Exercise sheet 0

These exercises are not to be handed in. They will be discussed in the tutorial sessions in week 2.

Exercise 1. Recall that for a pointed space X, there is a canonical group action of the fundamental group $\pi_1(X)$ on the homotopy groups $\pi_n(X)$ for all $n \geq 1$.

(a) For $n = 1$, show that the action is given by conjugation.

(b) Formulate an equivalent description of the action of $\pi_1(X)$ on $\pi_n(X)$ in terms of the universal cover of X, under the assumption that X admits a universal cover.

(c) For $X = \mathbb{R}P^2$, show that the above action is non-trivial for some n. In other words, $\mathbb{R}P^2$ is not a simple space.

Exercise 2. For an H-space (X, x_0) with multiplication $\mu\colon X \times X \to X$, show that the addition on $\pi_n(X, x_0)$ can also be defined by the rule

$$[f] + [g] = [\mu \circ (f, g)],$$

and that in this case, $\pi_n(X, x_0)$ is abelian also for $n = 1$.

Exercise 3. Let $f\colon S^n \times S^n \to S^{2n}$ be the quotient map collapsing $S^n \vee S^n$ to a point. Show that f induces the zero map on all homotopy groups but f is not nullhomotopic.

Exercise 4. Show that the set of basepoint-preserving homotopy classes $\langle X, Y\rangle$ is finite if X is a finite connected CW complex and $\pi_i(Y)$ is finite for $i \leq \dim X$.

Exercise 5. For $m, n \geq -1$, let X and Y be pointed CW complexes that are m-connected and n-connected, respectively, where "(-1)-connected" means "non-empty". Show that the join $X * Y$ is $(m + n + 2)$-connected.
Hint: For higher connectivity, show that there is a homotopy equivalence $\Sigma(X \wedge Y) \simeq X * Y$ and use Hurewicz.