. Sie hat wenigstens Knoten

	1	0	1	2	3	4	5	6
für $n =$	0	1	2	3	4	5	6	7.

Der Beweis scheint nicht leicht zu sein; wahrscheinlich wird dazu dienen, dass man die Curve ihrem Laufe nach in Theile abtheilt, deren Grenzen die Punkte sind, in denen ihre Richtung $= 90^0 (2n+1)$, dann eine unendliche gerade Linie, deren Richtung $= 90^0$, in der Richtung 0^0 durch die Fläche schiebt und die Folge der Stücke gehörig beachtet. So ist für diese Curve das Schema folgendes:

		—
A	1	*AB*
B	0	*DCAB*
C	1	*DCBA*
D	2	*DBCA*
A	3	*DA*
etc.		—

Figure 1: From the notes of C.-F. Gauß, probably between 1823 and 1827.

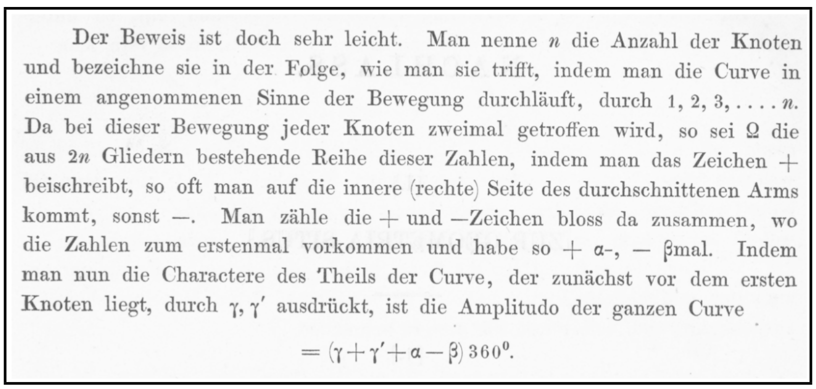
Exercise 1.1 ('Two-dimensional intermediate value theorem')
Let $f\colon \mathbb{D}^2 \to \mathbb{R}^2$ be a smooth function and $P \in \mathbb{R}^2$ a point such that $P \notin f(\mathbb{S}^1) = f(\partial \mathbb{D}^2)$. Recall the definition of the *umlauf number* $U(f,P)$ of the curve $f|_{\mathbb{S}^1}\colon \mathbb{S}^1 \to \mathbb{R}^2$ about the point P. Assuming that $U(f,P) \neq 0$, show that there exists a point $z \in \mathbb{D}^2$ such that $f(z) = P$.

Exercise 1.2 (Linking numbers of curves in $\mathbb{R}^3$)
(a) Let L be any one-dimensional affine subspace of $\mathbb{R}^3$ and let $f\colon \mathbb{S}^1 \to \mathbb{R}^3$ be a smooth curve which is disjoint from L. By considering a projection of f onto a plane orthogonal to L, define the *umlauf number* $U(f,L)$ of f about L.
(b) Show that if f and g are curves in the complement of L that are homotopic via a homotopy that never intersects L, then $U(f,L) = U(g,L)$.
(c) Fix a smooth curve $f\colon \mathbb{S}^1 \to \mathbb{R}^3$ and let $H\colon \mathbb{R} \times [0,1] \to \mathbb{R}^3$ be a continuous function that never intersects f and has the property that for each fixed t the function $H(-,t)\colon \mathbb{R} \to \mathbb{R}^3$ is an affine injection. Denote by L_t the one-dimensional affine subspace $H(\mathbb{R} \times \{t\})$ of $\mathbb{R}^3$. Show that $U(f,L_0) = U(f,L_1)$.
(d) We return to the situation where we have a fixed one-dimensional affine subspace L of $\mathbb{R}^3$ and a smooth curve $f\colon \mathbb{S}^1 \to \mathbb{R}^3$ in its complement. Consider a two-dimensional half-space with L as its boundary. (There are many possibilities – they are all related to each other by a rotation about L.) Define what it means for the half-space to *intersect f transversely*, and, if it does, the *sign* of the intersection. How can one calculate $U(f,L)$ using these signed intersections ?
(e)* Now let C be the unit circle in a two-dimensional affine subspace of $\mathbb{R}^3$. Find a continuous function $K\colon \mathbb{S}^3 \times [0,1] \to \mathbb{S}^3$ such that (i) each $K(-,t)$ is a diffeomorphism of $\mathbb{S}^3$, (ii) $K(-,0)$ is the identity on $\mathbb{S}^3$ and (iii) $K(C,1)$

is a one-dimensional affine subspace of $\mathbb{R}^3$ together with the point at infinity.
(f)* Using this, define the *umlauf number* $U(f, C)$ of any smooth curve $f\colon \mathbb{S}^1 \to \mathbb{R}^3$ which does not intersect C.
(g)* Consider a two-dimensional disc embedded into $\mathbb{R}^3$ in such a way that its boundary is C. Similarly to part (d), define a *transverse intersection* of f with the disc, the *sign* of such an intersection and show how one can compute $U(f, C)$ using these concepts.

Exercise 1.3 (Circumnavigating the world)
Let B be a sailing boat on the surface of the world $\mathbb{S}^2$, and assume that it never visits the north or the south pole. What does it mean for B to *circumnavigate the world* ? If you are given a map of its journey (which starts and ends at the basepoint Hamburg $\in \mathbb{S}^2$), how can you use lines of longitude to detect whether it has really circumnavigated the world ?

Exercise 1.4 (Local index of vector fields)
For each of the following sets of integers, give a vector field V on $\mathbb{S}^2$ with precisely k zeros $\zeta_1, \ldots, \zeta_k$ having the listed integers as their local indices $\mathrm{ind}(V, \zeta_i)$:
(a) $k = 2$ and indices $1, 1$,
(b) $k = 1$ and index 2,
(c) $k = 4$ and indices $1, 1, 1, -1$,
(d) $k = 2$ and indices $2, 0$,
(e) $k = 7$ and indices $1, 1, 1, 1, 1, -1, -2$.
(Do this by a drawing.)


Der Beweis ist doch sehr leicht. Man nenne n die Anzahl der Knoten und bezeichne sie in der Folge, wie man sie trifft, indem man die Curve in einem angenommenen Sinne der Bewegung durchläuft, durch $1, 2, 3, \ldots . n$. Da bei dieser Bewegung jeder Knoten zweimal getroffen wird, so sei Ω die aus $2n$ Gliedern bestehende Reihe dieser Zahlen, indem man das Zeichen $+$ beischreibt, so oft man auf die innere (rechte) Seite des durchschnittenen Arms kommt, sonst $-$. Man zähle die $+$ und $-$Zeichen bloss da zusammen, wo die Zahlen zum erstenmal vorkommen und habe so $+\ \alpha$-, $-\ \beta$mal. Indem man nun die Charactere des Theils der Curve, der zunächst vor dem ersten Knoten liegt, durch γ, γ' ausdrückt, ist die Amplitudo der ganzen Curve

$$= (\gamma + \gamma' + \alpha - \beta)\, 360^0.$$

Figure 2: A little later in the notes of C.-F. Gauß.

Exercise 1.5 (Symmetric vector fields on spheres)
In this exercise we will take the point of view that a tangent vector field on $\mathbb{S}^2 \subset \mathbb{R}^3$ is a continuous function $V\colon \mathbb{S}^2 \to \mathbb{R}^3$ with the property that x and $V(x)$ are orthogonal vectors for each $x \in \mathbb{S}^2$.
(a) Suppose that $V(x) = V(-x)$ for each $x \in \mathbb{S}^2$. If x is a zero of V, what is the relationship between the local index of V at x and the local index of V at $-x$?
(b) Now fix an angle $\theta \in [0, 2\pi]$. Consider the following equation (where $x \in \mathbb{S}^2$):

$$V(-x) \;=\; \cos(\theta)\, V(x) + \sin(\theta)\, (x \times V(x)), \tag{1}$$

where $\times$ denotes the cross product in $\mathbb{R}^3$. Show that (1) holds for x if and only if it holds for $-x$.
(c) Now assume that (1) holds for all $x \in \mathbb{S}^2$. If x is a zero of V, what is the relationship between the local index of V at x and the local index of V at $-x$? How does this relationship depend on θ?
(d) Now let V be an arbitrary vector field (not necessarily satisfying (1)) and define another vector field V_θ for $\theta \in [0, 2\pi]$ by

$$V_\theta(x) \;=\; \cos(\theta)\, V(x) + \sin(\theta)\, (x \times V(x)).$$

What are the zeros of V_θ in comparison to the zeros of V? How is the local index of a zero of V_θ determined by the local indices of the zeros of V?

(e)* Now let $F \subset \mathbb{R}^3$ be a torus, symmetrically embedded into $\mathbb{R}^3$ such that $x \in F$ implies that $-x \in F$ and the tangent spaces T_xF and $T_{-x}F$ are parallel. Let us say that it is the z-axis that passes through the 'hole' in the torus (so F is disjoint from the z-axis).
We will think of vector fields on F as continuous functions $V\colon F \to \mathbb{R}^3$ with the property that $V(x)$ is parallel to T_xF for each $x \in F$. Let $p\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the projection onto the (x, y)-axis. Show that, when $x \in F$, the vector $W(x) = p(x) \times (0, 0, 1)$ always lies in the tangent plane T_xF, where $\times$ is the cross product in $\mathbb{R}^3$. This is therefore an example of a non-vanishing vector field on F. Now suppose that V is any vector field on F satisfying the following property: for each $x \in F$,

$$V(-x) \;=\; \text{the reflection of } V(x) \text{ about the axis } \{\lambda.W(x) \mid \lambda \in \mathbb{R}\} \text{ in the tangent plane } T_xF.$$

If x is a zero of V, what is the relationship between the local index of V at x and the local index of V at $-x$? Give an example of a vector field V that has zeros and satisfies this property.

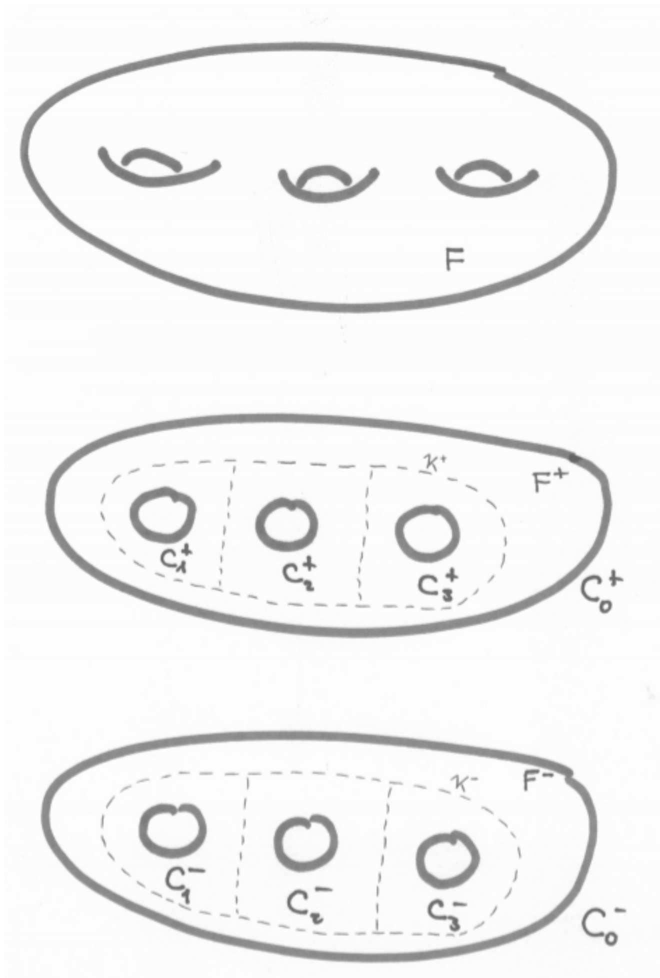


Figure 3: The surface F from Exercise 1.6.

Exercise 1.6 ('Satz vom Igel' for higher-genus surfaces)
The orientable surface F of genus g we imagine as smoothly embedded in $\mathbb{R}^3$ (see Figure 3 on page 3), symmetric with respect to the origin and to the (x, y)-plane. The intersection with this plane consists of an outer curve C_0 and g further curves $C_1, \ldots, C_g$. We have an upper resp. lower half-surface F^+ and F^-, both a disc with an outer

boundary curve C_0^+ resp. C_0^- and g inner boundary curves $C_1^+, \ldots, C_g^+$ resp. $C_1^-, \ldots, C_g^-$.
(a) Show that $T(F)$ is trivial over both F^+ and F^-. (Move a point ζ towards the point on the dotted core curve K^+ resp. K^- closest to ζ and slide the tangent plane along. The curves K^+ and K_- lie in planes parallel to the (x, y)-plane, where all tangent planes have a canonical isomorphism to $\mathbb{R}^2$.) In the end we have for each curve a clutching function $\phi_i \colon C_i \to \mathrm{GL}_2(\mathbb{R})$, $i = 0, 1, \ldots, g$.
(b) In the case of a torus, $g = 1$, conclude that the tangent bundle is trivial over the whole surface.
(c) Now assume a vector field is given. This vector field is a function $V \colon F \to \mathbb{R}^3$ such that $V(\zeta)$ is tangent to F at ζ, i.e., $V(\zeta) \in T_\zeta(F)$. What can we conclude about the zeroes of V? If we assume that V has no zeroes; then — since F is orientable — in each tangent plane we get by left-rotation of $V(\zeta)$ with angle $90°$ a second and linearly independent vector and thus a basis in each tangent space. Restricting to the curves C_i we have functions $\psi_i \colon C_i \to \mathrm{GL}_2(\mathbb{R})$. But these functions must satisfy relations. What are these relations, and what do they imply for non-vanishing vector fields on F?

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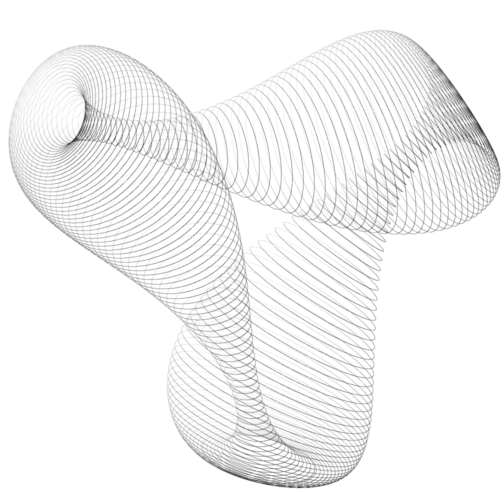
Week 2 — Covering spaces to be handed in on 09.11.2016 (before the lecture)



A 3-fold covering of the Klein bottle.

Exercise 2.1 (The higher spheres are simply-connected: $\pi_1(\mathbb{S}^n, x_0) = 1$ for $n \geq 2$.)

We prove this by showing in several steps, that any closed curve $\alpha\colon [0,1] \to \mathbb{S}^n$ with $\alpha(0) = \alpha(1) = x_0$ is contractible relative to $\{0,1\}$. In most steps we use a homeomorphism $(\mathbb{S}^n - \{P\}, x_0) \to (\mathbb{R}^n, 0)$ of pointed spaces.

(1) If α does not cover the entire sphere, then α is contractible relative to $\{0,1\}$. (N.B. There are curves (e.g. the Peano curves), which cover an entire sphere, even for $n > 1$.)

(2) There are finitely many $0 = t_0 < t_1 < \ldots < t_m = 1$, such that, for each curve $\alpha([t_k, t_{k+1}])$, there is (at least) one of the 2(n+1) open half-spheres $U_i^{\pm} := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \pm x_i > 0\}$, that entirely contains it, $k = 0, \ldots, m-1$, $i = 1, \ldots, n+1$. (This is an application of the Lemma of Lebesgue.)

(3) Any path $\beta\colon [a,b] \to \mathbb{S}^n$ that is contained in some half-sphere $U_i^{\pm}$ is homotopic, in $U_i^{\pm}$ and relative to $\{a,b\}$, to a path running along a section of the great circle from $\beta(a)$ to $\beta(b)$.

(4) Given (2), and using (3), there is a homotopy relative to $\{0,1\}$ between α and a closed path γ, which runs piecewise along sections of finitely many great circles.

(5) Since a path γ as in (4) satisfies the condition of (1), we are done. Where did we use $n > 1$?

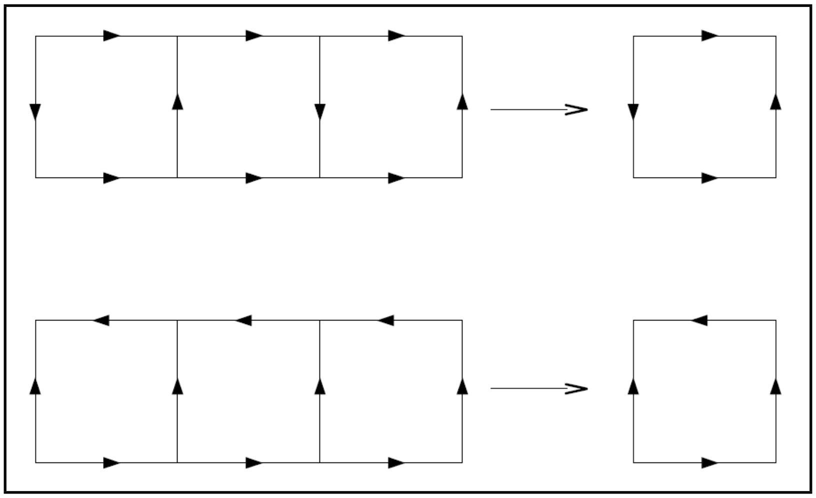
Exercise 2.2 (Coverings of the figure-eight space)

Find all 2-fold and 3-fold coverings of the figure-eight space $X = \mathbb{S}^1 \vee \mathbb{S}^1$: first classify all coverings, connected or disconnected, by giving two permutations; then sort by the number of connected components.

Exercise 2.3 (Sums, products and compositions of coverings)

(1) The sum $\tilde{X} \sqcup \tilde{Y}$ of two coverings $\tilde{X} \to X$ and $\tilde{Y} \to Y$ is a covering of the sum $X \sqcup Y$.

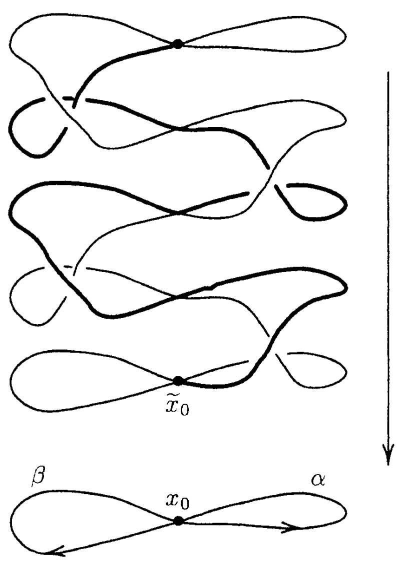
(2) The product $\tilde{X} \times \tilde{Y}$ of two coverings $\tilde{X} \to X$ and $\tilde{Y} \to Y$ is a covering of the product $X \times Y$.



Two 3-fold coverings of the Klein bottle.

(3) If $\tilde{X} \to X$ and $\tilde{\tilde{X}} \to \tilde{X}$ are two finite coverings, then the composition $\tilde{\tilde{X}} \to X$ is a covering.

Exercise 2.4 (Klein bottle covering itself.)
Show that the two maps in the figure above are 3-fold coverings of the Klein bottle K.



A 5-fold covering of the figure-eight space.
From A.Fomenko, D.Fuchs: *Homotopical Topology*, p.70.

Exercise 2.5 (A non-commutative fundamental group.)
The fundamental group $\pi_1(X, x_0)$ of the figure-eight space $X = \mathbb{S}^1 \vee \mathbb{S}^1$ is non-abelian. Assume it were commutative; consider the commutator $[\gamma] := [\alpha]\,[\beta]\,[\alpha]^{-1}\,[\beta]^{-1}$, where α resp. β is the closed curve running counter-clockwise along the right resp. clockwise along the left leaf of the bouquet $\mathbb{S}^1 \vee \mathbb{S}^1$, as in the figure above. If $[\gamma]$ were the trivial element, the lift $\widetilde{\gamma}$ of $\gamma = \alpha * \beta * \bar{\alpha} * \bar{\beta}$ in any covering $\tilde{X} \to X$ would be a closed curve.

Exercise 2.6 (Some ∞-fold coverings.)

As in the previous exercise, let $X = \mathbb{S}^1 \vee \mathbb{S}^1$ be the figure-eight space and let a and b be the closed curves described above. For each of the following subgroups G of $\pi_1(X, x_0)$, draw a covering $\tilde{X} \to X$ with the property that the image of the induced map of fundamental groups $\pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is G.

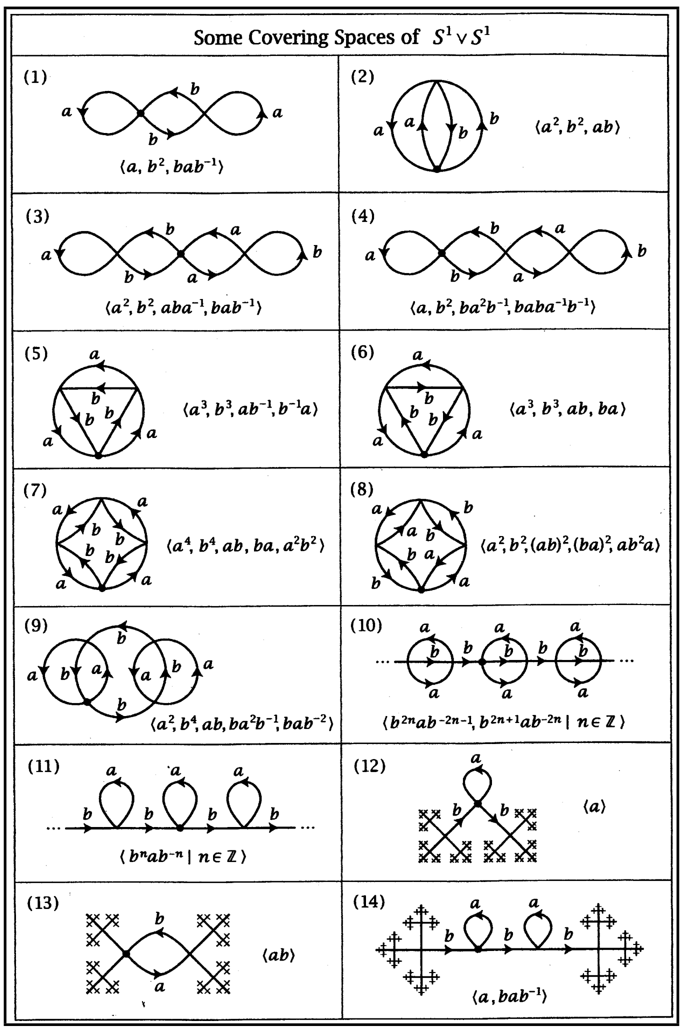
(a) $G =$ the normal subgroup generated by $[a]$.

(b) $G =$ the normal subgroup generated by the element $[c]$ defined in the previous exercise.

(c) $G =$ the subgroup generated by $[c]$.

(d) Now let w be any finite sequence of elements of the set $\{a, b, \bar{a}, \bar{b}\}$ and take $G =$ the subgroup generated by $[w]$.

In each case, once you have constructed a covering $\tilde{X} \to X$ which potentially corresponds to the correct subgroup G, what you need to check is that a based loop in X lifts to a *loop* (not just a path) in $\tilde{X}$ if and only if it represents an element of $\pi_1(X, x_0)$ that lies in G.



Some coverings of the figure-eight space.

From A.Hatcher: *Algebraic Topology*, Cambridge Univ. Press 2002, p. 58.

Aufgaben zur Topologie

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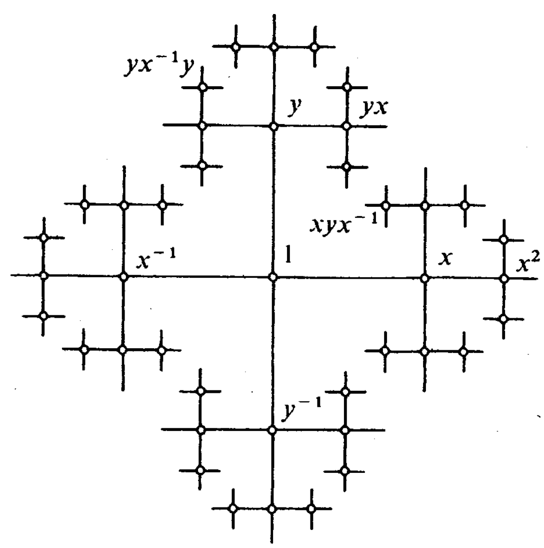
Wintersemester 2016/17

Week 3 — Coverings to be done by: 15.11.2016

Exercise 3.1 (Local properties of covering spaces)
A map $f\colon Y \to X$ is a *local homeomorphism* if, for each $y \in Y$, there exists an open neighbourhood U of y such that $f(U)$ is open in X and the restriction $f|_U\colon U \to f(U)$ of f to U is a homeomorphism. Suppose that $f\colon Y \to X$ is a local homeomorphism. Show for each of the following properties that if X has this property, then so does Y.
(a) locally connected,
(b) locally path-connected,
(c) locally compact.
Now let $\xi\colon \tilde{X} \to X$ be a covering.
(d) Show that ξ is a local homeomorphism.
Show for each of the following properties that if X has this property, so does $\tilde{X}$.
(e) Hausdorff,
(f) compact, if – in addition – the fibre is finite.

Exercise 3.2 (A covering space of a manifold is a manifold.)
Let $\xi\colon \tilde{M} \to M$ is a covering with finite or countable fibre. If M is a (differentiable, C^r, smooth, holomorphic, ...) manifold, then so is $\tilde{M}$. If M is orientable, then so is $\tilde{M}$ (however, the reverse implication does not hold).



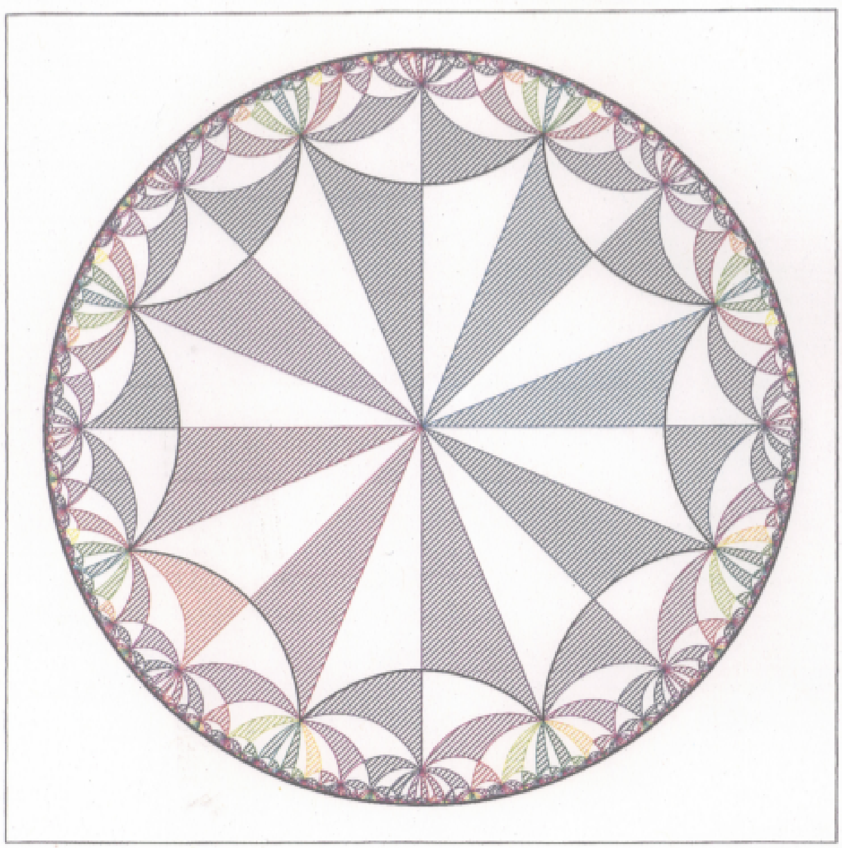
Universal cover of the figure-eight space $\mathbb{S}^1 \vee \mathbb{S}^1$

Exercise 3.3 (Properly discontinuous group actions)
Let the discrete group G act on a space Y and denote the action by $(g, y) \mapsto g\,.\,y$. The action is said to be *properly discontinuous*, if for each $y \in Y$ there is a neighbourhood U, such that $(g\,.\,U) \cap U = \emptyset$ for all but finitely many $g \in G$. If Y is Hausdorff and the action is free, then this is equivalent to the statement that, for each $y \in Y$, there

is a neighbourhood U, such that $(g \centerdot U) \cap U = \emptyset$ for all $g \in G$ except the identity element. (Why is this true?)

(a) For a free and properly discontinuous action on a Hausdorff space Y, the quotient map $\xi \colon Y \to X := Y/G$ is a covering with fibre G.
(b) Example: $G = \mathbb{Z}$ and $Y = \mathbb{C} - \{0\}$; for the action fix a complex number $\lambda \neq 0$ and set $n \centerdot z := \lambda^n z$. For which λ is this action free, for which is it properly discontinuous ? What is the quotient ?
(c) Example: Let $G = \mathrm{Fr}(2) = \langle x, y \mid \ \rangle$ denote the free (non-abelian) group on two letters x, y. The figure above shows its Cayley graph C, which is a tree and which we regard as a subspace of the plane. The vertices are reduced words w in the letters x, y (and their inverses) and we denote this vertex by (w); an edge between the vertices (w) and (w') we denote by (w, w') and such an edge exists iff $w^{-1} w'$ is x or y or x^{-1} or y^{-1}. Thus there is a vertex for each group element, and there are four edges emanating from each vertex. C is contractible and thus simply-connected. The right-action of G on C is described for a $g \in G$ as follows: for a vertex we set $(w) \centerdot g = (wg)$; for an edge we set $(w, w') \centerdot g = (wg, w'g)$. Note that as a continuous map the action by g must dilate the lengths of the edges. Show that the action is free and properly discontinuous. And show that the quotient C/G is the figure-eight space $\mathbb{S}^1 \vee \mathbb{S}^1$. Conclude $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) = \mathrm{Fr}(2)$.

Figure 12.3. A tiling of the hyperbolic plane modelled by the circular disk. From the hyperbolic point of view, all the 'triangles' are exactly the same shape. The (orientation-preserving) symmetries are Möbius maps which preserve this tiling and map any coloured triangle onto any other. This and the following figure were drawn by our DFS tiling program with the aid of an automaton provided by the program KBMAG by Derek Holt. The pattern of the colouring relates to the automaton.



A properly discontinuous and free action on the hyperbolic plane $\mathbb{H}^2$. Each semicircle S in the picture (together with a choice of orientation) defines an isometry of $\mathbb{H}^2$ given by translation parallel to S by an amount such that each vertex on S is moved forwards by two steps, i.e., it is sent to the next-but-one vertex in the direction given by the orientation. These hyperbolic translations generate a discrete subgroup G of the group $\mathrm{Isom}(\mathbb{H}^2)$ of all hyperbolic isometries of $\mathbb{H}^2$, which acts properly discontinuously and freely. Image credit: *Indra's Pearls*, David Mumford, Caroline Series and David Wright.

Exercise 3.4 (The pull-back of a covering is a covering.)
Let $\xi\colon \tilde{X} \to X$ be a covering and $f\colon Y \to X$ be any map. The *pull-back of ξ along f* consists of the space $\tilde{Y} = f^*(\tilde{X}) := \{(y,\tilde{x}) \in Y \times \tilde{X} \mid f(y) = \xi(\tilde{x})\}$ (with the subspace topology of the product topology) together with two maps $\tilde{\xi} = f^*(\xi)\colon \tilde{Y} = f^*(\tilde{X}) \to Y$, defined by $(y,\tilde{x}) \mapsto y$ and $\tilde{f}\colon \tilde{Y} = f^*(\tilde{X}) \to \tilde{X}$, defined by $(y,\tilde{x}) \mapsto \tilde{x}$. Thus we have a commutative square:

$$\begin{tikzcd} \tilde{Y} = f^*(\tilde{X}) \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\xi}=f^*(\xi)"'] & \tilde{X} \arrow[d, "\xi"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

(a) $\tilde{\xi}\colon \tilde{Y} \to Y$ is a covering with the same fibre as ξ.

(b) The following formulae hold, where $g\colon Z \to Y$ and $f\colon Y \to X$:

(1) $g^*(f^*(\tilde{X})) = (f \circ g)^*(\tilde{X})$ and $g^*(f^*(\xi)) = (f \circ g)^*(\xi)$;

(2) $\mathrm{id}^*(\tilde{X}) = \tilde{X}$ and $\mathrm{id}^*(\xi) = \xi$.

(c) If $\xi\colon \mathbb{S}^2 \to \mathbb{R}\mathrm{P}^2$ is the antipodal projection and $f\colon \mathbb{R}\mathrm{P}^2 - \{P\} \to \mathbb{R}\mathrm{P}^2$ the inclusion, then the pull-back $\tilde{\xi}$ is the covering of the Möbius-band by a band.
(d) Let $\xi_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ denote the n-fold covering $z \mapsto z^n$. What is $\xi_n^*(\xi_m)$? And what is $\xi_n \circ \xi_m$ from Exc. 2.3 ?

Exercise 3.5 (Maps out of spaces with finite fundamental group)
Let X be a path-connected (locally path-connected) space with basepoint x_0 such that $\pi_1(X, x_0)$ is finite.
(a) Show that any continuous map $X \to \mathbb{S}^1$ must be nullhomotopic.
(b) Using part (a), show that any continuous map $X \to \prod^k \mathbb{S}^1$ to a product of k copies of the circle $\mathbb{S}^1$ must be nullhomotopic.
(c) Show that any continuous map $X \to \bigvee^\ell \mathbb{S}^1$ to a wedge of copies of ℓ copies of the circle $\mathbb{S}^1$ must be nullhomotopic.
(Use Exercise 3.3(c) when $\ell = 2$. For larger values of ℓ you will need an appropriate generalisation of that exercise.)

Exercise 3.6* (The fundamental groupoid and fibre-transport)

A *groupoid* is a category in which every morphism is invertible. It earns its name from the following example. If G is a discrete group, then it may be considered as a groupoid $\mathcal{B}G$ with one object $\bullet$ and with the morphism set $\mathrm{Hom}_{\mathcal{B}G}(\bullet,\bullet) = \mathrm{Aut}_{\mathcal{B}G}(\bullet)$ equal to G. Another example that one may build from G is $\mathcal{E}G$, which has one object for each element of G, and exactly one morphism between any pair of objects (there is then only one possible way in which composition may be defined).

If we imagine a category $\mathcal{C}$ (and therefore in particular a groupoid) as having a vertex for each object, an edge for each morphism, a triangle for each pair (f,g) of morphisms such that source(f) = target(g), *etc.. One should think of f as a 'connection' between two objects, materialized as an edge, g likewise as another edge and $f \circ g$ as a third edge; and the fact that the third edge stands for the compostion $f \circ g$ and not just for any morphisms, we materilaize as a triangle — etc., etc.. The resulting space is called the classifying space* $\mathrm{B}\mathcal{C}$ *of the category $\mathcal{C}$. For $\mathcal{C} = \mathcal{E}G$ this space turns out to be a simplex of cardinality $|G|$ (which may be infinite).*

A third example, which may be built out of any topological space X, is the following. We take one object for each point $x \in X$. A morphism from x to y is then defined to be a homotopy class of paths in X starting at x and ending at y, where "homotopy" means homotopy of maps $[0,1] \to X$ relative to $\{0,1\}$. It is obvious, what the identity morphisms are and what the composition of two morphisms is. The result is a groupoid (why ?), denoted $\Pi(X)$.
(1) What is the group of automorphisms of the object $x \in X$?

(2) Show that any continuous map $f\colon X \to Y$ induces a functor $\Pi(f)\colon \Pi(X) \to \Pi(Y)$. We have $\Pi(g \circ f) = \Pi(g) \circ \Pi(f)$. Furthermore, if $F\colon X \times [0,1] \to Y$ is a homotopy between the maps $f_0(x) = F(x,0)$ und $f_1(x) = F(x,1)$, the F allows us to define a transformation from the functor $\Pi(f_0)$ to the functor $\Pi(f_1)$.

Now let $\xi\colon \tilde{X} \to X$ be a covering.
(3) Show that the unique path-lifting property allows us to define a "fibre-transport" functor

$$\mathrm{trans}\colon \Pi(X) \longrightarrow \mathrm{Sym},$$

where Sym is the category of sets and bijections, such that a point $x \in X$, i.e., an object of $\Pi(X)$, is taken to its fibre $\xi^{-1}(x)$.

Aufgaben zur Topologie

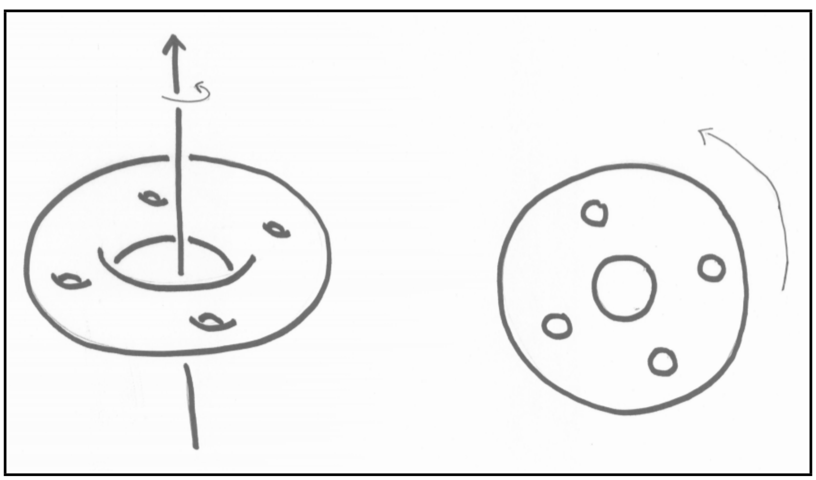
Prof. Dr. C.-F. Bödigheimer

Wintersemester 2016/17

Week 4 — Coverings and deck transformations Due: 23.11.2016 (before the lecture)

Exercise 4.1 (Fundamental group of the Klein bottle)
Consider $\tilde{X} = \mathbb{R}^2$ with the translation $A\colon (x, y) \mapsto (x + 1, y)$ and the glide reflection $B\colon (x, y) \mapsto (-x, y + 1)$. Let G denote the subgroup of affine motions of $\mathbb{R}^2$ generated by A and B.
(1) This action is free and properly discontinuous.
(2) A and B satisfy the relation $ABA = B$.
(3) G is isomorphic to the semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}$, where $b \in \mathbb{Z}$ acts on $\mathbb{Z}$ via $b \centerdot a := (-1)^b\, a$.
(4) The quotient $\xi\colon \tilde{X} \to X := \tilde{X}/G$ is the Klein bottle.
(5) Thus $\pi_1(K) \cong \mathbb{Z} \rtimes \mathbb{Z}$.

Exercise 4.2 (Connectivity and sections)
If $\xi_1\colon \tilde{X}_1 \to X$ and $\xi_2\colon \tilde{X}_2 \to X$ are two coverings of X, we denote by $\xi_1 \oplus \xi_2\colon \tilde{X}_1 \sqcup \tilde{X}_2 \to X$ their *sum over* X or *Whitney sum*. Show that a covering $\xi\colon \tilde{X} \to X$ has a section if and only if it is isomorphic to the sum $\xi' \oplus \mathrm{id}_X$ of some covering ξ' and the identity. Conclude that a connected covering with $k > 1$ leafs has no section. (Example: $\mathbb{S}^n \to \mathbb{R}\mathrm{P}^n$.)



The surface $F_5 \subset \mathbb{R}^3$ from Exercise 4.3(4c). The right-hand side is a bird's-eye view from above.

Exercise 4.3 (Change of fibres)
Let the discrete group G act (from the right) freely and properly discontinuously on the Hausdorff space $\tilde{X}$; thus we have a covering $\xi\colon \tilde{X} \to X := \tilde{X}/G$ with fibre G. Now assume that G also acts (from the left) on the discrete space F, not necessarily freely. We define on the space $\tilde{Y} = \tilde{X} \times F$ the "diagonal" action $g \centerdot (x, u) := (x \centerdot g^{-1}, g \centerdot u)$, and define Y to be the quotient space $\tilde{Y}/G$, which may also be denoted $\tilde{X} \times_G F$.
(1) The projection $\tilde{Y} \to \tilde{X}$ is G-equivariant, so we have a well-defined induced map $\xi_F\colon Y \to X$.
(2) This induced map is a covering with fibre F.
(3) If $F = G$ with the G-action given by left-multiplication, then $Y = \tilde{X}$ and $\xi_F = \xi$.

(4) Examples:

(a) If $\tilde{X} = \mathbb{S}^2$ with $G = \mathbb{Z}/2\mathbb{Z}$ acting by the antipodal map, then $\tilde{X} \to X$ is the two-fold covering $\mathbb{S}^2 \to \mathbb{RP}^2$. Now take $F = \mathbb{Z}/4\mathbb{Z}$ with the generator of G acting by multiplication by -1. Show that $Y \to X$ is the Whitney sum of two copies of the identity $\mathbb{RP}^2 \to \mathbb{RP}^2$ and one copy of $\mathbb{S}^2 \to \mathbb{RP}^2$. What about if $F = \mathbb{Z}/n\mathbb{Z}$ for other positive integers n, with G again acting by multiplication by -1?

(b) Let $\tilde{X} = \mathbb{R}$ with $G = \mathbb{Z}$ acting by translation; $F = \{e^{2\pi \mathrm{i} \frac{k}{n}} \mid k = 0, \ldots, n-1\}$ with $1 \in \mathbb{Z}$ acting by rotation by $\frac{360}{n}$ degrees. What is the covering $Y \to X$ in this case?

(c) If we imagine the surface F_{g+1} embedded into $\mathbb{R}^3$ in a symmetrical way, with one large hole and g smaller holes arranged around it at equal intervals, and with the z-axis running through the large hole – see the figure on the previous page for the case $g = 4$ – then there is an action of the cyclic group $\mathbb{Z}/g\mathbb{Z}$ on it given by rotation in the (x, y)-plane by $\frac{360}{g}$ degrees. This action is free and properly discontinuous and the quotient is homeomorphic to F_2. Denote the resulting covering by $\varphi_{g+1} \colon F_{g+1} \to F_2$.

Now take $g = 4$, let $\tilde{X} = F_5$ and let $G = \mathbb{Z}/4\mathbb{Z}$ acting as described above. So $\tilde{X} \to X$ is the covering $\varphi_5 \colon F_5 \to F_2$. Let F be the set $\{-1, 1\}$ with the action of $G = \mathbb{Z}/4\mathbb{Z}$ where a generator acts by multiplication by -1. Show that the new covering $Y \to X$ is $\varphi_3 \colon F_3 \to F_2$.

(d)* More generally, if m and n are positive integers, let $\tilde{X} = F_{mn+1}$ with $G = \mathbb{Z}/mn\mathbb{Z}$ acting as described above. So $\tilde{X} \to X$ is φ_{mn+1}. Let $F = \{e^{2\pi \mathrm{i} \frac{k}{n}} \mid k = 0, \ldots, n-1\}$ with the action of $G = \mathbb{Z}/mn\mathbb{Z}$ in which 1 acts by rotation by $\frac{360}{n}$ degrees. Then $Y \to X$ is φ_{n+1}.

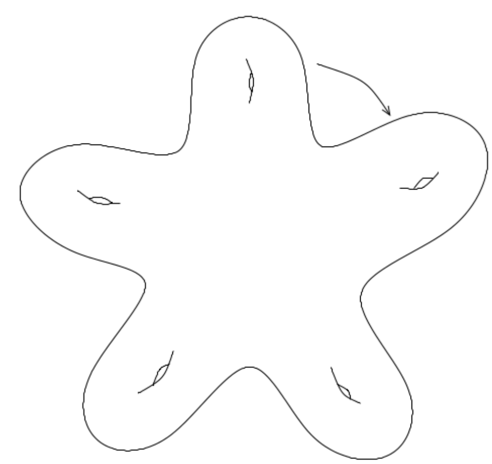


Figure 2.1 An order 5 element of $\mathrm{Mod}(S_5)$.

The surface F_5 has a 5-fold symmetry, as well as the 4-fold symmetry used in Exercise 4.3(4c).
Image credit: *A Primer on Mapping Class Groups*, B. Farb, D. Margalit, page 46.

Exercise 4.4 (Universal coverings of closed, orientable surfaces)

(a) Show that the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ given by addition in each coordinate is free and properly discontinuous. Deduce that the universal cover of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ is contractible. (See part (a) of Exercise 3.3.)

(b) Consider the diagram in the hyperbolic plane $\mathbb{H}^2$ on the next page, which is to be visualised using the Poincaré disc model. The tiling is constructed as follows. Start with an octagon, centred at the middle of the disc, with 90-degree internal angles. Reflecting in L and then in M as shown on the right-hand side of the figure results in another octagon, incident to the original octagon along one edge. The composite of reflection in L and then in M is a hyperbolic isometry. Continuing in this way indefinitely, using all possible hyperbolic isometries of this form (using the analogues of the lines L and M in the smaller octagons) produces the full tiling on the left-hand side of

the figure. Let G be the group of hyperbolic isometries generated by the ones that we used to construct the tiling. Then G acts properly discontinuously and freely on $\mathbb{H}^2$.
Show that each orbit of the G-action intersects the (closed) octagon in the centre of the picture. Moreover, show that each orbit either (i) intersects the octagon exactly once in its *interior*, (ii) intersects the octagon exactly twice on its *boundary* or (iii) intersects the octagon exactly eight times on its boundary. In the latter two cases, explain how the points of intersection are related. Deduce that the universal cover of the surface F_2 is contractible, where F_g denotes the closed, orientable surface of genus g.
(c) Sketch how to adapt this proof to show that the universal cover of F_g is contractible for all $g \geqslant 1$.
(d) An application (continuation of Exercise 3.5 from last week). Let X be a path-connected (locally path-connected) space with basepoint x_0 such that $\pi_1(X, x_0)$ is finite. Show that any continuous map $X \to F_g$ must be nullhomotopic. (You may assume the fact that every element of $\pi_1(F_g)$ has infinite order.)

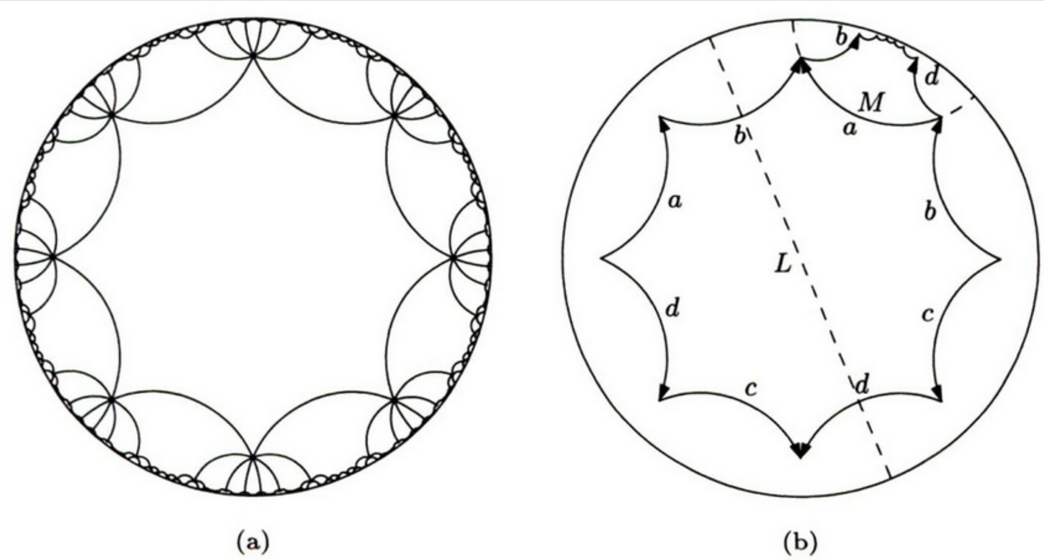


Figure 1.13. A tiling of the hyperbolic plane by regular octagons. (a) A tiling of the hyperbolic plane by identical regular octagons, seen in the Poincaré disk projection. **(b)** To get the small octagon from the big one, reflect in L, then in M.

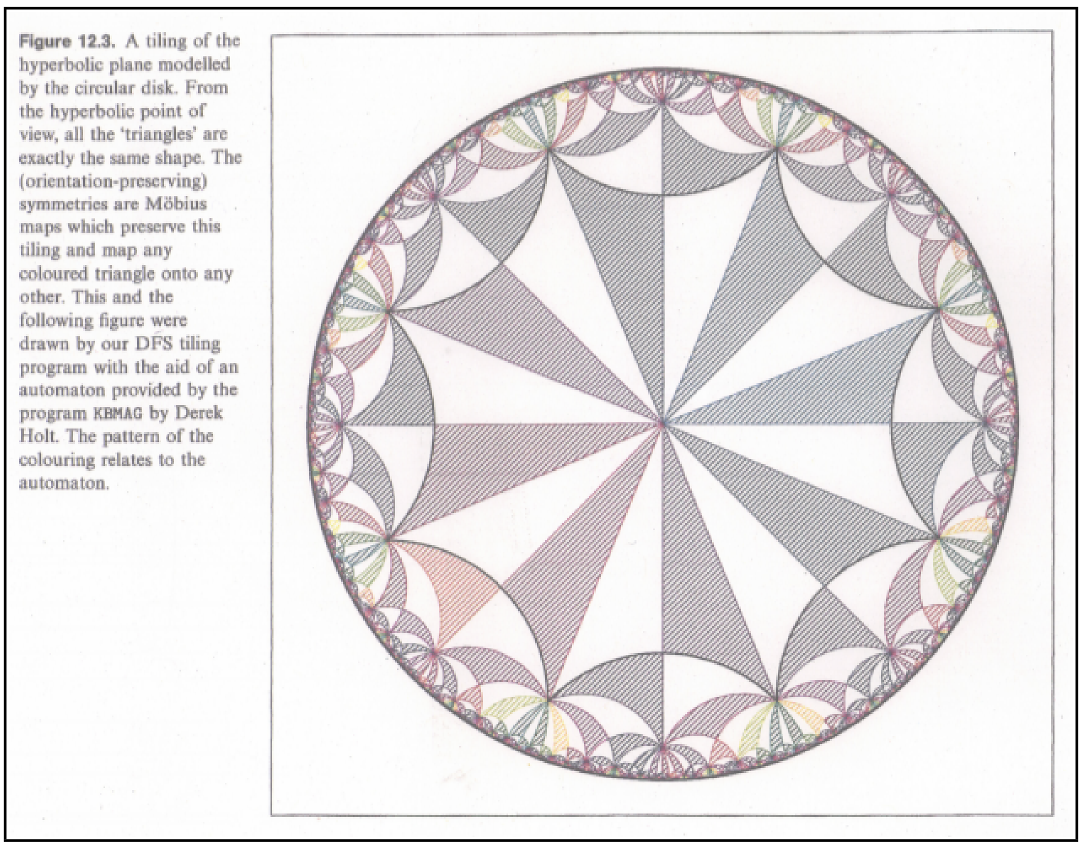
The tiling of the hyperbolic plane by regular octagons used in Exercise 4.4(b).
Image credit: *Three-Dimensional Geometry and Topology, Vol. 1*, William P. Thurston, page 16.

Exercise 4.5 (Conjugate covering space actions)

Let Y be a Hausdorff, locally path-connected, simply-connected space and let G and H be two discrete groups acting (on the right) on Y freely and properly discontinuously. Thus we have coverings $Y \to Y/G$ and $Y \to Y/H$ with deck transformation groups G and H respectively. Via their actions, we may consider G and H as subgroups of $\mathrm{Homeo}(Y)$, the group of all self-homeomorphisms of Y.
(1) If G and H are *conjugate* subgroups, then there is a homeomorphism $Y/G \cong Y/H$. (Suppose that $H = \phi G \phi^{-1}$. Then show that a homeomorphism may be defined by $y.G \mapsto y.\phi^{-1}.H$.)
(2) Conversely, show that if there is a homeomorphism $Y/G \cong Y/H$, then G and H must be conjugate.

Exercise 4.6* (Coverings of $SO(3) \times SO(3)$)

(1) We know from Exercise 2.1 that $\mathbb{S}^n$ is simply-connected for $n \geqslant 2$. The action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{S}^n$ via the antipodal map is free and (vacuously, since it is a finite group) properly discontinuous. Thus $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$.
(2) Recall that there is a homeomorphism $\mathbb{RP}^3 \cong SO(3)$ (using Euler angles, for example). Thus $\pi_1(SO(3)) \cong \mathbb{Z}/2\mathbb{Z}$.
(3) Note that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has exactly five subgroups. Therefore, $SO(3) \times SO(3)$ has exactly five coverings: four of these are simply products of $\mathbb{S}^3$ and/or $SO(3)$.
(4) The remaining covering of $SO(3) \times SO(3)$ is the 2-fold covering corresponding to the subgroup $\{(0,0),(1,1)\}$ of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The general theory tells us that it is the quotient of the universal cover $\mathbb{S}^3 \times \mathbb{S}^3$ by the involution corresponding to the element $(1,1)$. This involution is given by $(x,y) \mapsto (-x,-y)$.
(5) Consider $\mathbb{S}^3$ as the unit sphere in the quaternions $\mathbb{H}$. Each element $(x,y) \in \mathbb{S}^3 \times \mathbb{S}^3$ therefore gives us a map $f_{(x,y)} \colon \mathbb{H} \to \mathbb{H}$ defined by $f_{(x,y)}(z) = xzy^{-1}$. Show that $f_{(x,y)}$ is a rotation, i.e., an element of $SO(4)$.
(6) Show also that every rotation in $SO(4)$ is of this form for some $(x,y) \in \mathbb{S}^3 \times \mathbb{S}^3$ and that two elements (x,y) and (x',y') induce the same rotation if and only if $(x',y') = (-x,-y)$. Deduce that the fifth covering space of $SO(3) \times SO(3)$ is homeomorphic to $SO(4)$.



A subdivision of the tiling of $\mathbb{H}^2$ by regular octagons, in which each octagon has been subdivided into 16 triangles.
Image credit: *Indra's Pearls*, David Mumford, Caroline Series and David Wright.

Aufgaben zur Topologie

Prof. Dr. C.-F. Bödigheimer

Wintersemester 2016/17

Week 5 — Classification of coverings; Seifert-van-Kampen Theorem Due: 30. November 2016

Exercise 5.1 (Coverings and homeomorphisms.)
If $\xi\colon \tilde{X} \to X$ is a covering of a locally path-connected space X all of whose path-components are 1-connected, then $\tilde{X}$ is homeomorphic to a disjoint union of copies of X; if, in addition, $\tilde{X}$ is 0-connected, then ξ is a homeomorphism.

Exercise 5.2 (Connected coverings of Lie groups are Lie groups.)

Exercise 5.3 (Fundamental groups of topological groups are abelian.)
(1) Let G be a topological group; we denote its multiplication by $\mu = \centerdot\colon G \times G \to G, (x, y) \mapsto x \centerdot y$ and the inverse by ${}^{-1}\colon G \to G, x \mapsto x^{-1}$, and we take the neutral element 1 as basepoint. Consider for two pointed maps $a, b\colon \mathbb{S}^1 \to G$ the pointwise multiplication $(a \centerdot b)(t) := a(t) \centerdot b(t)$, and the pointwise inversion $a^{-1}(t) := a(t)^{-1}$. Show that this is a group structure on the set $M = \mathrm{maps}((\mathbb{S}^1, 1), (G, 1))$ of based maps. Convince yourself that all of this is continuous in the compact-open topology on M.
(2) Show that this group structure induces a well-defined group structure on the set of based homotopy classes, that is on $[(\mathbb{S}^1, 1), (G, 1)] = \pi_1(G, 1)$, by setting $[a] \centerdot [b] := [a \centerdot b]$ and $[a]^{-1} := [a^{-1}]$, where the homotopy class of the constant map $t \mapsto 1$ is the neutral element.
(3) Recall now the old group structure on $\pi_1(G, 1)$, denoted here by $[a] * [b] = [a * b]$ and $[a]^{-1} = [\bar{a}]$, where $a * b$ is the concatenation of two paths and $\bar{a}$ is the reverse path. Show (by pictures, not by formulae) that the multiplications satisfy the following exchange property: $([a] * [b]) \centerdot ([c] * [d]) = ([a] \centerdot [c]) * ([b] \centerdot [d])$.
(4) Assume that S is a set with two group structures $*$ and $\centerdot$ satisfying the exchange property. Then the group structures agree ($* = \centerdot$) and are abelian. (Not all of the group axioms are needed for the proof of this statement; which ones are used?)
(5) Thus the statement is proved.

Exercise 5.4 (Homotopy invariance of pull-backs.)
Let $\xi\colon \tilde{X} \to X$ be a covering and let $f_0, f_1\colon Y \to X$ be two maps. Denote the pullbacks (for $i = 0, 1$) of ξ by $\xi_i = f_i^*(\xi)\colon Y_i = f_i^*(\tilde{X}) \to Y$.
(1) If f_0 and f_1 are homotopic, there is a homeomorphism $\Phi\colon f_0^*(\tilde{X}) \to f_1^*(\tilde{X})$ with $\xi_1 \circ \Phi = \xi_0$.
(Hint: Consider the pull-back $F^*(\tilde{X}) \to Y \times [0, 1]$ of ξ along a homotopy F between f_0 and f_1, and lift the path $t \mapsto (y, t)$ with an arbitrary starting point in $f_0^*(\tilde{X}) \subset F^*(\tilde{X})$ over $(y, 0)$.)
(2) If $f\colon Y \to X$ is null-homotopic, then $f^*(\xi)$ is a trivial covering.
(3) Application: The inclusions $\iota_n\colon \mathbb{S}^1 \cong \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^n$ are not null-homotopic; even better, ι_n induces isomorphisms on fundamental groups.

Exercise 5.5 (Branched coverings and polynomials.) Let $p(z) = z^n + a_{n-1}z^{n-1} + \ldots a_1 z + a_0$ be a non-constant complex polynomial. Denote by S the set of all critical points, i.e., points z with $p'(z) = 0$ and V the set of all critical values $v = p(\zeta)$ for $\zeta \in S$.
(1) Show that $p\colon \mathbb{C} - S \to \mathbb{C} - V$ is an n-fold covering.
(Hint: $\mathbb{C}$ is locally path-connected and V is a closed subset (why?), so for each $z \in \mathbb{C} - V$ you may find a connected open neighbourhood U of z in $\mathbb{C} - V$. Study the preimage $p^{-1}(U)$ and use the Inverse Function Theorem.)

Now we consider the special cases $p_n(z) = z^n$ for $n \geq 2$ as maps from the disc $B_r(0)$ of radius $r > 0$ around 0 to the disc $B_{r^n}(0)$. We just saw that the restriction $p_n|\colon B_r(0) - 0 \to B_{r^n}(0) - 0$ is an n-fold covering. But what is $p_n\colon B_r(0) \to B_{r^n}(0)$, where $\zeta = 0$ has only one point in its pre-image and not n points (as do all other points) ? The

map p_n is a prototypical example of a so-called *branched covering*; we will not define this notion in all generality, but want to prove that a non-constant complex polynomial p is a branched covering in the following sense:
(2) Show that for each critical value $v \in V$ there is a neighbourhood $U \subset \mathbb{C}$ and a partition $k_1+k_2+\cdots+k_l = n$, such that the restriction $p|\colon p^{-1}(U) \to U$ is homeomorphic to the Whitney sum of l branched coverings $p_{k_1}, p_{k_2}, \ldots, p_{k_l}$ as considered above.
(Note: we call two coverings $\xi\colon \tilde{X} \to X$ and $\nu\colon \tilde{Y} \to Y$ homeomorphic if there are homeomorphisms $\phi\colon X \to Y$ and $\tilde{\phi}\colon \tilde{X} \to \tilde{Y}$ such that $\nu \circ \tilde{\phi} = \phi \circ \xi$.)
(3)* What happens in a neighbourhood of ∞ if we extend p to $\mathbb{S}^2 \to \mathbb{S}^2$ by setting $p(\infty) = \infty$?

Die Schar der konzentrischen Kreise $r =$ konst. geht also in die Schar der konzentrischen Kreise $\varrho =$ konst. über, und die Winkel φ werden verdoppelt. Läßt man also φ bei festem $r > 0$ von 0 bis 2π laufen, so variiert ϑ monoton zwischen 0 und 4π, und der Bildpunkt ζ durchwandert den Kreis $\varrho =$ konst. zweimal, wenn der Punkt z den Kreis $r =$ konst. einmal durchläuft. Somit entsprechen einem Werte von ζ zwei Werte von z; d. h. die Umkehrfunktion

$$z = \sqrt{\zeta}$$

unserer Funktion $\zeta = z^2$ ist eine „zweideutige" Funktion von ζ.

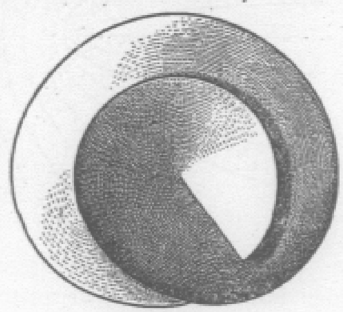


Abb. 95.

Zur Veranschaulichung dieser Verhältnisse denkt man sich die ζ-Ebene in zwei Exemplaren, zwei „*Blättern*", übereinandergelegt. Dann führt man in beiden Blättern von $\zeta = 0$ bis $\zeta = \infty$ kongruente Schnitte, etwa längs der positiven reellen Achse, und heftet die vier entstehenden Schnittufer unter wechselweiser Durchdringung der beiden Blätter kreuzweise aneinander (Abb. 95).

A discussion of the function $z \mapsto z^2$ and its inverse $\zeta \mapsto \sqrt{\zeta}$ from *Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen* by A. Hurwitz and R. Courant, using polar coordinates $z = re^{i\varphi}$ and $\zeta = z^2 = \varrho e^{i\vartheta}$.

Exercise 5.6* (Spaces with fundamental group $\mathbb{Z}/n\mathbb{Z}$.)
Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a map of degree $\mathrm{grad}(f) = n$. Consider the space $M(n) = \mathbb{S}^1 \cup_f \mathbb{D}^2$ obtained by attaching a 2-disc to a circle along its boundary using the map f — i.e., the quotient $(\mathbb{S}^1 \sqcup \mathbb{D}^2)/\sim$ where $\sim$ is the equivalence relation generated by the relations $\zeta \sim f(\zeta)$ for $\zeta \in \partial\mathbb{D}^2 = \mathbb{S}^1$.
(1) Make a sketch of this identification.
(2) Show that $\pi_1(M(n)) \cong \mathbb{Z}/n\mathbb{Z}$.

Exercise 5.7* (Any group is the fundamental group of some space.)
(1) Let G be a group with finite presentation $\langle s_1, \ldots, s_n \mid r_1, \ldots, r_k \rangle$. Using a similar idea to Exercise 5.6, and the Seifert-van-Kampen Theorem multiple times, construct a space X such that $\pi_1(X) \cong G$. First find a space Y_0 whose fundamental group is the free group $\langle s_1, \ldots, s_n \mid \rangle$, then attach a 2-disc to form a space Y_1 with fundamental group $\langle s_1, \ldots, s_n \mid r_1 \rangle$, and so on, until you find $Y_k = X$.
(2) Now suppose that G is any group, not necessarily possessing a finite presentation, or even a finite generating set (think of $G = \mathbb{Q}$, for example, or $G = S^1$, considered as an abstract (uncountable!) group). Using a limit argument, show that there is nevertheless a space X with fundamental group G. You may use the following facts:
(a) Suppose that X is path-connected and is the union of a family of path-connected open subspaces X_α. Assume that each intersection $X_\alpha \cap X_\beta$ is X_γ for some γ. Also assume that X and each X_α are "nice" (i.e., locally 0-connected and semi-locally 1-connected). Let $x \in \bigcap_\alpha X_\alpha$. Then $\pi_1(X, x)$ is the direct limit of the subgroups $\pi_1(X_\alpha, x)$.
(b) Any group is the direct limit of the family of all of its finitely presentable subgroups.

Note: There is a subtlety with the limit argument if one tries to use fact (b): it is not possible to pick a finite presentation for each finitely present*able* subgroup of G in a way that is compatible with all inclusions between them. To rectify this, you can instead use the following modification of fact (b):
(c) Any group G is the direct limit of the diagram of groups whose objects are all finitely presentable subgroups of G equipped with a choice of presentation, and whose morphisms are just those inclusions $\langle s_1, \ldots, s_n \mid r_1, \ldots, r_k\rangle \hookrightarrow \langle s'_1, \ldots, s'_m \mid r'_1, \ldots, r'_l\rangle$ for which $\{s_1, \ldots, s_n\}$ is a subset of $\{s'_1, \ldots, s'_m\}$ and $\{r_1, \ldots, r_k\}$ is a subset of $\{r_1, \ldots, r'_l\}$.

Exercise 5.8* (Addendum to the classification theorem for coverings.)
Let X be a 0-connected, locally 0-connected and semi-locally 1-connected space. We denote by

$$\mathrm{char}\colon \mathcal{C}ov^0(X, x_0) \longrightarrow \mathcal{G}(\pi_1(X, x_0))$$

the function, which associates to an isomorphism class $[\xi] = [\xi\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)]$ of based and connected coverings the characteristic subgroup $\mathrm{char}[\xi] = \xi_*(\pi_1(X, x_0))$ of $\pi_1(X, x_0)$.
(1) $\mathrm{char}[\xi_1] \le \mathrm{char}[\xi_2] \iff$ There is a unique morphism $\xi_1 \to \xi_2$.
(2) $\mathrm{char}[\xi]$ is normal $\iff \xi$ is regular (i.e., "fibre-transitive").
(3) Let $H \le \pi_1(X, x_0)$ be a subgroup and denote by $\bar{\xi}\colon \bar{X} \to X$ the universal covering of X. In the commutative diagram

$$\begin{tikzcd} \bar{X} \arrow[dd, "\bar{\xi}"'] \arrow[dr, "q_H"] & \\ & X_H \arrow[dl, "\xi_H"] \\ X & \end{tikzcd}$$

we have:
(3.1) $q_H\colon \bar{X} \to X_H := \bar{X}/H$ is a universal covering; thus $\mathcal{D}(q_H) \cong H$ for the group of deck transformations.
(3.2) $\mathcal{D}(\xi_H) \cong \mathrm{Weyl}(H) = N_G(H)/H$, the Weyl group of H in $G = \pi_1(X, x_0)$.
(3.3) In particular, $\mathcal{D}(\xi_H) \cong \pi_1(X, x_0)/H$ if H is normal.

Aufgaben zur Topologie

Prof. Dr. C.-F. Bödigheimer

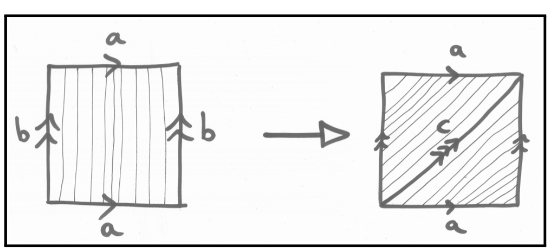
Wintersemester 2016/17

Week 6 — Fundamental groups and first homology groups Due: 7. December 2016

Exercise 6.1 (Gradients, rotation and divergence: grad, rot and div.)
(1) Let X be an open subset of $\mathbb{R}^2$ and define real vector spaces as follows.
∘ $C_0 = C^\infty(X)$, the space of smooth real-valued functions on X.
∘ $C_{-1} = C^\infty(X) \times C^\infty(X)$, to be thought of as the space of smooth vector fields $v = (v_1, v_2)$ on X, in coordinates.
∘ $C_{-2} = C^\infty(X)$, to be thought of as the space of volume forms on X.
There are linear maps $\mathrm{grad}\colon C_0 \to C_{-1}$ and $\mathrm{rot}\colon C_{-1} \to C_{-2}$ defined by $\mathrm{grad}(f) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ and $\mathrm{rot}(v_1, v_2) = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$.
(a) Show (using vector calculus) that this is a chain complex (over the field $\mathbb{R}$), where all undefined chain modules are 0.
(b) For arbitrary X, find the dimension of $H_0(C_\bullet)$, i.e., the kernel of grad.
(c) When X is not simply-connected, give an example of a vector field that has zero rotation, but is not the gradient of any smooth function on X, thus showing that the homology group $H_{-1}(C_\bullet)$ is non-trivial in this case.

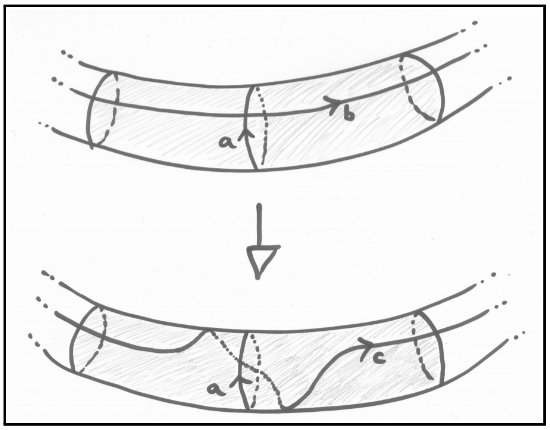
(2) Now let X be an open subset of $\mathbb{R}^3$ and define real vector spaces as follows.
∘ $C_0 = C^\infty(X)$.
∘ $C_{-1} = C^\infty(X) \times C^\infty(X) \times C^\infty(X)$.
∘ $C_{-2} = C^\infty(X) \times C^\infty(X) \times C^\infty(X)$.
∘ $C_{-3} = C^\infty(X)$.
(a) Recall the definitions of the linear operators $\mathrm{grad}\colon C_0 \to C_{-1}$, $\mathrm{rot}\colon C_{-1} \to C_{-2}$ and $\mathrm{div}\colon C_{-2} \to C_{-3}$ in this setting, and show that these form a chain complex.
(b) As above, compute the dimension of $H_0(C_\bullet)$.
(c) Show that the homology group $H_{-1}(C_\bullet)$ is non-trivial when X is not simply-connected by finding a vector field with zero rotation and which is not the gradient of any smooth function on X.
(d)* Find an X such that $H_{-2}(C_\bullet)$ is non-trivial, i.e., we need a vector field defined on X with zero divergence and which is not the rotation of any other vector field on X. (A first case to consider is $X = \mathbb{R}^3 - \{(0,0,0)\}$.)

Exercise 6.2 (Induced maps on π_1 and the abelianisation of π_1.)
Recall: If $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a map of degree k, then the induced map $\pi_1(f)\colon \pi_1(\mathbb{S}^1, 1) \to \pi_1(\mathbb{S}^1, 1)$ is the multiplication by k in $\mathbb{Z}$.
(1) Consider the following map $T_a\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$, called the *Dehn twist along the curve a*.



The loop a is taken to itself, whereas the loop b is taken to the diagonal loop c pictured on the right-hand side. In general, each vertical loop on the left-hand side is skewed to the right as it travels upwards, so that it becomes one of the 45-degree diagonal loops on the right-hand side.
Describe the induced homomorphism $\pi_1(T_a)$ on the fundamental group $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z} \times \mathbb{Z}$.

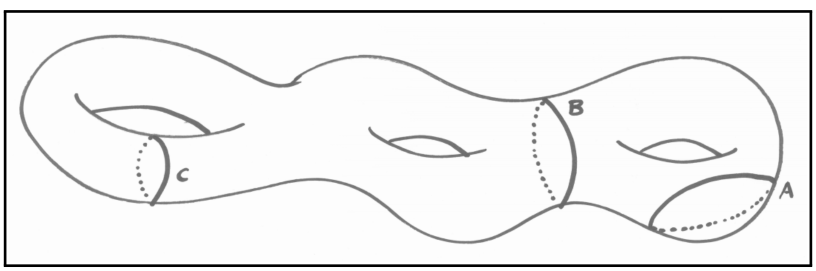
(2) Dehn twists may be defined more generally for surfaces. Given a piece of a surface, homeomorphic to a cylinder, one may define the Dehn twist T_a along a as follows:



(it acts by the identity outside of the shaded region). Taking a to be one of the standard generators for the fundamental group of F_2 (recall this from lectures), describe the induced homomorphism $\pi_1(T_a)\colon \pi_1(F_2) \to \pi_1(F_2)$.

A result that will soon appear in lectures is the fact that the first homology $H_1(X)$ of a path-connected space X is isomorphic to the abelianisation of its fundamental group $\pi_1(X, x)$ based at any point $x \in X$.
(3)* Using the computations of the fundamental groups of orientable and non-orientable surfaces from the lecture, compute their first homology groups.
(4)* Let $F_{g,n}$ be the orientable surface of genus g with $n > 0$ points removed. Compute its fundamental group und its first homology group. (Hint: Write $F_{g,n}$ "in normal form", that means as a quotient space of a regular $4g$-gon; draw $n-1$ extra (not necessarily straight) edges from one corner to another or the same corner; now remove in each of the n "compartments" one interior point; find a retraction onto the subspace which consists of the $4g$ edges on the boundary and the $n-1$ extra edges.)

Exercise 6.3 (Nullhomotopies and nullhomologies.)
Consider the following three curves on the surface F_3.



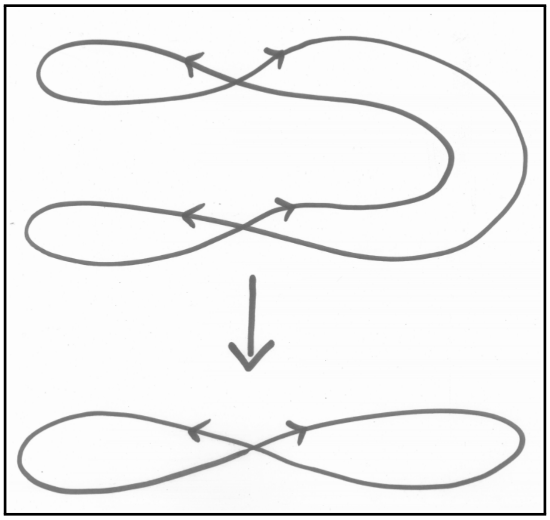
(a) Observe that the curve A is nullhomotopic.
(b) Construct a 2-chain whose boundary is equal to the 1-cycle represented by the curve B. Thus, B is nullhomologous. (Write F_3 in normal form as above and use the obvious triangulation.)
(c)* However, B is not nullhomotopic (show this using your knowledge of $\pi_1(F_3)$; this is harder than one expects).
(d)* Show that the curve C is neither nullhomotopic nor nullhomologous. (Consider the commutator subgroup of $\pi_1(F_3)$, which also gives an alternative way to deduce that B is nullhomologous.)

Exercise 6.4 (Disjoint unions of spaces.)
Let X be a topological space which splits as the topological disjoint union of subspaces $X = \bigsqcup_\alpha X_\alpha$. Show that the singular chain complex $S_\bullet(X)$ of X splits into a direct sum of summands indexed by α, and that the boundary operator ∂ preserves the summands. Deduce that the subcomplexes of cycles and of boundaries also split with

respect to α, and therefore so does the homology of X, in other words we have, for each n,

$$H_n\Big(\bigsqcup_\alpha X_\alpha\Big) \;\cong\; \bigoplus_\alpha H_n(X_\alpha).$$

Exercise 6.5 (Coverings and H_1.)
Let $\xi\colon \tilde{X} \to X$ be a covering. Recall from the lecture that the map of fundamental groups $\pi_1(\xi)\colon \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x)$ is injective. Consider the covering



of $X = \mathbb{S}^1 \vee \mathbb{S}^1$.
(a) Show that $H_1(\tilde{X}) \cong \mathbb{Z}^3$, whereas $H_1(X) \cong \mathbb{Z}^2$.
(b) Compute the homomorphism $\pi_1(\xi)$ of fundamental groups induced by ξ, then abelianise this to compute the homomorphism $H_1(\xi)$ that it induces on first homology. Deduce that coverings do not always induce injective maps on homology.

Exercise 6.6* (Multi-valued functions: integrating on non-simply-connected domains.)
Let $\Omega \subset \mathbb{C}$ be a region (i.e., open and connected) and $z_0 \in \Omega$, and consider a holomorphic function $f\colon \Omega \to \mathbb{C}$; we assume that $f'(z) \neq 0$ for all $z \in \Omega$.
We would like to define a new function

$$z \mapsto \int_w f(\zeta) d\zeta \;:=\; \int_0^1 f(w(t))\dot{w}(t) dt,$$

where w is a path in Ω from z_0 to z; but this path integral depends on the path w and not just on its endpoint $w(1) = z$; so we would get a multi-valued function. However, — since f is holomorphic —, it depends only on the homotopy class $[w]$, not on the actual path. This is our chance: If $\xi\colon \tilde{\Omega} \to \Omega$ denotes the universal covering of Ω, we define a function

$$\tilde{F}\colon \tilde{\Omega} \to \mathbb{C}, \quad \tilde{F}([w], z) := \int_w f(\zeta) d\zeta = \int_0^1 f(w(t))\dot{w}(t) dt.$$

(1) $\tilde{F}$ is well-defined.
(2) $\tilde{F}$ is holomorphic. (N.B: $\tilde{\Omega}$ is a holomorphic manifold, or a Riemann surface; cf. Exercise 3.2.)
(3) Now define the *period homomorphism* $\mathrm{Per}_f\colon \pi_1(\Omega, z_0) \to \mathbb{C}$ as follows:

$$\mathrm{Per}_f([w]) = \int_w f(\zeta) d\zeta.$$

Convince yourself of the formulae:
$\mathrm{Per}_f(\alpha\beta) = \mathrm{Per}_f(\alpha) + \mathrm{Per}_f(\beta)$, $\mathrm{Per}_f(\alpha^{-1}) = -\mathrm{Per}_f(\alpha)$, $\mathrm{Per}_f(1) = 0$, which say that Per_f is a homomorphism.
There are more formulae like:
$\mathrm{Per}_{f+g}(\alpha) = \mathrm{Per}_f(\alpha) + \mathrm{Per}_g(\alpha)$, $\mathrm{Per}_{\lambda f}(\alpha) = \lambda\,\mathrm{Per}_f(\alpha)$, $\mathrm{Per}_{\overline{f}}(\alpha) = \overline{\mathrm{Per}_f(\alpha)}$, which say what ?

Next conclude, that the kernel $K := \ker(\mathrm{Per}_f) \leqslant \pi_1(\Omega, z_0)$ of Per_f contains at least the commutator subgroup of $\pi_1(\Omega, z_0)$. Now let $\xi_f\colon \Omega_f \to \Omega$ be the covering corresponding to that subgroup K, i.e., the quotient of $\tilde{\Omega}$ by the action of K by deck transformations. Denote this quotient map by q_f.
Show that

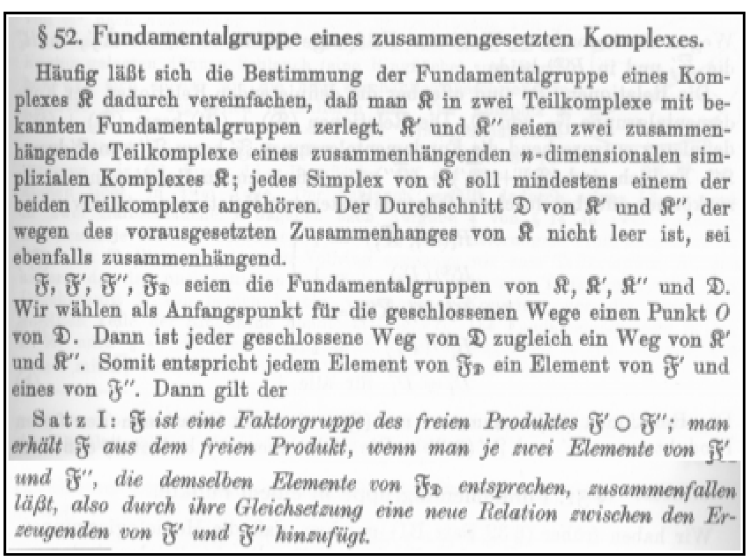
$$\tilde{F}([a * w], z) = \tilde{F}([w], z) + \mathrm{Per}_f([a]),$$

where a is a closed loop based at z_0 and w is any path from z_0 to z. Conclude that $\tilde{F}$ factors as the composite of q_f followed by a well-defined map $F\colon \Omega_f \to \mathbb{C}$. Summarising, we have the diagram:

$$\begin{tikzcd}
\tilde{\Omega} \arrow[dd, "\xi"'] \arrow[dr, "q_f"] \arrow[drr, bend left, "\tilde{F}"] & & \\
& \Omega_f \arrow[dl, "\xi_f"] \arrow[r, "F"'] & \mathbb{C} \\
\Omega & &
\end{tikzcd}$$

Thus we have found the natural domain of (*well*-)definition of the multi-valued function $z \mapsto \int_{z_0}^{z} f$.

(5) Examples.
In each example, describe $\pi_1(\Omega, z_0)$, compute the period homomorphism and describe the covering $\xi_f\colon \Omega_f \to \Omega$ and the function F.
(5.1) : Take $\Omega = \mathbb{C} - \{0\}$ and $f(z) = \frac{1}{z}$.
(5.2) : Take $\Omega = \mathbb{C} - \{-1, 1\}$ and $f(z) = \frac{1}{1+z} + \frac{1}{1-z}$.
(5.3) : Take $\Omega = \mathbb{C} - \{-1, 1\}$ and $f(z) = \frac{a}{1+z} + \frac{b}{1-z}$, for integers $a, b \in \mathbb{Z}$.
(5.4) : Take $\Omega = \mathbb{C} - \{-1, 1\}$ and $f(z) = \frac{1}{1+z} + \frac{\pi}{1-z}$.
In the last three examples, feel free to build a model of the covering Ω_f as demonstrated in lectures.



§ 52. Fundamentalgruppe eines zusammengesetzten Komplexes.

Häufig läßt sich die Bestimmung der Fundamentalgruppe eines Komplexes $\mathfrak{K}$ dadurch vereinfachen, daß man $\mathfrak{K}$ in zwei Teilkomplexe mit bekannten Fundamentalgruppen zerlegt. $\mathfrak{K}'$ und $\mathfrak{K}''$ seien zwei zusammenhängende Teilkomplexe eines zusammenhängenden n-dimensionalen simplizialen Komplexes $\mathfrak{K}$; jedes Simplex von $\mathfrak{K}$ soll mindestens einem der beiden Teilkomplexe angehören. Der Durchschnitt $\mathfrak{D}$ von $\mathfrak{K}'$ und $\mathfrak{K}''$, der wegen des vorausgesetzten Zusammenhanges von $\mathfrak{K}$ nicht leer ist, sei ebenfalls zusammenhängend.

$\mathfrak{F}$, $\mathfrak{F}'$, $\mathfrak{F}''$, $\mathfrak{F}_{\mathfrak{D}}$ seien die Fundamentalgruppen von $\mathfrak{K}$, $\mathfrak{K}'$, $\mathfrak{K}''$ und $\mathfrak{D}$. Wir wählen als Anfangspunkt für die geschlossenen Wege einen Punkt O von $\mathfrak{D}$. Dann ist jeder geschlossene Weg von $\mathfrak{D}$ zugleich ein Weg von $\mathfrak{K}'$ und $\mathfrak{K}''$. Somit entspricht jedem Element von $\mathfrak{F}_{\mathfrak{D}}$ ein Element von $\mathfrak{F}'$ und eines von $\mathfrak{F}''$. Dann gilt der

Satz I: *$\mathfrak{F}$ ist eine Faktorgruppe des freien Produktes $\mathfrak{F}' \circ \mathfrak{F}''$; man erhält $\mathfrak{F}$ aus dem freien Produkt, wenn man je zwei Elemente von $\mathfrak{F}'$ und $\mathfrak{F}''$, die demselben Elemente von $\mathfrak{F}_{\mathfrak{D}}$ entsprechen, zusammenfallen läßt, also durch ihre Gleichsetzung eine neue Relation zwischen den Erzeugenden von $\mathfrak{F}'$ und $\mathfrak{F}''$ hinzufügt.*

The Seifert-van-Kampen Theorem, from *Lehrbuch der Topologie*, H. Seifert and W. Threlfall.

Aufgaben zur Topologie

Prof. Dr. C.-F. Bödigheimer

Wintersemester 2016/17

Week 7 — Chain complexes Due: 14. December 2016

Exercise 7.1 (Decomposition of chain complexes.)
(1) Let

$$0 \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_{N-1} \leftarrow C_N \leftarrow 0$$

be a bounded chain complex of finitely generated free abelian groups. Show that it splits as a direct sum of finitely many subcomplexes, each of which is of the form

$$0 \leftarrow \mathbb{Z} \leftarrow 0 \qquad \text{or} \qquad 0 \leftarrow \mathbb{Z} \xleftarrow{k} \mathbb{Z} \leftarrow 0$$

for some non-zero $k \in \mathbb{Z}$, up to shifts to the left and right.
(Hint: Use the Elementarteilersatz (Smith normal form) for integer matrices.)

(2) Show that, if we had started with a bounded chain complex of finite-dimensional vector spaces over a field $\mathbb{K}$ instead, then it splits as a direct sum of finitely many subcomplexes of just two types, namely $0 \leftarrow \mathbb{K} \leftarrow 0$ and $0 \leftarrow \mathbb{K} \xleftarrow{\mathrm{id}} \mathbb{K} \leftarrow 0$.

(3) Thus any bounded chain complex of finite-dimensional vector spaces is isomorphic to one with chain modules of the form $C_n = B_n \oplus H_n \oplus B_{n-1}$, where B_n denotes the boundaries of degree n, and where the boundary operator

$$\partial \colon C_n = B_n \oplus H_n \oplus B_{n-1} \twoheadrightarrow B_{n-1} \hookrightarrow B_{n-1} \oplus H_{n-1} \oplus B_{n-2} = C_{n-1}$$

is the projection of C_n onto B_{n-1} composed with the inclusion of B_{n-1} into C_{n-1}. It follows that the homology is $H_n(C_\bullet) \cong H_n$.

Exercise 7.2 (Homology of some small chain complexes.)
(a) Compute the homology of each of the following chain complexes.
(b) Take the tensor product with $\mathbb{Q}$ and compute the homology of the resulting chain complex.
(c) Take the tensor product with $\mathbb{F}_p$ (for a prime p) and compute the homology of the resulting chain complex.

$$(A_\bullet) \qquad 0 \to \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right)} \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right)} \quad \cdots \quad \xrightarrow{\left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right)} \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right)} \mathbb{Z}^2 \to 0.$$

$$(B_\bullet) \qquad 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \quad \cdots \quad \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

$$(C_\bullet) \qquad 0 \to \mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}} \mathbb{Z}^6 \xrightarrow{\left(\begin{smallmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 & -1 \end{smallmatrix}\right)} \mathbb{Z}^2 \to 0.$$

(d) Use Exercise 7.1 part (1) to show that (in general – not just for these examples) the computations in (b) and (c) may in fact be deduced directly from the computations in (a), without knowledge of the original chain complex.

Exercise 7.3 (Chain homotopy is an equivalence relation.)
Recall that a *chain homotopy* $h \colon f_0 \simeq f_1$ between two chain maps $f_0, f_1 \colon C_\bullet \to D_\bullet$ is a collection of homomorphisms

$h_n\colon C_n \to D_{n+1}$ such that $h_{n-1} \circ \partial + \partial' \circ h_n = f_0 - f_1$.
Suppose that $C_\bullet, D_\bullet, E_\bullet$ are chain complexes, $f_0, f_1, f_2\colon C_\bullet \to D_\bullet$ and $g_0, g_1\colon D_\bullet \to E_\bullet$ are chain maps and $h\colon f_0 \simeq f_1$, $\hat{h}\colon f_1 \to f_2$ and $k\colon g_0 \simeq g_1$ are chain homotopies. Show that there are chain homotopies
(a) $f_0 \simeq f_0$,
(b) $f_1 \simeq f_0$,
(c) $f_0 \simeq f_2$,
(d) $g_0 f_0 \simeq g_1 f_1$,
given by 0, $-h$, $h + \hat{h}$ and $g_0 h + k f_1$ respectively. Conclude that chain homotopy is an equivalence relation and is preserved by composition.

Exercise 7.4 (Tensor products of chain complexes)
Let $\mathbb{K}$ denote a principal ideal domain. For two chain complexes $A_\bullet$ and $B_\bullet$ over $\mathbb{K}$ with boundary operator ∂^A resp. ∂^B we define a new complex $C_\bullet = A_\bullet \otimes B_\bullet$ by setting $C_n := \sum_{n=k+l} A_k \otimes B_l$ and defining the boundary operator $\partial^\otimes\colon C_n \to C_{n-1}$ by setting (Leibniz-like)

$$\partial^\otimes(a \otimes b) := \partial^A(a) \otimes b \;+\; (-1)^k a \otimes \partial^B(b)$$

for a generator $a \in A_k$ and $b \in B_l$ with $k + l = n$.
(1) Show that this is a chain complex.
(2) Assume $A_n = B_n = 0$ for $n < 0$ and $A_0 = B_0 = \mathbb{K}$. Can you define chain maps $\iota_A\colon A_\bullet \to C_\bullet$ and $\iota_B\colon B_\bullet \to C_\bullet$ and $\pi_A\colon C_\bullet \to A_\bullet$ and $\pi_B\colon C_\bullet \to B_\bullet$ such that $\pi_A \circ \iota_A = \mathrm{id}$, $\pi_B \circ \iota_B = \mathrm{id}$, and $\pi_A \circ \iota_B = 0 = \pi_B \circ \iota_A$?

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5. Die Euler-Poincarésche Formel. Die Untersuchung der ganzzahligen Bettischen Gruppen führt zu einer berühmten Identität, die den elementaren Eulerschen Polyedersatz verallgemeinert: Unter der *Eulerschen Charakteristik des Komplexes K* versteht man die Zahl

$$\chi(K) = \sum_{r=0}^{n} (-1)^r \alpha^r,$$

wobei α^r die Anzahl der r-dimensionalen Simplexe von K ist; die Identität, um die es sich handelt, ist die sog. *Euler-Poincarésche Formel*

$$\chi(K) = \sum_{r=0}^{n} (-1)^r p^r. \qquad (1)$$

Figure 1: From *Topologie I*, by P. Alexandroff and H. Hopf (1935), page 214. What they term *Betti groups* are the homology groups with integer coefficients of a space. In the first formula, α^r is the number of r-dimensional simplices of the n-dimensional simplicial complex K (cf. Exercise 7.6 on page 4), whereas, in the second formula, p^r denotes the *rank* of the r-th homology group $H_r(K; \mathbb{Z})$. (The *rank* of an abelian group is defined exactly analogously to the *dimension* of a vector space.)

Exercise 7.5 (Euler characteristic)
If $C_\bullet$ is a bounded chain complex of finite-dimensional vector spaces over a field $\mathbb{K}$, we can define its *Euler characteristic* by

$$\chi(C_\bullet) := \sum_n (-1)^n \dim_{\mathbb{K}}(C_n).$$

Show the following formulae:
(1) $\chi(C_\bullet) = \sum_n (-1)^n \dim_{\mathbb{K}} H_n(C_\bullet)$. (Hint: use Exercise 7.1 part (3).)
(2) $\chi(A_\bullet \oplus B_\bullet) = \chi(A_\bullet) + \chi(B_\bullet)$.
(3) $\chi(A_\bullet \otimes B_\bullet) = \chi(A_\bullet)\,\chi(B_\bullet)$.

(Comment to (1): This is the famous formula of Euler-Poincaré-Hopf (see Figure 1 above): the Euler characteristic depends only on the homology. This is true in general, not only over fields, as we will see later.)

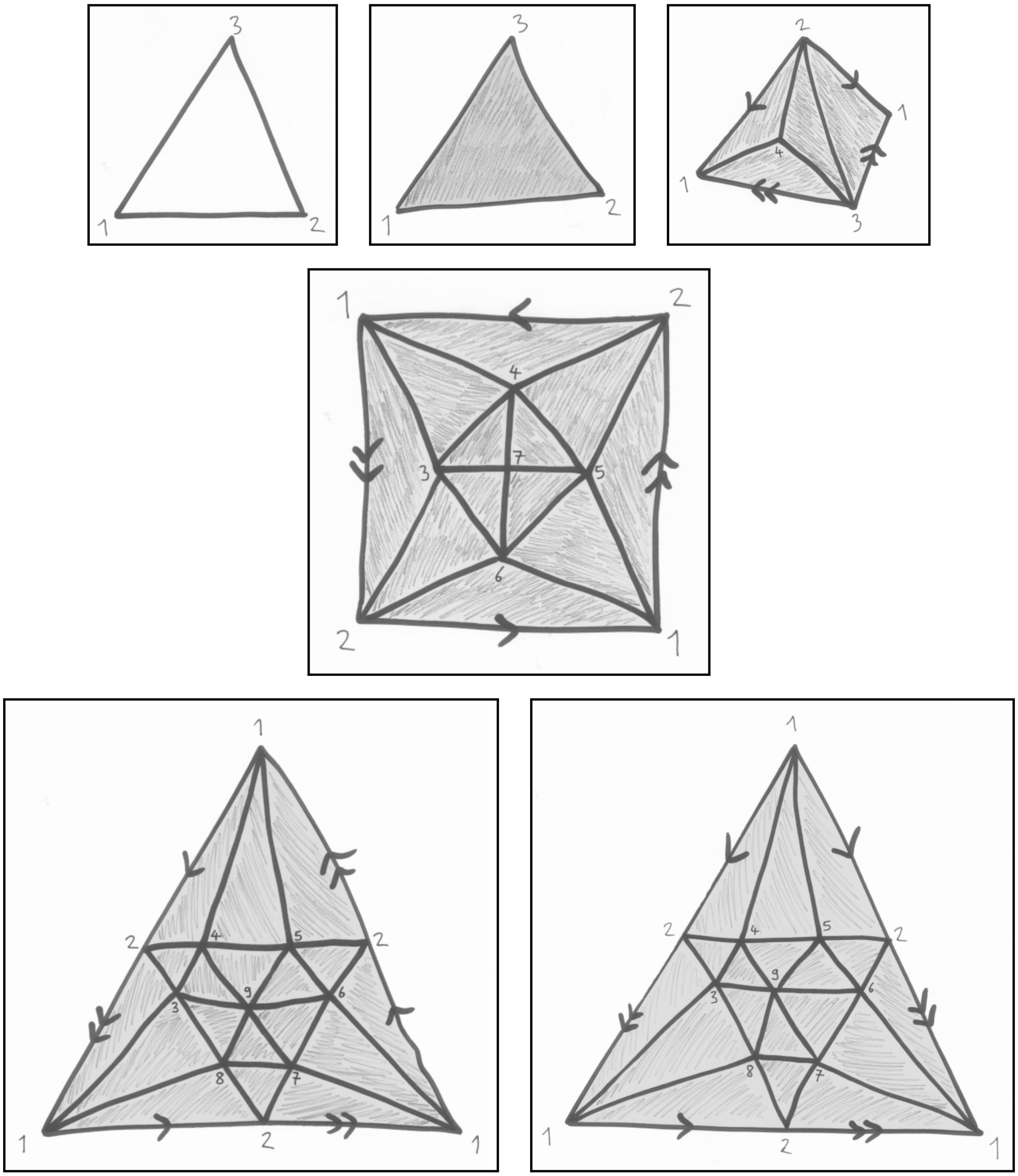


Figure 2: The simplicial complexes from Exercise 7.6 part (4). Note about the 2-simplices in these figures: if an (innermost) triangle is shaded, then the corresponding 2-simplex is present; otherwise, it is not.

Exercise 7.6* (Simplicial chain complexes)
Let $\mathcal{X}$ be a set of non-empty, finite subsets of some fixed set X_0 such that $\sigma \in \mathcal{X}$ implies $\tau \in \mathcal{X}$ for any non-empty subset τ of σ. We denote by X_n all elements σ of $\mathcal{X}$ with exactly $n+1$ elements of X_0. There is an obvious reason why we call the elements of X_0 *vertices*, the elements of X_1 *edges* or *1-simplices*, those in X_2 *triangles* or *2-simplices* and so on. We assume that X_0 is a linearly ordered set; thus any $\sigma \in X_n$ is an ordered set of $n+1$ vertices, which we number $v_0 < v_1 < \ldots < v_n$ from 0 to n. Denote now, for $i = 0, 1, \ldots, n$, by $d'_i(\sigma)$ the set σ with its i-th element v_i removed; this defines functions $d'_i \colon X_n \to X_{n-1}$ for $n > 0$.
(1) Show that $d'_i \circ d'_i = d'_i \circ d'_{i+1}$ and for $i < j$ that $d'_i \circ d'_j = d'_{j-1} \circ d'_i$.
(2) Denote by $C_n(\mathcal{X})$ the free module over the principal ideal domain $\mathbb{K}$ generated by the set X_n. Consider the homomorphisms $d_i \colon C_n(\mathcal{X}) \to C_{n-1}(\mathcal{X})$ determined by d'_i by linear extension. Show that the formulae from (1) hold also for the d_i.
(3) If we set $\partial := \sum_{i=0}^{n} (-1)^i d_i$ show that $\partial \circ \partial = 0$ holds.
(4) In each of the examples depicted in Figure 2 on the previous page, the figure depicts a *triangulation* of a certain space. The vertices of the triangulation form the set X_0 and a subset σ of X_0 belongs to $\mathcal{X}$ if and only if there exists a simplex (in the figures, this means either an edge, a triangle or a vertex) whose vertices are precisely the vertices corresponding to σ. In each case, write down the chain complex $C_\bullet(\mathcal{X})$ and compute its homology groups $H_n(C_\bullet(\mathcal{X}))$ for all n and its Euler characteristic.

Aufgaben zur Topologie

Prof. Dr. C.-F. Bödigheimer

Wintersemester 2016/17

Week 8 — Homological algebra and homotopy invariance of homology Due: 21. December 2016

§ 5. — Homologies.

Considérons une variété V à p dimensions; soit maintenant W une variété à q dimensions ($q \leq p$) faisant partie de V. Supposons que la frontière complète de W se compose de λ variétés continues à $q-1$ dimensions

$$v_1, \quad v_2, \quad \ldots, \quad v_\lambda.$$

Nous exprimerons ce fait par la notation

$$v_1 + v_2 + \ldots + v_\lambda \backsim 0.$$

Plus généralement la notation

$$k_1 v_1 + k_2 v_2 \backsim k_3 v_3 + k_4 v_4,$$

où les k sont des entiers et les v des variétés à $q-1$ dimensions, signifiera qu'il existe une variété W à q dimensions faisant partie de V et dont la frontière complète se composera de k_1 variétés peu différentes de v_1, de k_2 variétés peu différentes de v_2, de k_3 variétés peu différentes de la variété opposée à v_3 et de k_4 variétés peu différentes de la variété opposée à v_4.

Les relations de cette forme pourront s'appeler des *homologies*.

The birth of homology, from *Analysis Situs*, H. Poincaré (1895).

Exercise 8.1 (The five-lemma.)
Prove the famous *five-lemma*. Let R be a ring and suppose we have the following diagram of modules over R:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

Assume that this diagram is commutative and that the two horizontal rows of homomorphisms are exact. Moreover, assume that α is surjective, β and δ are bijective and ε is injective. Prove that γ is bijective.

Application: let $\phi\colon B \to B'$ be a homomorphism of R-modules taking a submodule $A \subseteq B$ to a submodule $A' \subseteq B'$, so that we have restricted and induced homomorphisms $\phi|_A\colon A \to A'$ and $\bar{\phi}\colon B/A \to B'/A'$. If $\phi|_A$ and $\bar{\phi}$ are both isomorphisms then so is ϕ.

Exercise 8.2 (Mapping cones and mapping cylinders of chain complexes.)
Let A and B be chain complexes with differential ∂_A resp. ∂_B and let $f\colon A \to B$ be a chain map.
(i) Define a new chain complex $\mathrm{Cone}(f)$ by $\mathrm{Cone}(f)_n = A_{n-1} \oplus B_n$ and setting its differential $\partial\colon A_{n-1} \oplus B_n \to A_{n-2} \oplus B_{n-1}$ to be the sum of the four maps

$$\partial_A\colon A_{n-1} \to A_{n-2} \qquad \partial_B\colon B_n \to B_{n-1} \qquad 0\colon B_n \to A_{n-2} \qquad (-1)^n.f_{n-1}\colon A_{n-1} \to B_{n-1},$$

or as a formula

$$\partial(a,b) := (\partial_A(a), (-1)^n.f_{n-1}(a) + \partial_B(b)).$$

(1) Prove that this is indeed a chain complex.
(2) Construct chain maps $B \to \mathrm{Cone}(f)$ and $\mathrm{Cone}(f) \to A[1]$, where $A[1]$ simply means the chain complex A with the modified grading $A[1]_n = A_{n-1}$, and show that you have constructed a short exact sequence

$$0 \to B \longrightarrow \mathrm{Cone}(f) \longrightarrow A[1] \to 0.$$

(ii) Now define a chain complex $\mathrm{Cyl}(f)$ by $\mathrm{Cyl}(f)_n = A_{n-1} \oplus B_n \oplus A_n$ with differential $\partial\colon \mathrm{Cyl}(f)_n \to \mathrm{Cyl}(f)_{n-1}$ given in block form by the matrix

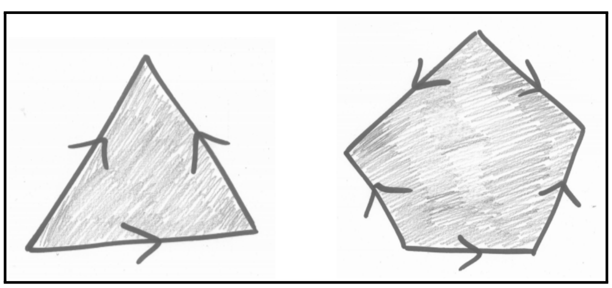
$$\begin{pmatrix} \partial_A & 0 & 0 \\ (-1)^n.f_{n-1} & \partial_B & 0 \\ (-1)^{n+1}.\mathrm{id} & 0 & \partial_A \end{pmatrix}.$$

(3) Prove that this is a chain complex.
(4) Construct a chain homotopy equivalence $\mathrm{Cyl}(f) \simeq B$.

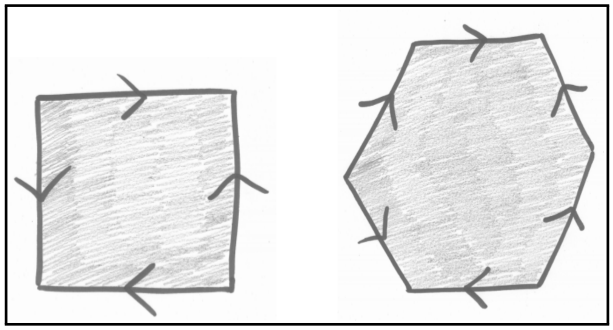
Exercise 8.3 (Cones of continuous maps and dunce caps.)
Let Z be a space with subspace $A \subseteq Z$ and let $f\colon A \to Y$ be a continuous map. Recall (cf. Exercise 5.6) that the space $Z \cup_f Y$ is defined to be the quotient of the disjoint union $Z \sqcup Y$ by the smallest equivalence relation $\sim$ such that $a \sim f(a)$ for all $a \in A$.
Now let $g\colon X \to Y$ be a continuous map and define its *mapping cylinder* to be $\mathrm{Cyl}(g) = Z \cup_f Y$ where $Z = X \times [0,1]$, $A = X \times \{0\}$ and f is g composed with the obvious identification $X \times \{0\} \cong X$. Define its *mapping cone* to be $\mathrm{Cone}(g) = \mathrm{Cyl}(g)/{\sim}$, where $\sim$ is the smallest equivalence relation such that $(x,1) \sim (x',1)$ for all $x, x' \in X$.

(1) Draw a picture to show what is going on geometrically in these constructions.
(2) Construct an embedding $Y \to \mathrm{Cone}(g)$ and a projection $\mathrm{Cone}(g) \to \Sigma X$, where ΣX is the *suspension* of X, defined to be $X \times [0,1]/{\sim}$, where $\sim$ is the smallest equivalence relation such that $(x,1) \sim (x',1)$ and $(x,0) \sim (x',0)$ for all $x, x' \in X$.
(3) Show that $\mathrm{Cyl}(g)$ is homotopy equivalent to Y.
(We will see later in the course that these constructions yield those of the previous exercise after applying the singular chain functor.)

(4) Now let $A \subseteq X$ be a closed subspace for which there exists an open neighbourhood $U \supseteq A$ that deformation retracts onto A. Let $f, g\colon A \to Y$ be two continuous maps which are homotopic. Prove that $X \cup_f Y$ and $X \cup_g Y$ are homotopy equivalent. Thus the operations $\mathrm{Cyl}(-)$ and $\mathrm{Cone}(-)$ are homotopy invariant.
(5) Thus show that the following two spaces are contractible:



(Hint: realise each of them as $\mathbb{D}^2 \cup_f \mathbb{S}^1$ for some map $f\colon \partial\mathbb{D}^2 \to \mathbb{S}^1$, and consider the degree of this map.)
The left-hand space above is often called the “dunce cap”. There are many *generalised dunce caps* like the right-hand space above – each of them is the quotient of a polygon with an odd number of sides, which are all identified with certain choices of orientations.
(6) Using a similar trick to above, show that the following two spaces each have fundamental group isomorphic to $\mathbb{Z}$, and draw a generator in each case:
(Note: in the left-hand space on the next page, in addition to the depicted identifications of edges, we also identify all four (not just two) vertices to a single point.)



Exercise 8.4 (Mapping tori of chain complexes.)
Let R be a ring, C a chain complex of R-modules and $f\colon C \to C$ be a chain map from C to itself. We can formally adjoin an invertible indeterminate t to C to obtain a chain complex $\bar{C}$ of $R[t^{\pm 1}]$-modules by first setting $\bar{C}_n = C_n \otimes_R R[t^{\pm 1}]$ and then defining $\bar{\partial}$ to be ∂ extended by linearity in t (more formally: $\bar{\partial} = \partial \otimes \mathrm{id}$, where id is the identity map $R[t^{\pm 1}] \to R[t^{\pm 1}]$). Here, $R[t^{\pm 1}]$ is the ring of *Laurent polynomials* in t with coefficients in R, or, equivalently, the *group-ring* $R[\mathbb{Z}]$ of the group $\mathbb{Z}$ with coefficients in R. The chain map f extends by linearity in t to a chain map $\bar{f}\colon \bar{C} \to \bar{C}$. There is also a canonical chain map $t\colon \bar{C} \to \bar{C}$ where each $t_n\colon \bar{C}_n \to \bar{C}_n$ is just multiplication by t. Define:

$$\mathrm{Torus}(f) = \mathrm{Cone}(\bar{f} - t).$$

(1) Describe this explicitly in terms of the C_n, ∂_C and f_n.
(2) Suppose that we have a commutative square

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\alpha"'] & C \arrow[d, "\alpha"] \\ D \arrow[r, "g"] & D \end{tikzcd}$$

Define a chain map $(\alpha, \alpha)_\sharp\colon \mathrm{Torus}(f) \to \mathrm{Torus}(g)$ and show that your construction satisfies the two functoriality properties $(\mathrm{id}, \mathrm{id})_\sharp = \mathrm{id}$ and $(\alpha, \alpha)_\sharp \circ (\alpha', \alpha')_\sharp = (\alpha \circ \alpha', \alpha \circ \alpha')_\sharp$.
(3) Show that, for any chain map $f\colon C \to C$, the chain map $(f, f)_\sharp\colon \mathrm{Torus}(f) \to \mathrm{Torus}(f)$ induces isomorphisms on all homology groups.
(Hint: construct a chain homotopy from $(f, f)_\sharp$ to the "multiplication by t" chain map from $\mathrm{Torus}(f)$ to itself. Then show that this "multiplication by t" chain map induces isomorphisms on all homology groups and use the homotopy-invariance property of homology to deduce that the same is true for $(f, f)_\sharp$.)
(4) Deduce that, for chain maps $f\colon C \to D$ and $g\colon D \to C$, the chain complexes $\mathrm{Torus}(f \circ g)$ and $\mathrm{Torus}(g \circ f)$ have the same homology groups.
(Hint: consider the chain maps $(f \circ g, f \circ g)_\sharp$ and $(g \circ f, g \circ f)_\sharp$.)

The 6-point space Σ^2 "would be homeomorphic to a 2-sphere if it were only Hausdorff." More precisely, consider the following conditions on a topological space X : (1) *The complement of each point in X is acyclic* (*in singular homology*); (2) $H_2(X) \neq 0$. We have seen that the T_0 space Σ^2 satisfies these two conditions. However, simply by adding the extra condition (3) *X is Hausdorff*, one can conclude that X is homeomorphic to the 2-sphere. (See [5].)

From *Singular homology groups and homotopy groups of finite topological spaces* by M. C. McCord (1966). In his notation, Σ^2 is the 6-point space considered in Exercise 8.5(c) on the next page. Thus you have a strong hint as to what the homology of that space "should" be!

Exercise 8.5 (Finite topological spaces.)
(a) There are three topological spaces X having exactly two points. In each case, compute the singular chain complex $S_\bullet(X)$ and the homology $H_n(X)$ for all n.
(b) Consider the 4-point topological space $\{a, b, c, d\}$ whose topology is generated by the base

$$\{a\}; \{b\}; \{a, b, c\}; \{a, b, d\}$$

and calculate its homology.
(c)* Do the same for the 6-point space $\{a, b, c, d, e, f\}$ whose topology is generated by the base

$$\{a\}; \{b\}; \{a, b, c\}; \{a, b, d\}; \{a, b, c, d, e\}; \{a, b, c, d, f\}.$$

(d)* In general, there is a $(2n + 2)$-point space $\{a_1, b_1, \ldots, a_{n+1}, b_{n+1}\}$ whose topology is generated by the base

$$\{a_1\}; \{b_1\}; \{a_1, b_1, a_2\}; \{a_1, b_1, b_2\}; \ \ldots\ldots\ ; \{a_1, b_1, \ldots, a_n, b_n, a_{n+1}\}; \{a_1, b_1, \ldots, a_n, b_n, b_{n+1}\}.$$

Make a conjecture about its homology, and about which (more familiar!) space it is homotopy equivalent to.
(e)** Prove your conjecture.

Exercise 8.6* (A chain complex of chain maps.)
Let C and D be chain complexes of R-modules. We define a *chain map of degree* d to be a collection $f = \{f_n\}_{n\in\mathbb{Z}}$ of homomorphisms of R-modules $f_n \colon C_n \to D_{n+d}$ such that $\partial^D_{n+d} \circ f_n = f_{n-1} \circ \partial^C_n$ for all n. We define a *pre-chain map of degree* d to be simply a collection $f = \{f_n\}_{n\in\mathbb{Z}}$ of homomorphisms $f_n \colon C_n \to D_{n+d}$, with no condition.
(1) Show that the set of all pre-chain maps of a fixed degree d forms an R-module, denoted $\mathrm{PreChain}_d(C, D)$.
(2) Given $f = \{f_n\} \in \mathrm{PreChain}_d(C, D)$, show that the formula

$$(df)_n = \partial^D_{n+d} \circ f_n - (-1)^d f_{n-1} \circ \partial^C_n$$

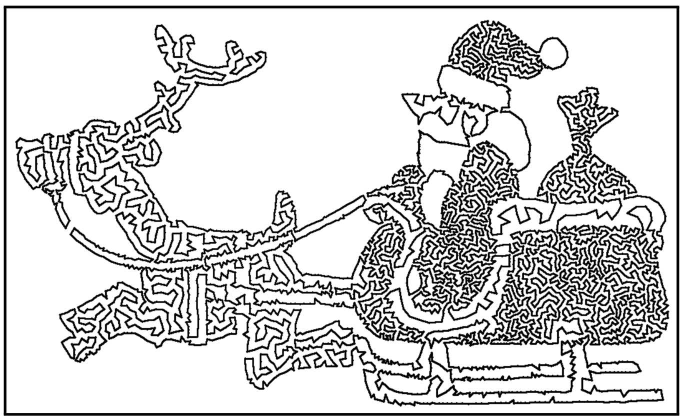
defines a pre-chain map $df \in \mathrm{PreChain}_{d-1}(C, D)$.
(3) Show that $ddf = 0$, and hence that $\mathrm{PreChain}_\bullet(C, D)$ is a chain complex.
(4) Prove that there is a natural isomorphism between $H_0(\mathrm{PreChain}_\bullet(C, D))$ and the set of chain-homotopy-classes of chain maps (of degree 0) from C to D. (Hint: first note that $Z_0(\mathrm{PreChain}_\bullet(C, D))$ is naturally isomorphic to the set of chain maps of degree 0.)

Aufgaben zur Topologie

Prof. Dr. C.-F. Bödigheimer

Wintersemester 2016/17

Week 9 — Long exact homology sequences Due: 11. January 2017



Henry Poincaré and his red-nosed reindeer called Rudolph Homology.

Exercise 9.1 (The connecting homomorphism I)
Let

$$0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$$

be a short exact sequence of chain complexes over $\mathbb{K}$ and denote by $\partial_*\colon H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ the connecting homomorphism.
(a) ∂_* is a homomorphism of $\mathbb{K}$-modules.
(b) ∂_* is natural.

Exercise 9.2 (Induced maps on homology in degree zero.)
Let X be a space and denote by $\pi_0(X)$ the set of path-components of X.

(1) Show that there is an isomorphism $H_0(X) \cong \mathbb{Z}\langle\pi_0(X)\rangle := \bigoplus_{\pi_0(X)} \mathbb{Z}$, the free abelian group generated by the set $\pi_0(X)$.

(2) Show that any continuous map $f\colon X \to Y$ induces a well-defined function $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$.

(3) If we identify $H_0(X)$ with $\mathbb{Z}\langle\pi_0(X)\rangle$ and $H_0(Y)$ with $\mathbb{Z}\langle\pi_0(Y)\rangle$ as above, then the homomorphisms $H_0(f)$ and $\mathbb{Z}\langle\pi_0(f)\rangle$ are equal.

Exercise 9.3 (Relative homology in degree zero.)
Let A be a subspace of X; we denote by $i\colon A \to X$ the inclusion and by $q\colon X \to X/A$ the quotient map identifying A to a point. (What does this mean, when $A = \emptyset$?). For the sets of path-components we have induced functions $\pi_0(i)\colon \pi_0(A) \to \pi_0(X)$ and $\pi_0(q)\colon \pi_0(X) \to \pi_0(X/A)$.

(1) Show that the relative homology group $H_0(X, A)$ is isomorphic to $\mathbb{Z}\langle\pi_0(X/A)\rangle/\mathbb{Z}\langle[A/A]\rangle$, where $[A/A]$ denotes the path-component of X/A containing the point A/A. [*Retrospective correction*: this is not true for arbitrary pairs (X, A); one needs some extra hypothesis. Assuming that X is locally path-connected suffices, for example.]

(2) When A is non-empty, show that $H_0(q)\colon H_0(X) \to H_0(X, A)$ is surjective and that $H_0(X, A)$ is isomorphic to

the quotient module $H_0(X)/\mathrm{im}(H_0(i))$.
(3) Describe $H_1(X,A)$ and the connecting homomorphism $\partial_*\colon H_1(X,A) \to H_0(A)$.

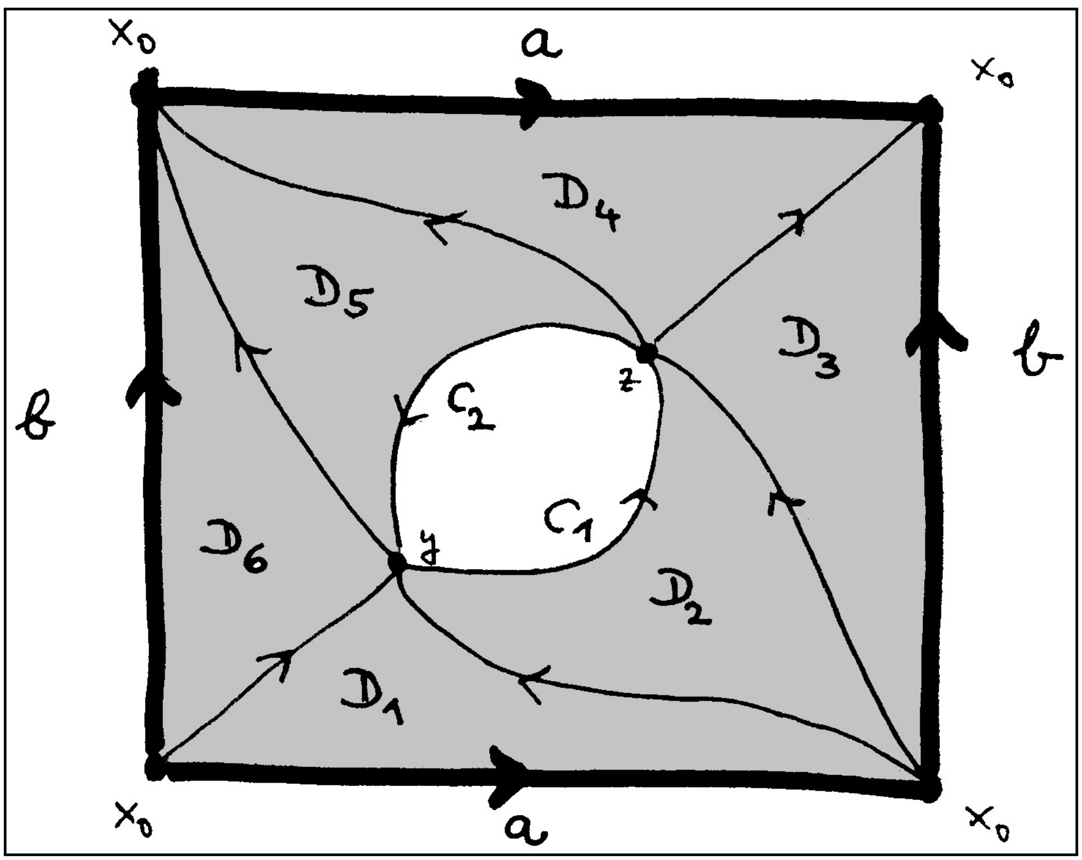
Exercise 9.4 (Torus with one boundary curve.)
Let X be a torus with one boundary curve, as shown in the figure. We denote by A the boundary curve, by $i\colon A \to X$ the inclusion of spaces, by $S_\bullet(i)\colon S_\bullet(A) \to S_\bullet(X)$ the inclusion of singular chain complexes, and by $q_\bullet\colon S_\bullet(X) \to S_\bullet(X,A) = S_\bullet(X)/S_\bullet(A)$ the quotient homomorphism of singular chain complexes; thus we have the short exact sequence of chain complexes

$$0 \to S_\bullet(A) \xrightarrow{S_\bullet(i)} S_\bullet(X) \xrightarrow{q_\bullet} S_\bullet(X,A) \to 0$$

leading to a long exact sequence of homology groups:

$$\ldots \to H_2(A) \xrightarrow{i_*} H_2(X) \xrightarrow{q_*} H_2(X,A) \xrightarrow{\partial_*} H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{q_*} H_1(X,A) \xrightarrow{\partial_*} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{q_*} H_0(X,A) \to 0$$

(1) Compute all homology groups in degrees 0 and 1 and all homomorphisms between them.
(2) Show that $H_2(X,A)$ contains at least a summand which is isomorphic to $\mathbb{Z}$ and generated by the relative cycle $D = -D_1 + D_2 + D_3 + D_4 + D_5 - D_6$.
(3) Compute for the connecting homomorphism $\partial_*([D]) = [c_1 + c_2]$.



A torus, written as a square with dark edges a and b identified. Some singular 1- and 2-simplices, used in Exercise 9.4, are shown.

Exercise 9.5 (Calculating with long exact sequences of abelian groups.)

(1) Show that, if $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \to 0$ is an exact sequence, then we also have exact sequences:

(a) $0 \to B/\alpha(A) \xrightarrow{\bar{\beta}} C \xrightarrow{\gamma} D \to 0$ and

(b) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} \mathrm{image}(\beta) \to 0$.

(2) More generally, if we have an exact sequence $\cdots \to A \xrightarrow{e} B \xrightarrow{f} C \xrightarrow{g} D \xrightarrow{h} E \to \cdots$, then we may "localise" it at C, meaning that there is a short exact sequence:

$$0 \to \mathrm{coker}(e) \xrightarrow{\bar{f}} C \xrightarrow{g} \ker(h) \to 0,$$

where $\mathrm{coker}(e)$ means $B/e(A)$.
(3) Show that, if $0 \to A \to B \to \mathbb{Z}^k \to 0$ is exact, then B is isomorphic to $\mathbb{Z}^k \oplus A$.
(Hint: start by constructing a homomorphism $\mathbb{Z}^k \to B$, which is a right-inverse for the given homomorphism $B \to \mathbb{Z}^k$.)
(4) Give an example of a short exact sequence $0 \to A \to B \to \mathbb{Z}/k\mathbb{Z}$ (for any $k \neq -1, 0, +1$) for which B is not isomorphic to $A \oplus \mathbb{Z}/k\mathbb{Z}$.

Exercise 9.6* (Connecting homomorphisms II)
Assume we have a short exact sequence

$$(*) \quad 0 \to A_\bullet \xrightarrow{\alpha_\bullet} B_\bullet \xrightarrow{\beta_\bullet} C_\bullet \to 0$$

of free chain complexes; we know that $B_n \cong A_n \oplus C_n$, since C_n is free. But the boundary operator $\partial^B \colon B_n \to B_{n-1}$ may not be the direct sum of ∂^A und ∂^C, i.e., not of diagonal block form; but it must have the form

$$\partial^B = \begin{pmatrix} \partial^A & \varphi_n \\ 0 & \partial^C \end{pmatrix}$$

for some family of homomorphisms $\varphi_n \colon C_n \to A_{n-1}$.
(1) Which condition must this family satisfy, such that $\partial^B \circ \partial^B = 0$?
(2) Compute the connecting homomorphism in the long exact homology sequence of (*).

Aufgaben zur Topologie

Prof. Dr. C.-F. Bödigheimer

Wintersemester 2016/17

Week 10 — Excision and mapping tori Due: 18. January 2017

Exercise 10.1 (Reduction Lemma.)
Let $A_\bullet$ be a subcomplex of a chain complex $B_\bullet$ and assume the following two hypotheses:
(R1) Each cycle $b \in B_\bullet$ is homologous (in $B_\bullet$) to a cycle in $A_\bullet$.
(R2) If two cycles a, a' in $A_\bullet$ are homologous in $B_\bullet$, then they are homologous in $A_\bullet$.
Prove that the inclusion $\iota_\bullet \colon A_\bullet \to B_\bullet$ induces an isomorphism $\iota_* \colon H_n(A_\bullet) \to H_n(B_\bullet)$.

In addition show that (R1) and (R2) follow from:
(R3) Each chain in $B_\bullet$ is homologous (in $B_\bullet$) to a chain in $A_\bullet$.

Exercise 10.2 (Transformators.)
A *transformator* is a natural self-transformation $\tau_\bullet$ of the functor $S_\bullet$ with $\tau_0 = \mathrm{id}$. Spelled out in detail this is, for each space X and each $n \geq 0$, a homomorphism $\tau_n^X : S_n(X) \to S_n(X)$ such that
(1) $\tau_0^X = \mathrm{id}_{S_0(X)}$,
(2) $\partial \circ \tau_n^X = \tau_{n-1}^X \circ \partial$,
(3) $S_n(f) \circ \tau_n^X = \tau_n^Y \circ S_n(f)$ for any map $f \colon X \to Y$.
Examples:
(i) $\tau_n^X = \mathrm{id}_{S_n(X)}$ is a trivial example.
(ii) The composition of two transformators is a transformator.
(iii) The barycentric subdivision $\tau = B$ (as defined in the lectures) is the fundamental example.
(iv) If $\omega_n \colon \Delta^n \to \Delta^n$ is the affine homeomorphism permuting the vertices $e_0, e_1, \ldots, e_n$ of Δ^n like $\omega_n(e_i) = e_{n-i}$, then $\tau_n^X(a) := \epsilon(n).a \circ \omega_n$ for a simplex $a \colon \Delta^n \to X$, where the sign $\epsilon(n)$ is -1 when $n \equiv 1, 2 \bmod 4$ and is $+1$ when $n \equiv 0, 3 \bmod 4$, defines a transformator.

Prove the following:
(a) If $\tau_\bullet$ is a transformator, then its image $S_\bullet^\tau(X) := \mathrm{im}(\tau_\bullet^X \colon S_\bullet(X) \to S_\bullet(X))$ defines a subcomplex of $S_\bullet(X)$, which satisfies the two hypotheses (R1) and (R2) of the Reduction Lemma (Exercise 10.1).
(b) $\tau_\bullet^X$ induces the identity in homology, since there is a natural chain homotopy between $\tau_\bullet^X$ and the identity.
(c)* (A generalisation of part (a).) If $\Lambda_n \colon \mathfrak{B}_n(X) \to \mathbb{N}$ is a family of functions from the basis $\mathfrak{B}_n(X)$ of $S_n(X)$, denote by $S_n^\Lambda(X)$ the subgroup of $S_n(X)$ generated by the elements $\tau^k(a)$ for all $a \in \mathfrak{B}_n(X)$ and all $k \geq \Lambda_n(a)$. Assume that the functions satisfy the following property: whenever $a \in \mathfrak{B}_n(X)$ and b is a basis element in $\partial(a)$, we have $\Lambda_n(a) \geq \Lambda_{n-1}(b)$. Then $S_\bullet^\Lambda(X)$ is a subcomplex of $S_\bullet(X)$ satisfying (R1) and (R2) of the Reduction Lemma. (Cf. the preparation for the Excision Theorem.)

Exercise 10.3 (Local homology.)
For a space X and a point $x \in X$, the *local homology groups* of X at x are by definition the relative homology groups $H_n(X, X - \{x\})$, in other words, the homology groups of the quotient chain complex $S_\bullet(X)/S_\bullet(X - \{x\})$. The name *local* comes from the following property of these groups:
(1) Let U be an open neighbourhood of x in X. Assume that the one-point subspace $\{x\}$ is closed in X (for example, this is always true if X is Hausdorff). Using the Excision Theorem, show that there are isomorphisms

$$H_n(X, X - \{x\}) \cong H_n(U, U - \{x\}).$$

Thus the local homology of a space X at a point depends only on its topology arbitrarily close to that point.
(2) If F is a surface, show that for any point $x \in F$ the relative homology $H_n(F, F - \{x\})$ is trivial for $n \neq 2$ and isomorphic to $\mathbb{Z}$ for $n = 2$. There are three steps:

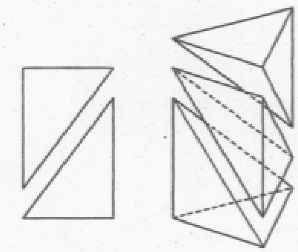


Fig. 59 Triangulations of cylinders over simplices

From: A. Fomenko, D. Fuchs, *Homotopical Topology.*

(i) $H_n(F, F - \{x\}) \cong H_n(\mathbb{D}^2, \mathbb{D}^2 - \{0\})$
(ii) $H_n(\mathbb{D}^2, \mathbb{D}^2 - \{0\}) \cong H_n(\mathbb{D}^2, \partial\mathbb{D}^2)$
(iii) $H_n(\mathbb{D}^2, \partial\mathbb{D}^2)$ is isomorphic to $H_{n-1}(\mathbb{S}^1)$ for $n \geq 2$ and is zero for $n = 0, 1$. (cf. Exercise 9.3)
(You may assume the fact that the homology $H_n(\mathbb{S}^1)$ of $\mathbb{S}^1$ is isomorphic to $\mathbb{Z}$ for $n = 0, 1$ and is trivial for $n \geq 2$.)
(3)* If F is a surface with boundary and x is a point on the boundary ∂F, show that the relative homology groups $H_n(F, F - \{x\})$ are all trivial, including for $n = 2$.

Exercise 10.4 (Quotient Theorem : Relative homology and collapsing a subspace to a point.)
Let X be a space and let $C \subseteq U \subseteq X$ be subspaces, where C is closed in X and U is open in X.
(1) Using inclusions and (restrictions of) the quotient map $X \to X/C$ that collapses C to a point, construct a commutative diagram of pairs of spaces as follows:

$$\begin{tikzcd} (X - C, U - C) \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & (X, U) \arrow[d, "\delta"] \\ (X/C - C/C, U/C - C/C) \arrow[r, "\beta"] & (X/C, U/C) \end{tikzcd} \tag{$\odot$}$$

(2) Show that, for any subspace $A \subseteq X - C$, the restriction to A of the quotient map $X \to X/C$ is a homeomorphism onto its image.
(3) The quotient map $X \to X/C$ induces a map of long exact sequences from the long exact sequence associated to the pair $(X - C, U - C)$ to the long exact sequence associated to the pair $(X/C - C/C, U/C - C/C)$.
(4) Using the five-lemma (cf. Exercise 8.1), show that the map γ in $(\odot)$ induces isomorphisms on homology.
(5) Applying excision to the maps α and β, show that $H_*(X, U)$ is isomorphic to $H_*(X/C, U/C)$ via δ_*.
(6)* Now assume that U deformation retracts onto C. Show that U/C deformation retracts onto the point C/C.
(7)* Under the same assumption, use the long exact sequences associated to the pairs (X, U) and $(X/C, U/C)$ to show that

$$H_*(X/C, C/C) \cong H_*(X/C, U/C) \cong H_*(X, U) \cong H_*(X, C).$$

So the relative homology of the pair (X, C) is isomorphic to the homology of the quotient space X/C relative to a point.

Exercise 10.5 (Homology of mapping tori.)
Given a continuous map $f \colon X \to X$, the *mapping torus* $T(f)$ of f is defined to be the quotient space $X \times [0, 1]/\sim$, where $\sim$ is the equivalence relation generated by the relations $(x, 0) \sim (f(x), 1)$ for all $x \in X$.
(a) Using Exercise 8.3, explain why $f \simeq f'$ implies that $T(f) \simeq T(f')$.
In Exercise 10.6* you will (optionally) construct the associated long exact sequence, which is of the form

$$\cdots H_{i+1}(T(f)) \to H_i(X) \to H_i(X) \to H_i(T(f)) \to H_{i-1}(X) \to \cdots, \tag{$\star$}$$

where the map $H_i(X) \to H_i(X)$ is defined by $H_i(f) - \mathrm{id}$. Explain why this is the zero map for $i = 0$ when X is path-connected.

(b) Now take $X = \mathbb{S}^1$ and let f be a self-map of degree $d \neq 1$. Using the long exact sequence ($\star$) and the techniques from Exercise 9.5, together with the fact (which you may assume) that $H_i(\mathbb{S}^1) \cong \mathbb{Z}$ for $i = 0, 1$ and $H_i(\mathbb{S}^1) = 0$ for $i \neq 0, 1$, compute the homology of the mapping torus $T(f)$.
Also show that $T(f)$ is homotopy equivalent to the Klein bottle when $d = -1$ and to $\mathbb{S}^1$ when $d = 0$. (In fact, for any space X, if f is nullhomotopic, then $T(f)$ is homotopy equivalent to $\mathbb{S}^1$.)
(c) Now let X be any space and take f to be the identity $X \to X$. Show that the homology of $T(f)$ is related to that of X via short exact sequences $0 \to H_i(X) \to H_i(T(f)) \to H_{i-1}(X) \to 0$. Therefore, if the homology of X is free abelian in each degree, we have isomorphisms $H_i(T(f)) \cong H_i(X) \oplus H_{i-1}(X)$. As a special case, we obtain the homology of the torus $\mathbb{S}^1 \times \mathbb{S}^1$.
(d)* (The homology of some 3-manifolds.) If X is a (smooth) surface and $f \colon X \to X$ is a homeomorphism (resp. diffeomorphism) then $T(f)$ is a (smooth) 3-manifold. In this part we take X to be the genus-2 surface F_2.
(i) First suppose that f is a Dehn twist. Recall (cf. Exercise 6.2) that, if one chooses the generators of $H_1(F_2) \cong \mathbb{Z}^4$ in an appropriate way, the induced homomorphism $H_1(f)$ is the elementary matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

You may assume the following facts: $H_2(F_2) \cong \mathbb{Z}$ and $H_i(F_2) = 0$ for $i \geqslant 3$. Moreover, any self-homeomorphism of F_2 which is the identity on some open subset induces the identity homomorphism on $H_2(F_2)$. Compute the homology of the mapping torus $T(f)$.
(ii) Now suppose that F_2 is embedded into $\mathbb{R}^3$ in such a way that it is symmetric with respect to reflection in a plane P that cuts it into two punctured tori, and let $f \colon F_2 \to F_2$ be the self-homeomorphism induced by reflection in P. Show (by a picture) that one may choose generators for $H_1(F_2)$ in such a way that $H_1(f)$ is given by the permutation matrix

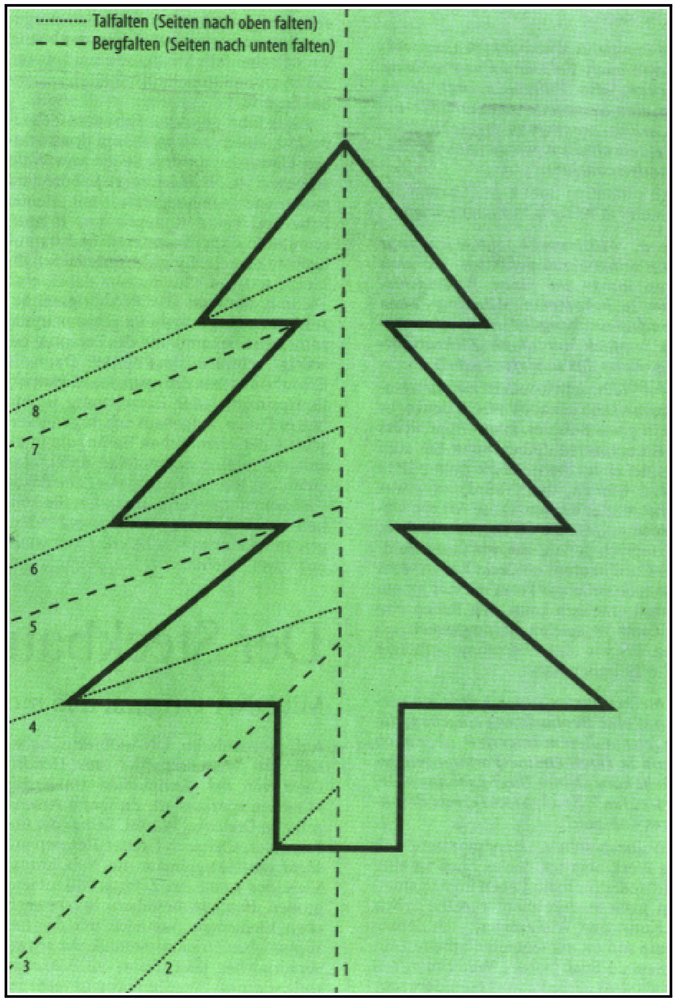
$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

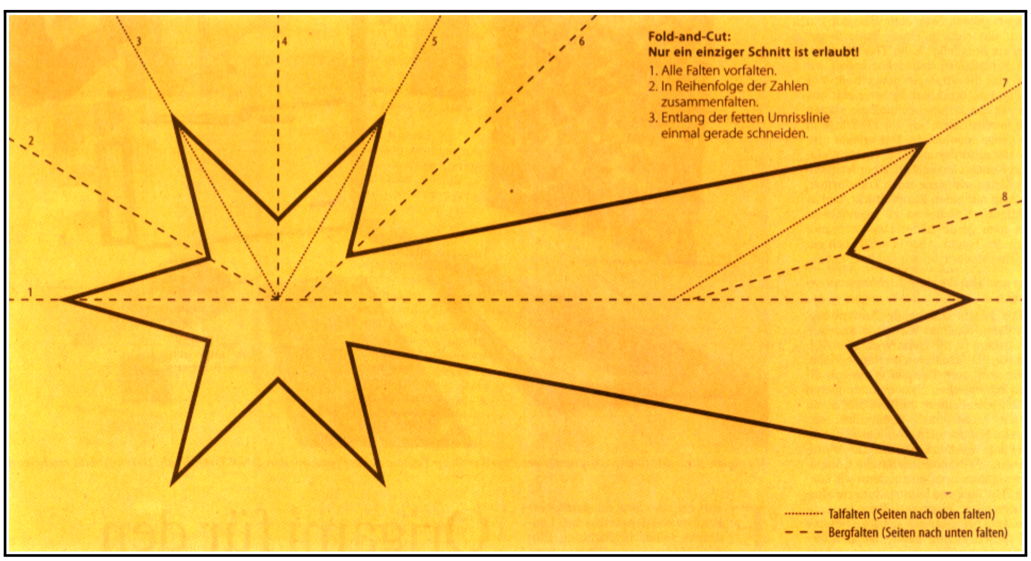
with respect to these generators. Calculate the homology of $T(f)$ in this case. (You may use the fact that, in this case, f acts on $H_2(F_2)$ by multiplication by -1.)

Exercise 10.6* (The long exact sequence associated to a mapping torus.)
Recall the definition of the *mapping torus* $T(f)$ of a map $f \colon X \to X$ from Exercise 10.5, and let $q \colon X \times [0,1] \to T(f)$ denote the quotient map.
(1) Show that the image of $X \times \{0,1\}$ under q is a subspace of $T(f)$ homeomorphic to X. Moreover, if we identify $X \times \{0,1\}$ with $X \sqcup X$ and $q(X \times \{0,1\})$ with X, then the restriction of q to $X \times \{0,1\}$ is $f \sqcup \mathrm{id}$.
(2) We therefore have a map of pairs of topological spaces of the form $(X \times [0,1], X \times \{0,1\}) \to (T(f), X)$. Each pair of spaces has an associated long exact sequence of homology groups; draw a diagram of the *map* of long exact sequences induced by the map of pairs.
(3) The desired long exact sequence ($\star$) is very similar to the long exact sequence associated to the pair $(T(f), X)$. To derive ($\star$) from this sequence, you just need to explain why the terms $H_i(T(f), X) = H_i(C_\bullet(T(f))/C_\bullet(X))$ that appear in the long exact sequence for the pair $(T(f), X)$ can be replaced by $H_{i-1}(X)$, and why the map $H_i(X) \to H_i(X)$ after doing this replacement is given by $H_i(f) - \mathrm{id}$. Do this using the following steps:
(a) The map $H_i(X \times \{0,1\}) \to H_i(X \times [0,1])$ in your diagram is surjective (for every i).
(b) The map $H_{i+1}(X \times [0,1], X \times \{0,1\}) \to H_i(X \times \{0,1\})$ is an isomorphism onto a subgroup which is isomorphic to $H_i(X)$.
(c) The map $H_i(X \times [0,1], X \times \{0,1\}) \to H_i(T(f), X)$ is an isomorphism. (Hint: each pair of spaces has the property that the subspace has a neighbourhood in the larger space that deformation retracts onto it.)
(d) Thus there is an isomorphism $H_i(X) \cong H_{i+1}(T(f), X)$. Using commutativity of your diagram, and part (1), it follows that the map $H_i(X) \cong H_{i+1}(T(f), X) \to H_i(X)$ is given by $H_i(f) - H_i(\mathrm{id}_X) = H_i(f) - \mathrm{id}_{H_i(X)}$.

Der Meister der Origami-Technik

Erik Demaine wurde 1981 in Halifax (Kanada) geboren. Seit 2001 ist er Professor am Massachusetts Institute of Technology und interessiert sich für fast alle Bereiche der Mathematik und Informatik, die mit Algorithmen zu tun haben, insbesondere für Faltungsprobleme der diskreten und rechnerbasierten Geometrie. Sein Vater, der Künstler Martin Demaine, nahm ihn mit sieben Jahren aus der Schule, um mit ihm vier Jahre lang durch die Vereinigten Staaten zu reisen. In dieser Zeit entdeckte Demaine seine Begeisterung für das Programmieren, die ihn mit zwölf Jahren ein Studium der Informatik beginnen ließ. Am MIT arbeiten Vater und Sohn nach wie vor eng zusammen. Im Jahr 1999 bewiesen beide das „Fold-and-Cut-Theorem", das besagt, dass man jede polygone Form mit nur einem einzigen geraden Schnitt aus einem gefalteten Papier schneiden kann. Als mathematisches Rätsel wurde dieses Problem in Japan bereits 1721 beschrieben. Einer Anekdote zufolge war die einfache Erzeugung fünfzackiger Sterne per Fold-and-Cut-Technik auch der Grund dafür, dass die amerikanische Flagge diese anstatt sechszackiger Sterne verwendet. Wir haben für Sie hier zwei Faltanleitungen abgedruckt, mit deren Hilfe Sie diese Technik zur Produktion von Weihnachtsmotiven nutzen können. (sian)





For those who missed Christmas completely: an origami from FAZ 21.12.2016.

Aufgaben zur Topologie

Prof. Dr. C.-F. Bödigheimer

Wintersemester 2016/17

Week 11 — Mayer-Vietoris sequence, reduced homology, mapping degree Due: 25. January 2017

Exercise 11.1 (Spaces with finite type homology; homological dimension.)
The homology of a space X is called *of finite type* if there exists an integer $N \geq 0$ such that
(F1) $H_n(X) = 0$ for $n > N$,
(F2) $H_n(X)$ is finitely generated for $0 \leq n \leq N$.

(1) Show, for a space $X = X_1 \cup X_2$, that if X_1 and X_2 are open subspaces of X having homology of finite type, and the homology of $X_1 \cap X_2$ is also of finite type, then so is the homology of X.

The smallest integer N such that (F1) holds is called the *homological dimension* $\mathrm{dimhom}(X)$ of X.
(2) Using the proof of part (1), give an upper bound for $\mathrm{dimhom}(X)$ in terms of $\mathrm{dimhom}(X_1)$, $\mathrm{dimhom}(X_2)$ and $\mathrm{dimhom}(X_1 \cap X_2)$.

Exercise 11.2 (Reduced homology and augmentation.)
Recall that the reduced homology of a space X is defined as $\widetilde{H}_n(X) = \ker\big(e_*\colon H_n(X) \to H_n(P)\big)$, where P is a one-point space and $e\colon X \to P$ the obvious map. Consider the singular chain complex $S_\bullet(X)$ of X and the *augmentation homomorphism*

$$\epsilon\colon S_0(X) \longrightarrow \mathbb{Z} \qquad \epsilon\Big(\sum_i \lambda_i x_i\Big) = \sum_i \lambda_i,$$

where we wrote, instead of a 0-simplex $a_i\colon \Delta^0 \to X$ with $a_i(e_0) = x_i$, just the value x_i. We call

$$S_\bullet^{\mathrm{aug}}(X): \qquad 0 \leftarrow \mathbb{Z} \xleftarrow{\epsilon} S_0(X) \xleftarrow{\partial} S_1(X) \xleftarrow{\partial} \cdots$$

with $S_{-1}^{\mathrm{aug}}(X) = \mathbb{Z}$ the *augmented singular chain complex* of X.

(1) Show that $\epsilon \circ \partial = 0$. So one may regard $S_\bullet^{\mathrm{aug}}(X)$ as a chain complex with one negatively indexed chain group.
(2) There is a chain map $E_\bullet\colon S_\bullet^{\mathrm{aug}}(X) \to S_\bullet(X)$ with $E_n = \mathrm{id}$ for $n \geq 0$ and $E_{-1} = 0$:

$$\begin{tikzcd}
0 & \mathbb{Z} \arrow[l] \arrow[d] & S_0(X) \arrow[l, "\epsilon"'] \arrow[d, equal] & S_1(X) \arrow[l, "\partial"'] \arrow[d, equal] & \cdots \arrow[l] \\
0 & 0 \arrow[l] & S_0(X) \arrow[l] & S_1(X) \arrow[l, "\partial"'] & \cdots \arrow[l]
\end{tikzcd}$$

(3) The chain map $E_\bullet$ induces homomorphisms

$$E_*\colon H_n(S_\bullet^{\mathrm{aug}}(X)) \longrightarrow H_n(S_\bullet(X))$$

for each n. Show that E_* is injective for $n \geq 0$, an isomorphism for $n \geq 1$ and $H_{-1}(S_\bullet^{\mathrm{aug}}(X)) = 0$ for non-empty X.
Hint: Regard ϵ as a chain map

$$\begin{tikzcd}
0 & S_0(X) \arrow[l] \arrow[d, "\epsilon"'] & S_1(X) \arrow[l, "\partial"'] \arrow[d] & S_2(X) \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l] \\
0 & \mathbb{Z} \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

(4)* Let $A_\bullet$ be any chain complex and "split" it into a negative part ($A_n^- = A_{n-1}$ for $n \leq 0$ and $A_n^- = 0$ for $n > 0$) and a non-negative part ($A_n^+ = 0$ for $n < 0$ and $A_n^+ = A_n$ for $n \geq 0$). We have a chain map $D_\bullet\colon A_\bullet^+ \to A_\bullet^-$ as follows (where $A_\bullet$ is "bent"):

$$\begin{tikzcd}
\cdots & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & A_0 \arrow[l] \arrow[d, "\partial"'] & A_1 \arrow[l, "\partial"'] \arrow[d] & A_2 \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & A_{-3} \arrow[l, "\partial"] & A_{-2} \arrow[l, "\partial"] & A_{-1} \arrow[l, "\partial"] & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

with $D_n = 0$ for $n \neq 0$ and $D_0 = \partial\colon A_0 \to A_{-1}$. Thus we have induced homomorphisms

$$D_*\colon H_n(A_\bullet^+) \longrightarrow H_n(A_\bullet^-).$$

(a) $H_n(A_\bullet^+) = 0$ for $n < 0$
(b) $H_n(A_\bullet^+) = H_n(A_\bullet)$ for $n \geq 1$
(c) $H_n(A_\bullet^-) = 0$ for $n > 0$
(d) $H_n(A_\bullet^-) = H_{n-1}(A_\bullet)$ for $n \leq -1$
(e) $H_0(A_\bullet) \cong \ker\bigl(D_*\colon H_0(A_\bullet^+) \to H_0(A_\bullet^-)\bigr)$
(f) $H_{-1}(A_\bullet) \cong \operatorname{coker}\bigl(D_*\colon H_0(A_\bullet^+) \to H_0(A_\bullet^-)\bigr)$
What if we "bend" $A_\bullet$ several times?

Exercise 11.3 (Moore spaces.)
(a) Let $f\colon X \to Y$ be a continuous map. Recall from Exercise 8.3 the definition of the *cone* of f, denoted $\mathrm{Cone}(f)$, and that there is an embedding $Y \hookrightarrow \mathrm{Cone}(f)$ of Y as a closed subspace of $\mathrm{Cone}(f)$.
(b) Show that the image of Y under this embedding has an open neighbourhood that deformation retracts onto it. Therefore use the long exact sequence for the pair $(\mathrm{Cone}(f), Y)$ and Exercise 10.4 (the Quotient Theorem) to construct a long exact sequence of the form:

$$\cdots \to H_n(Y) \to H_n(\mathrm{Cone}(f)) \to \widetilde{H}_n(\Sigma X) \to H_{n-1}(Y) \to H_{n-1}(\mathrm{Cone}(f)) \to \cdots.$$

(c) Use the Suspension Isomorphism, and analyse the above sequence carefully in degree zero, to show that we have a long exact sequence of the form

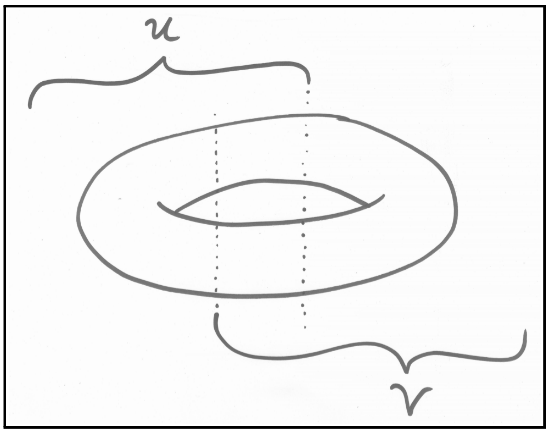
$$\cdots \to \widetilde{H}_n(Y) \to \widetilde{H}_n(\mathrm{Cone}(f)) \to \widetilde{H}_{n-1}(X) \to \widetilde{H}_{n-1}(Y) \to \widetilde{H}_{n-1}(\mathrm{Cone}(f)) \to \cdots,$$

where the map $\widetilde{H}_{n-1}(X) \to \widetilde{H}_{n-1}(Y)$ in the sequence is the one induced by f.
(d) Apply this sequence to a map $f_{n,k}\colon \mathbb{S}^n \to \mathbb{S}^n$ of degree k, where $n > 0$ and $k \neq 0$, to show that $H_i(\mathrm{Cone}(f_{n,k}))$ is isomorphic to $\mathbb{Z}/k\mathbb{Z}$ for $i = n$ and is zero for $i \neq n$.
(e) Now let $n > 0$ and let A be any finitely generated abelian group. Construct a space Y such that $\widetilde{H}_n(Y) \cong A$ and $\widetilde{H}_i(Y) = 0$ for $i \neq n$. Such a space is called a *Moore space* for the pair (A, n).
(*Hint*: look at Exercise 11.5 part (1) below.)
In fact a similar construction works for any (not necessarily finitely generated) abelian group A.
(f) Now let $(A_0, A_1, A_2, \ldots)$ be any sequence of finitely generated abelian groups with A_0 free. Construct a space Y such that $H_i(Y) \cong A_i$ for all $i \geq 0$.

Exercise 11.4 (The degree of a rotation.)
Let $A \in O(n+1)$. The restriction $f = A|_{\mathbb{S}^n}$ of $A\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ to the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a self-map of $\mathbb{S}^n$, and therefore has a degree, defined as in the lectures to be the unique integer $\deg(f)$ such that $f_*(x) = \deg(f).x$ where x is any non-zero element of $H_n(\mathbb{S}^n) \cong \mathbb{Z}$. (We assume that $n > 0$ in this exercise.)
(1) Explain why $f = A|_{\mathbb{S}^n}$ must have degree either $+1$ or -1.
(2) Show that, in fact, $\deg(A|_{\mathbb{S}^n}) = \det(A)$.
(*Hint*: first construct an explicit singular n-cycle on $\mathbb{S}^n$ representing a non-zero element of $H_n(\mathbb{S}^n)$.)
(3) In particular, if A is a reflection in a k-dimensional subspace in $\mathbb{R}^{n+1}$, $\deg(A|_{\mathbb{S}^n}) = (-1)^{n-k+1}$.

Exercise 11.5 (Applications of Mayer-Vietoris.)
(1) Show that for a wedge of spaces $X \vee Y$ we have $H_n(X \vee Y) \cong H_n(X) \oplus H_n(Y)$ for $n > 0$. What happens in degree zero? Thus compute the homology of $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$.
(2) Decompose $\mathbb{S}^1 \times \mathbb{S}^1$ into two open subsets, each homeomorphic to the open annulus $\mathbb{S}^1 \times (0,1)$, such that their

intersection is the disjoint union of two open annuli (see the figure on the next page). Using the Mayer-Vietoris sequence for this decomposition, compute the homology of $\mathbb{S}^1 \times \mathbb{S}^1$. (Cf. your answer to Exercise 10.5(c).)
(3) Note that these two spaces have isomorphic homology in each degree. Explain (using π_1) why, nevertheless, they are *not* homotopy equivalent spaces.
(4) Describe the universal coverings of $\mathbb{S}^1 \times \mathbb{S}^1$ and of $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ and compute the homology groups of each.
(5)* Let Y be the torus $\mathbb{S}^1 \times \mathbb{S}^1$ and let Z be the Möbius band, with $A \subset Z$ its boundary. Let $\phi \colon A \cong \mathbb{S}^1$ be a parametrisation of A. Define an embedding $g \colon A \hookrightarrow Y$ by $g(a) = (\phi(a), 1)$, where $1 \in \mathbb{S}^1$ is the basepoint. Let $X = Z \cup_g Y$, as defined in Exercise 8.3. Using an open cover by "small" open neighbourhoods of $Y \subset X$ and $Z \subset X$, compute the homology of X in all degrees.
(6)* Repeat the previous exercise, but instead of the embedding g, use the embedding $h_k \colon A \hookrightarrow Y$ defined by $h_k(a) = (\phi(a), \phi(a)^k)$, where k is a fixed integer. (Note that $h_0 = g$.) Compute the homology of $X_k = Z \cup_{h_k} Y$.



The decomposition of the torus in Exercise 11.5(2).

Exercise 11.6* (The Brouwer Fixed Point Theorem and PageRank.)
(a) Let n be a positive integer and let $f \colon \mathbb{D}^n \to \mathbb{D}^n$ be a self-map of the n-dimensional disc. Show that f has a fixed point.
(*Hint*: the one-dimensional case can be proved using the Intermediate Value Theorem. You have already seen a proof of the two-dimensional case, using π_1. The proof for $n > 2$ is very similar, using homology instead.)
(b) So any self-map of a space homeomorphic to $\mathbb{D}^n$ has (at least) one fixed point.
(c) (Theorem of Perron-Frobenius) Let A be a real $n \times n$ matrix. If all of its entries are non-negative, then it has a non-negative eigenvalue and a corresponding eigenvector all of whose entries are non-negative. Prove this statement in two cases: if there is a non-zero vector x with non-negative entries such that $Ax = 0$ then we are done; otherwise consider the self-map f of $\Delta^{n-1} \subseteq \mathbb{R}^n$ defined by

$$f(x) = \tfrac{y}{y_1 + \cdots + y_n} \qquad \text{for} \qquad y = (y_1, \ldots, y_n) = Ax.$$

(First explain why this is well-defined.)
(d) (Stronger version for column-sum-one matrices.) Suppose that A has the property that the sum of the entries in each column is 1. Using similar methods to part (c), show that, in this case, we can take the eigenvector to lie in Δ^{n-1} and the eigenvalue to be equal to 1.
(e) (Solution to the PageRank problem.) Let G be a directed graph with a vertex for every webpage and an edge from w_1 to w_2 if the webpage w_1 contains a link to the webpage w_2. We want to extract from this a measure of the relative "importance" of each webpage. This will assign to each webpage w a non-negative real number $a(w)$, such that the sum of $a(w)$ over all webpages w is 1. In other words, a is a point on the $(n-1)$-simplex $\Delta^{n-1} \subseteq \mathbb{R}^n$, where n is the number of webpages on the internet.

The first idea is to simply set $a(w)$ equal to the number of other webpages that link to w, and then normalise so that the sum is equal to 1. But then webpages that link to millions of other webpages would have a disproportionate influence on the results compared to those with relatively few links. So to make the measure more democratic, we say that each webpage has one "vote", which it distributes to other webpages by linking to them: if it links to 5 other webpages, each of those webpages earns $\frac{1}{5}$ of its vote. Then $a(w)$ is equal to the total number of votes received by w from other websites – again we have to normalise this. The final version of the idea is to give a higher weight to the votes from websites that are themselves more "important" according to the measure a. This sounds circular, but what it means is just that, instead of a direct definition of the vector $a \in \Delta^{n-1} \subseteq \mathbb{R}^n$, we have a system of linear equations that it must satisfy. Namely:

$$a(w_i) \ = \ \sum_{j=1}^{n} A(i,j).a(w_j),$$

where $A(i,j) = 0$ if there is no edge $w_j \to w_i$ and $A(i,j) = \frac{1}{k}$ if there is an edge $w_j \to w_i$ and k is the total number of edges leaving the vertex w_j (i.e., the total number of distinct links on the webpage w_j). If we let A be the matrix whose (i,j)th entry is $A(i,j)$, then these equations are equivalent to:

$$a = Aa.$$

The PageRank problem is to find such a vector $a \in \Delta^{n-1} \subseteq \mathbb{R}^n$. This gives us our measure of the relative "importance" of each webpage, and (ignoring the issue that some entries of the vector a might be equal!) we may use this to rank all webpages by importance.
To do: show that the PageRank problem has a solution.
(f) (Uniqueness of the solution in a strongly connected internet.) Let us make the unreasonable assumption that the internet is strongly connected: for any pair of webpages (w_1, w_2) there exists a sequence of links taking you from w_1 to w_2. Prove that the vector $a \in \Delta^{n-1}$ such that $a = Aa$ is unique in this case.
(*Hint*: Do this by contradiction. Non-uniqueness of this solution means that the 1-eigenspace of A intersects Δ^{n-1} in more than one point, so it must have dimension at least 2. But then it must also intersect the boundary of Δ^{n-1}, so there exists a solution a such that $a(w_i) = 0$ for some website w_i. Show that this implies that $a(w_j) = 0$ for every website w_j that links to w_i.)
(g)** What happens for a less well-connected internet?

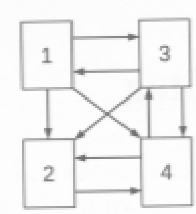


Abb. 1.1 Verknüpfungsstruktur in einem „4-Seiten-Internet“

From the introduction to the book *Lineare Algebra* by J. Liesen and V. Mehrmann. Note that this 4-page internet is strongly connected (cf. part (f) above), so there is a unique solution to the PageRank problem in this case. Can you compute it?

Aufgaben zur Topologie

Prof. Dr. C.-F. Bödigheimer

Wintersemester 2016/17

Week 12 — Suspensions, coefficient rings, (co)invariants, surgery. Due: 1. February 2017

Exercise 12.1 (Sums of maps.)
Let $\tilde{\Sigma}X$ denote the reduced suspension $\Sigma X/\Sigma x_0$ of a based space X with x_0 the basepoint; and denote by

$$\nabla\colon \tilde{\Sigma}X \longrightarrow \tilde{\Sigma}X \vee \tilde{\Sigma}X$$

the so-called *co-multiplication* defined by

$$\nabla([x,t]) = \begin{cases} [x,2t] \text{ in the left summand,} & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ [x,2t-1] \text{ in the right summand,} & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Here we denote the points of $\tilde{\Sigma}X = X \times [0,1]/A$ with $A = (X \times \{0,1\}) \cup (\{x_0\} \times [0,1])$ by $[x,t]$, using their X-coordinate and their height t in the double cone. We denote by $p_i\colon \tilde{\Sigma}X \vee \tilde{\Sigma}X \to \tilde{\Sigma}X$ for $i = 1,2$ the projection onto the left resp. right summand, which collapses the other summand to a point. And by $\iota_i\colon \tilde{\Sigma}X \to \tilde{\Sigma}X \vee \tilde{\Sigma}X$ we denote the inclusions of the left resp. right summand.

(1) Show that $p_i \circ \nabla \simeq \mathrm{id}_{\tilde{\Sigma}X}$ for $i = 1,2$.

(2) Show that $(a,b) \mapsto \iota_{1*}(a) + \iota_{2*}(b)$ is an isomorphism $\Phi\colon H_n(\tilde{\Sigma}X) \oplus H_n(\tilde{\Sigma}X) \to H_n(\tilde{\Sigma}X \vee \tilde{\Sigma}X)$, for $n > 0$.

(3) Conclude that the homomorphism $\Phi^{-1} \circ \nabla_*\colon H_n(\tilde{\Sigma}X) \to H_n(\tilde{\Sigma}X) \oplus H_n(\tilde{\Sigma}X)$ is the diagonal.

Now we write the n-sphere $\mathbb{S}^n = \Sigma\mathbb{S}^{n-1}$ (for $n \geq 1$) as a suspension of $\mathbb{S}^{n-1}$. For two based maps $f, g\colon \mathbb{S}^n \to \mathbb{S}^n$ we declare their sum $f + g\colon \mathbb{S}^n \to \mathbb{S}^n$ by $f + g := \mathrm{F} \circ (f \vee g) \circ \nabla$, where $\mathrm{F}\colon \tilde{\Sigma}X \vee \tilde{\Sigma}X \to \tilde{\Sigma}X$ is the folding map.

(4) Prove the formula:

$$\deg(f+g) = \deg(f) + \deg(g).$$

(5)* More generally, prove that $(f+g)_* = f_* + g_*\colon H_n(\tilde{\Sigma}X) \to H_n(\tilde{\Sigma}X)$.

Exercise 12.2 (An application of mapping degree: fixed and antipodal points of self-maps of spheres.)
Let $n \geqslant 1$ and let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be a self-map of the n-sphere.
(a) If n is even, show that f must have either a fixed point or an antipodal point ($x \in \mathbb{S}^n$ such that $f(x) = -x$).
(b) More generally, if n is even, any two self-maps f, g of $\mathbb{S}^n$ must have either an incidence point ($x \in \mathbb{S}^n$ such that $f(x) = g(x)$) or an opposite point ($x \in \mathbb{S}^n$ such that $f(x) = -g(x)$), unless they both have degree 0.
(c) For n odd, give an example of a self-map $f\colon \mathbb{S}^n \to \mathbb{S}^n$ with no fixed point and no antipodal point.
(d) More generally, given a self-map $f\colon \mathbb{S}^n \to \mathbb{S}^n$, construct (when n is odd) another self-map $g\colon \mathbb{S}^n \to \mathbb{S}^n$ such that f and g have no incidence points and no opposite points.

Exercise 12.3 (Coefficient rings.)
Let Z be the space $\mathbb{D}^2 \times \{0,1\}$, i.e., the disjoint union of two closed 2-discs, and let $A \subset Z$ be its boundary $\mathbb{S}^1 \times \{0,1\}$. Let $Y = \mathbb{S}^1$ and consider a map $f\colon A \to Y$ that sends $\mathbb{S}^1 \times \{0\}$ to Y by a map of degree m and sends $\mathbb{S}^1 \times \{1\}$ to Y by a map of degree n. See the figure on the next page.
(a) Using the Mayer-Vietoris sequence for an appropriate open covering of $X = Z \cup_f Y$, show that there is an exact sequence

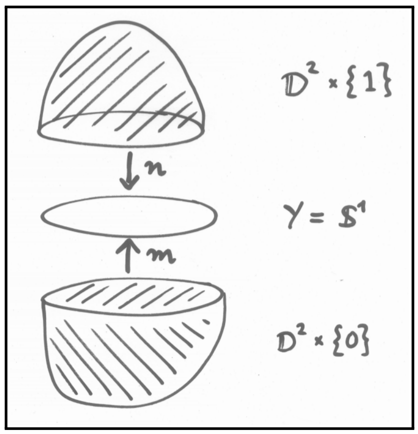
$$0 \to H_2(X) \to \mathbb{Z}^2 \xrightarrow{\phi} \mathbb{Z} \to H_1(X) \to 0,$$

where ϕ is given by the matrix $(m\ n)$, and hence that $H_2(X) \cong \mathbb{Z}$ (unless $m = n = 0$, in which case $H_2(X) \cong \mathbb{Z}^2$) and $H_1(X) \cong \mathbb{Z}/h\mathbb{Z}$, where $h = \gcd(m,n)$ is the greatest common divisor of m and n if they are both non-zero,

and is $\max(|m|,|n|)$ otherwise.
(b) What happens when we compute homology not with $\mathbb{Z}$ coefficients, but rather with R coefficients, where
(i) $R = \mathbb{Q}$,
(ii) $R = \mathbb{F}_p$, for a prime p,
(iii) $R = \mathbb{Z}[\frac{1}{p}]$, for a prime p, where $\mathbb{Z}[\frac{1}{p}] = \{\frac{a}{b} \in \mathbb{Q} \mid a \text{ and } b \text{ are coprime and } b = p^c \text{ for some integer } c \geqslant 0\}$?
(c)* Now take $Y' = \mathbb{S}^1 \vee \mathbb{S}^1$ and consider a map $g\colon A \to Y'$ that sends $\mathbb{S}^1 \times \{0\}$ to Y' as a loop that winds 3 times around the left-hand circle of the "figure-of-eight" and twice around the right-hand circle, and sends $\mathbb{S}^1 \times \{1\}$ to Y' as a loop that winds 5 times around the left-hand circle and 7 times around the right-hand circle. Let $X' = Z \cup_g Y'$. Similarly to part (a), show that there is an exact sequence

$$0 \to H_2(X') \to \mathbb{Z}^2 \xrightarrow{\psi} \mathbb{Z}^2 \to H_1(X') \to 0,$$

where ψ is given by the matrix $\left(\begin{smallmatrix} 3 & 5 \\ 2 & 7 \end{smallmatrix}\right)$, and hence that $H_2(X') = 0$ and $H_1(X') \cong \mathbb{Z}/11\mathbb{Z}$.
(d)* What happens if we change the ring of coefficients as in part (b)?



The attaching map f for the space $X = Z \cup_f Y$ in Exercise 12.3(a).

Exercise 12.4 (Invariants and coinvariants.)
Let X be a space with an action of a group G. We write X^G for the subspace of X consisting of all fixed points under the action (the *invariants*) and X/G for the quotient space $\{x.G \mid x \in X\}$ (the *orbit space*).
Fix a commutative ring R with unit. If M is an R-module with a G-action by R-linear automorphisms, we define the *invariants* M^G, as above, to be the submodule of all elements that are fixed under the action. The module of *coinvariants* M_G is the quotient of M by the submodule generated by the set $\{m - m.g \mid m \in M, g \in G\}$.
(a) If X is a space with a G-action, then $M = H_n(X;R)$ is an R-module with a G-action. There are inclusion and quotient maps $X^G \hookrightarrow X \twoheadrightarrow X/G$ and also $M^G \hookrightarrow M \twoheadrightarrow M_G$. Complete the following commutative diagram by defining the dotted arrows:

$$\begin{tikzcd}
H_i(X^G) \arrow[dr] \arrow[dd, dotted, "f_i"'] & & H_i(X)_G \arrow[dd, dotted, "g_i"] \\
& H_i(X) \arrow[ur] \arrow[dr] & \\
H_i(X)^G \arrow[ur] & & H_i(X/G)
\end{tikzcd}$$

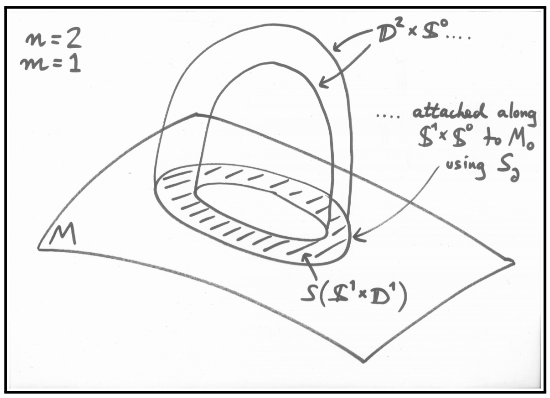
(b) Take $R = \mathbb{Z}$ and let $X = \mathbb{S}^n$ ($n \geqslant 2$) with $G = \mathbb{Z}/2\mathbb{Z}$ acting by a reflection. Show that
(i) g_i is an isomorphism for $i < n$, but g_n is not injective;
(ii) f_{n-1} is also not injective.
(c) Now consider the same set-up, except that $G = \mathbb{Z}/2\mathbb{Z}$ acts by the antipodal map instead. Show that
(i) f_0 is not surjective;

(ii) f_n is an isomorphism if and only if n is even; whereas g_n is injective but not surjective for n odd and is surjective but not injective for n even;
(iii) in degrees $0 < i < n$ we have: g_i is an isomorphism if and only if i is even.
(d) In part (c), replace $R = \mathbb{Z}$ with $R = \mathbb{Q}$ or $R = \mathbb{F}_p$ for an odd prime p. Now g_i is an isomorphism for all i.

Exercise 12.5 (Surgery on a manifold.)
Recall that an n-dimensional topological manifold is a Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$. A *framed embedded sphere* S in M of dimension m is an embedding $S\colon \mathbb{S}^m \times \mathbb{D}^{n-m} \hookrightarrow M$. Write S_∂ to denote the restriction of S to $\mathbb{S}^m \times \partial\mathbb{D}^{n-m} = \mathbb{S}^m \times \mathbb{S}^{n-m-1} = \partial\mathbb{D}^{m+1} \times \mathbb{S}^{n-m-1}$. We then define

$$M(S) \;=\; \big(\mathbb{D}^{m+1} \times \mathbb{S}^{n-m-1}\big) \cup_{S_\partial} M_\circ, \qquad \text{where} \qquad M_\circ \;=\; M - S(\mathbb{S}^m \times \mathring{\mathbb{D}}^{n-m}),$$

and call this *the result of surgery on* M *along* S. For example, the result of surgery along a framed embedded 1-sphere in a surface look like the following:



(a) Draw a sketch to show why this is again a manifold.
(b) Explain why we have a diagram of the form

$$\begin{array}{ccccccc}
 & & H_{i+1}(M(S), M_\circ) & & & & \\
 & & \downarrow & & & & \\
H_{i+1}(M, M_\circ) & \longrightarrow & H_i(M_\circ) & \longrightarrow & H_i(M) & \longrightarrow & H_i(M, M_\circ) \\
 & & \downarrow & & & & \\
 & & H_i(M(S)) & & & & \\
 & & \downarrow & & & & \\
 & & H_i(M(S), M_\circ) & & & &
\end{array} \tag{1}$$

with one exact row and one exact column. To relate $H_i(M(S))$ to $H_i(M)$, it is important to understand the relative homology groups appearing in (1).
(c) Using Excision, show that

$$H_i(M, M_\circ) \cong H_i(\mathbb{S}^m \times \mathbb{D}^{n-m}, \mathbb{S}^m \times \mathbb{S}^{n-m-1})$$
$$H_i(M(S), M_\circ) \cong H_i(\mathbb{D}^{m+1} \times \mathbb{S}^{n-m-1}, \mathbb{S}^m \times \mathbb{S}^{n-m-1}).$$

(d) Explain why the inclusion map $\mathbb{S}^a \times \mathbb{S}^b \hookrightarrow \mathbb{S}^a \times \mathbb{D}^{b+1}$ induces surjections on homology in every degree.
(*Hint*: Apart from degree 0, it is enough to show that a certain homology class in $H_*(\mathbb{S}^a \times \mathbb{D}^{b+1})$ is in the image.)
(e) You may from now on assume the following fact:

$$\widetilde{H}_i(\mathbb{S}^a \times \mathbb{S}^b) \;\cong\; \mathbb{Z}^{\delta_{i,a}} \oplus \mathbb{Z}^{\delta_{i,b}} \oplus \mathbb{Z}^{\delta_{i,(a+b)}},$$

where $\delta_{i,j}$ is the Kronecker delta function: $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ for $i \neq j$.
(If you like, try to prove this inductively using the Mayer-Vietoris sequence. Find an open cover $\{U, V\}$ of $\mathbb{S}^a \times \mathbb{S}^b$

such that $U \simeq V \simeq \mathbb{S}^a$ and $U \cap V \simeq \mathbb{S}^a \times \mathbb{S}^{b-1}$. For the base case, note that $\mathbb{S}^a \times \mathbb{S}^0 = \mathbb{S}^a \sqcup \mathbb{S}^a$.)
(f) Using parts (c)–(e), compute:

$$H_i(M, M_\circ) \cong \begin{cases} \mathbb{Z}^2 & i = n \text{ and } m = 0 \\ \mathbb{Z} & (i = n \text{ or } i = n - m) \text{ and } m \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H_i(M(S), M_\circ) \cong \begin{cases} \mathbb{Z}^2 & i = n \text{ and } m = n - 1 \\ \mathbb{Z} & (i = n \text{ or } i = m + 1) \text{ and } m \neq n - 1 \\ 0 & \text{otherwise} \end{cases}$$

(g) Assume that $n \geqslant 1$ and $m \leqslant \frac{n}{2}$. Use these calculations and (1) to show that in degrees $i \leqslant m - 2$,

$$H_i(M(S)) \;\cong\; H_i(M).$$

(h)* Now we consider a more specific example. Let M be a 7-manifold and let $S\colon \mathbb{S}^4 \times \mathbb{D}^3 \hookrightarrow M$ be a framed embedded 4-sphere. Show that

$$H_4(M(S)) \;\cong\; H_4(M)/\langle [c] \rangle,$$

where $[c]$ is the image under S_* of a generator of $H_4(\mathbb{S}^4 \times \mathbb{D}^3) \cong \mathbb{Z}$.

Exercise 12.6* (H-spaces and co-H-spaces.)
A based space C is called a *co-H-space*, if there is a map $\nabla\colon C \to C \vee C$ such that

$$p_i \circ \nabla \simeq \mathrm{id}_C \quad \text{for} \quad i = 1, 2$$

where p_1 and p_2 are the projections onto the first resp. second summand, which collapse the other summand to a point. One calls C *co-associative*, if

$$(\nabla \vee \mathrm{id}_C) \circ \nabla \simeq (\mathrm{id}_C \vee \nabla) \circ \nabla.$$

Example: the reduced suspension $C = \tilde{\Sigma} X$ of a based space X is a co-associative co-H-space.

By $\iota_i\colon C \to C \vee C$ for $i = 1, 2$ we will denote the inclusions of the left resp. right summand. With the same proof as in Exercise 12.1 we see that $(a, b) \mapsto \iota_{1*}(a) + \iota_{2*}(b)$ is an isomorphism $\Phi\colon H_n(C) \oplus H_n(C) \to H_n(C \vee C)$, for $n > 0$.

(1) Show that $\Phi^{-1} \circ \nabla_*\colon H_n(C) \to H_n(C) \oplus H_n(C)$ is the diagonal map.

For any two based maps $f, g\colon C \to C$ we can define their sum by $f + g := F \circ (f \vee g) \circ \nabla$, where $F\colon C \vee C \to C$ is the folding map.

(2) We have $(f + g)_* = f_* + g_*\colon H_n(C) \to H_n(C)$.

A based space M is called an *H-space*, if there is a map $\mu\colon M \times M \to M$, such that

$$\mu \circ \iota_i \simeq \mathrm{id}_M \quad \text{for} \quad i = 1, 2,$$

where $\iota_1\colon M \to M \times M$ sends m to (m, m_0), where m_0 is the basepoint of M, and similarly ι_2 sends m to (m_0, m). One calls M *associative*, if

$$\mu \circ (\mathrm{id}_M \times \mu) \simeq \mu \circ (\mu \times \mathrm{id}_M).$$

Example: A topological group, in particular a Lie group, is an H-space.

For a co-H-space C and a based space Y, we set $M := \mathrm{maps}_0(C, Y)$, the space of all based maps $f\colon C \to Y$. These are important spaces when C is a sphere.

(3) Show that M is an H-space, and it is associative if C is co-associative.

Aufgaben zur Topologie

Prof. Dr. C.-F. Bödigheimer

Wintersemester 2016/17

Week 13 — Products of spheres, linking number, cellular homology. Not due for handing-in. (Non-compulsory)

Exercise 13.1 (Homology of products of spheres.)
Recall from the lectures that, for spaces X and Y with "good" basepoints (this means that the basepoint is closed as a subset and also has an open neighbourhood that deformation retracts onto it), there are split short exact sequences

$$0 \to \widetilde{H}_i(X) \oplus \widetilde{H}_i(Y) \longrightarrow \widetilde{H}_i(X \times Y) \longrightarrow \widetilde{H}_i(X \wedge Y) \to 0$$

for $i \geqslant 0$. Note that the smash product $\mathbb{S}^n \wedge X$ is homeomorphic to the n-fold reduced suspension $\widetilde{\Sigma}^n X = \widetilde{\Sigma} \cdots \widetilde{\Sigma} X$.
(a) Use this fact, the above short exact sequences and the Suspension Theorem to show that

$$\widetilde{H}_i(\mathbb{S}^n \times X) \cong \begin{cases} \widetilde{H}_i(X) & 0 \leqslant i \leqslant n-1 \\ \mathbb{Z} \oplus \widetilde{H}_n(X) \oplus \widetilde{H}_0(X) & i = n \\ \widetilde{H}_i(X) \oplus \widetilde{H}_{i-n}(X) & i \geqslant n+1. \end{cases}$$

(b) Calculate the homology of a product of two spheres $\mathbb{S}^k \times \mathbb{S}^\ell$.
(c)* More generally, what is the homology of an iterated product of spheres $\mathbb{S}^{k_1} \times \cdots \times \mathbb{S}^{k_i}$?

Exercise 13.2 (Homology of knot complements.)
Let $f\colon \mathbb{S}^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{R}^3$ be a framed knot in Euclidean space, i.e., an embedding of $\mathbb{S}^1 \times \mathbb{D}^2$ into $\mathbb{R}^3$. The complement of its image, $M = \mathbb{R}^3 \smallsetminus f(\mathbb{S}^1 \times \mathbb{D}^2)$, is then a non-compact 3-manifold.
(a) Describe an open covering $\{U, V\}$ of $\mathbb{R}^3$ such that $U \simeq M$, $V \simeq \mathbb{S}^1$ and $U \cap V \simeq \mathbb{S}^1 \times \mathbb{S}^1$.
(b) Using the Mayer-Vietoris sequence for this covering, calculate the homology of the knot-complement M, in particular concluding that $H_1(M) \cong \mathbb{Z}$.
(c) Draw a 1-cycle μ representing a generator of $H_1(M)$.

Exercise 13.3 (Linking number.)
As in the previous exercise, let $f\colon \mathbb{S}^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{R}^3$ be a framed knot, write $K = f(\mathbb{S}^1 \times \mathbb{D}^2)$ and $M = \mathbb{R}^3 \smallsetminus K$. Fix a generator $[\mu]$ of $H_1(M) \cong \mathbb{Z}$ as in part (c) of the previous exercise. For any curve $c\colon \mathbb{S}^1 \to M$ we may define its *linking number with* K, denoted $L(c, K)$ or just $L(c)$, to be the unique integer such that

$$c_*([\omega_1]) = L(c, K).[\mu],$$

where $[\omega_1] \in H_1(\mathbb{S}^1)$ is a generator. Note that $L(c, K)$ depends on the choices of μ and ω_1. See the figure on the next page for an example.

Show:
(a) $L(c_1) = L(c_2)$, if $c_1 \simeq c_2 \colon \mathbb{S}^1 \to M$.
(b) $L(c) = 0$, if the image of c and K may be separated by a plane in $\mathbb{R}^3$.
(c) Suppose that $\Phi\colon \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ is an ambient isotopy, i.e., each $\Phi_t = \Phi(-, t)$ is a self-homeomorphism of $\mathbb{R}^3$ and Φ_0 is the identity. Then $L(c, K) = L(c', K')$, where $c' = \Phi_1 \circ c$ and $K' = \Phi_1(K)$, and we use the generator $[\mu'] = (\Phi_1)_*([\mu])$ of $H_1(\mathbb{R}^3 \smallsetminus K')$.
(d)* Now let $f_1, f_2\colon \mathbb{S}^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{R}^3$ be two non-intersecting framed knots in $\mathbb{R}^3$. Let $K_i = f_i(\mathbb{S}^1 \times \mathbb{D}^2)$ and $c_i = f_i \circ c$, where $c\colon \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{D}^2$ is defined by $c(t) = (t, 0)$ (so c_i is the "core" of the framed knot f_i). Then we may define a difference map

$$D\colon \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{S}^2$$

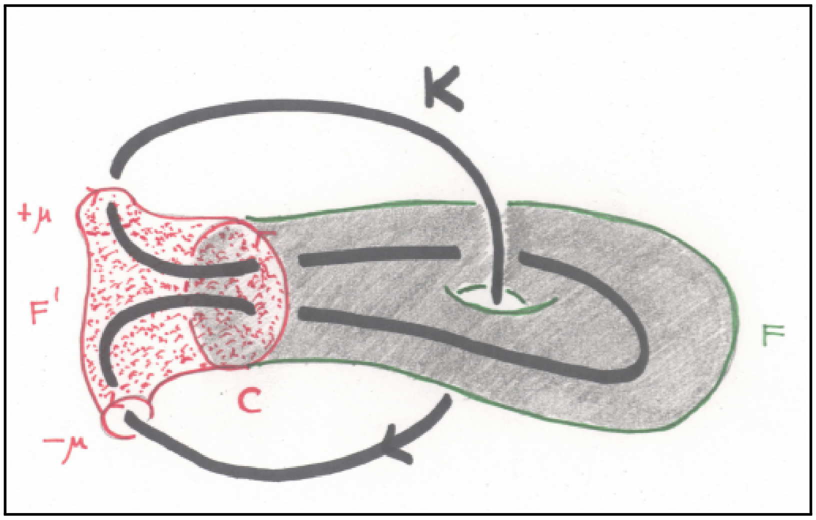


Figure for Exercise 13.3: The union of the curves K and C is the Whitehead link. Note that $L(C, K) = 0$. The surface F shows that C is nullhomologous in $\mathbb{R}^3 \smallsetminus K$, and F' shows that C is homologous to $\mu + (-\mu) = 0$. (But still, K and C cannot be isotoped to curves separated by a plane.)

by the formula

$$D(t_1, t_2) \;=\; \frac{c_1(t_1) - c_2(t_2)}{\|c_1(t_1) - c_2(t_2)\|}.$$

Consider the induced homomorphism $D_* \colon H_2(\mathbb{S}^1 \times \mathbb{S}^1) \to H_2(\mathbb{S}^2)$. Prove that $D_* = 0$ if $L(c_1, K_2) = L(c_2, K_1) = 0$. See the figure above for an example.

Exercise 13.4 (Mapping degree for tori.)
We know that for the torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ we have $H_2(\mathbb{T}) \cong \mathbb{Z}$. So let us choose a generator $[\tau]$ of $H_2(\mathbb{T})$ and define the *mapping degree* of a self-map f of the torus to be the unique integer $\deg(f)$ such that

$$f_*([\tau]) = \deg(f).[\tau].$$

(a) This definition is independent of whether we choose $[\tau]$ or $-[\tau]$ as our generator.
(b) If f and g are homotopic, then $\deg(f) = \deg(g)$.
(c) If $f_1, f_2 \colon \mathbb{S}^1 \to \mathbb{S}^1$ are two self-maps of the circle, and $f = f_1 \times f_2 \colon \mathbb{T} \to \mathbb{T}$ is their product – a self-map of the torus – then we have:

$$\deg(f) \;=\; \deg(f_1) \cdot \deg(f_2).$$

Show this using the following steps (or via another argument if you prefer):

(i) We may assume without loss of generality that f_1 is the map $z \mapsto z^m$ and f_2 is $z \mapsto z^n$ for some $m, n \in \mathbb{Z}$.
(ii) Recall the comultiplication $\nabla \colon \mathbb{S}^2 \to \mathbb{S}^2 \vee \mathbb{S}^2$ and the fold map $F \colon \mathbb{S}^2 \vee \mathbb{S}^2 \to \mathbb{S}^2$ from Exercise 12.1. These may be iterated, leading to maps $\nabla_k \colon \mathbb{S}^2 \to \bigvee^k \mathbb{S}^2$ and $F_k \colon \bigvee^k \mathbb{S}^2 \to \mathbb{S}^2$. On second homology groups, we have

$$(\nabla_k)_*(1) = (1, \ldots, 1) \in \mathbb{Z}^k$$
$$(F_k)_*(0, \ldots, 0, 1, 0, \ldots, 0) = 1 \in \mathbb{Z}.$$

(iii) Let $A = \mathbb{S}^1 \vee \mathbb{S}^1 \subset \mathbb{S}^1 \times \mathbb{S}^1$. Then the quotient map $q \colon \mathbb{S}^1 \times \mathbb{S}^1 \to (\mathbb{S}^1 \times \mathbb{S}^1)/A \cong \mathbb{S}^2$ induces an isomorphism on $H_2(-)$.
(iv) Let $B \subset \mathbb{S}^1 \times \mathbb{S}^1$ be an $(m \times n)$ rectangular grid in the usual picture of the torus as a square with edge identifications. Then $(\mathbb{S}^1 \times \mathbb{S}^1)/B$ is homeomorphic to a wedge sum of mn copies of $\mathbb{S}^2$. Under this identification, the quotient map $\mathbb{S}^1 \times \mathbb{S}^1 \to (\mathbb{S}^1 \times \mathbb{S}^1)/B \cong \bigvee^{mn} \mathbb{S}^2$ is homotopic to $\nabla_{mn} \circ q$.
(v) The following diagram is commutative up to homotopy, and so the result follows.

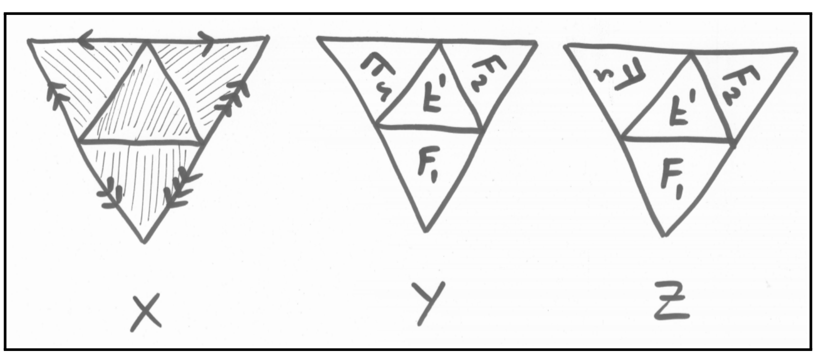
$$\begin{tikzcd}
\mathbb{S}^1 \times \mathbb{S}^1 \arrow[rr, "f = f_1 \times f_2"] \arrow[d, "q"'] & & \mathbb{S}^1 \times \mathbb{S}^1 \arrow[d, "q"] \\
\mathbb{S}^2 \arrow[r, "\nabla_{mn}"] & \bigvee^{mn} \mathbb{S}^2 \arrow[r, "F_{mn}"] & \mathbb{S}^2
\end{tikzcd}$$

Exercise 13.5 (Cellular homology of quotients of the 3-simplex.)
Let X be the 3-simplex, the 2-skeleton of which is depicted on the left-hand side in the figure below, and identify its four faces in two pairs, as indicated in the middle part of the figure, to obtain a quotient space Y.
(a) Describe the natural cell complex structure on X and the induced structure on Y, with two 0-cells $\{P, Q\}$, three 1-cells $\{a, b, c\}$, two 2-cells $\{F_1, F_2\}$ and one 3-cell ω.
(b) Compute the differentials in the cellular chain complex

$$0 \leftarrow \mathbb{Z}\langle P, Q\rangle \leftarrow \mathbb{Z}\langle a, b, c\rangle \leftarrow \mathbb{Z}\langle F_1, F_2\rangle \leftarrow \mathbb{Z}\langle \omega\rangle \leftarrow 0 \leftarrow 0 \leftarrow \cdots$$

of Y, and thus compute its homology.
(c) Now identify the faces of the 3-simplex as indicated on the right-hand side of the figure, to obtain a quotient space Z. Describe the induced cell structure on Z, with one 0-cell, two 1-cells, two 2-cells and one 3-cell.
(d) Compute the cellular homology of Z.

Exercise 13.6* (Cellular homology of a quotient of the dodecahedron.)
Let X be the dodecahedron, which has a cell structure with 20 zero-cells, 30 one-cells, 12 two-cells and one three-cell. For each face, imagine pushing it through the interior of the dodecahedron until it lies in the same plane as the opposite face, and then rotating it by $\frac{\pi}{5}$ radians. This gives a homeomorphism between each pair of opposite faces. Let $\sim$ be the equivalence relation generated by $x \sim \phi(x)$, where ϕ is one of these homeomorphisms. Describe the induced cell structure on the quotient space $X/\sim$ and its cellular chain complex. Prove that the (cellular) homology of $X/\sim$ is the same as the homology of $\mathbb{S}^3$. This is the famous *Poincaré homology sphere.*



Figures for Exercise 13.5: The left-hand figure is the 2-skeleton (the union of all cells of dimension at most 2) of the 3-simplex X. The middle figure describes how to identify two pairs of faces of X to obtain the quotient space Y. Similarly, the right-hand figure shows how to identify the same two pairs of faces – in a *different* way – to obtain the quotient space Z.

Dodekaederraum.

Von Poincaréschen Räumen mit endlicher Fundamentalgruppe sind uns zwei bekannt. Ihre Fundamentalgruppen stimmen mit denen des Dodekaederraumes überein. Wir wissen nicht, ob die beiden Räume untereinander und mit dem Dodekaederraum homöomorph sind[31]). Dagegen gibt es außer der Fundamentalgruppe noch eine andere Eigenschaft, die Poincarés Poincaréscher Raum mit dem Dodekaederraum gemein hat, nämlich die Zerlegbarkeit in zwei Doppelringe.

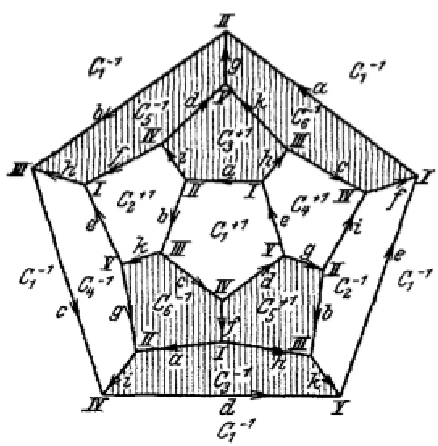


Fig. 11.

Wir benennen die Kanten und Flächenstücke des Diskontinuitätsbereiches, der uns den Dodekaederraum liefert, wie in der Fig. 11 (schematisches Netz der Dodekaederfläche) angegeben ist, und schreiben die wesentlichen Relationen an, die man aus den Kantenumläufen gewinnt. Wir sind dann nach § 10 sicher, daß die von den $\alpha_2 = 6$ Erzeugenden $C_1, \ldots, C_6$ und den $\alpha_1 = 10$ (den Kanten a), ..., k) entsprechenden) wesentlichen Relationen definierte Gruppe mit der binären Ikosaedergruppe (§ 6, S. 26)

From W. Threlfall, H. Seifert, *Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes*, Math. Ann. (1931).