Exercise sheet 2

Due before the lecture on Monday, 29 October 2018.

Exercise 1. (4 points) Let $f\colon X \to Y$ be a weak equivalence and let A be a based CW-complex. Show that f induces bijections $[A, X] \to [A, Y]$ and $\langle A, X\rangle \to \langle A, Y\rangle$. *Hint:* use the Compression Lemma.

Exercise 2. (6 points) Find a map of CW-complexes that induces isomorphisms on integral homology and on π_1, but is not a homotopy equivalence. *Hint:* For $n \geqslant 2$, recall that $\pi_n(S^1 \vee S^n) \cong \mathbb{Z}\left[t, t^{-1}\right]$ by looking at the universal cover. Construct a CW-complex X from $S^1 \vee S^n$ by attaching a single $(n+1)$-cell along a map representing $2t-1$ and consider the inclusion of the 1-skeleton into X.

Exercise 3. (5 points) Let $f\colon A \to X$ be a cofibration. Show:

(a) f is an embedding.

(b) f is a closed map if X is Hausdorff.

Exercise 4. (5 points)

(a) Show that $E \to B$ is a Serre fibration if its restriction $X \to B$ is a Serre fibration for each path-component X of E.

(b) Consider the subspace $E \subseteq \mathbb{R}^2$ given by

$$E := \big([0,1] \times \{\tfrac{1}{n} \,|\, n \in \mathbb{N}\}\big) \cup \{(x, x-1) \,|\, 0 \leqslant x \leqslant 1\}.$$

Show that the map

$$p\colon E \to [0,1]\,,\ (x,y) \mapsto x$$

is a Serre fibration, but not a Hurewicz fibration, i.e. that p satisfies the homotopy lifting property with respect to all CW-complexes, but not with respect to arbitrary spaces.
(*Remark:* One can show that a Serre fibration between CW-complexes is always a Hurewicz fibration.)