Exercise sheet 6

Due before the lecture on Monday, 26 November 2018.

Exercise 1. (5 points) Recall that we introduced the *Whitehead product* on the previous exercise sheet.

(a) Show that $[\mathrm{id}_{S^2}, \mathrm{id}_{S^2}] = 2 \cdot [\eta]$ in $\pi_3(S^2)$, where $\eta\colon S^3 \to S^2$ is the Hopf map.

(*Hint:* Consider the cohomology ring of the homotopy cofibre of a map representing $[\mathrm{id}_{S^2}, \mathrm{id}_{S^2}]$ and use the fact that the Hopf invariant (introduced in Topology II, exercise 8.2) defines an isomorphism $\pi_3(S^2) \cong \mathbb{Z}$.)

(b) Show that all Whitehead products of classes $\alpha \in \pi_i(X)$, $\beta \in \pi_j(X)$ lie in the kernel of the suspension homomorphism

$$\Sigma\colon \pi_{i+j-1}(X) \longrightarrow \pi_{i+j}(\Sigma X).$$

(*Hint:* First show that the suspension of the inclusion $S^i \vee S^j \hookrightarrow S^i \times S^j$ admits a retraction up to homotopy.)

Together, (a) and (b) give an alternative proof of the fact that the group $\pi_4(S^3)$ has at most two elements.

Exercise 2. (4 points) Show that if X is a path-connected H-space, then all Whitehead products on $\pi_*(X)$ vanish.

Exercise 3. (6 points) For $p, q \geqslant 1$, let X and Y be well-based spaces that are p- and q-connected, respectively.

(a) Show that for $i \geqslant 2$, the long exact sequence of $(X \times Y, X \vee Y)$ gives rise to a split short exact sequence

$$0 \to \pi_{i+1}(X \times Y, X \vee Y) \to \pi_i(X \vee Y) \to \pi_i(X \times Y) \to 0.$$

(b) Show that the composite $\pi_i(X) \to \pi_i(X \vee Y) \to \pi_i(X \vee Y, Y)$ is an isomorphism for $i \leqslant p + q$ (and similarly for X and Y switched). (*Hint:* Compose with the projection to show injectivity and use the Blakers-Massey theorem for surjectivity.)

(c) Show that the inclusions of wedge summands induce an isomorphism

$$\pi_i(X) \times \pi_i(Y) \longrightarrow \pi_i(X \vee Y)$$

for $i \leqslant p+q$ and conclude from (a) that $\pi_i(X \times Y, X \vee Y) = 0$ for $i \leqslant p+q+1$.

(d) Compute $\pi_n(S^n \vee S^n)$ for $n \geqslant 2$.

Exercise 4. (5 points) Let X be a based CW-complex. Show that the contravariant functor $\langle - , X \rangle$ from based CW-complexes to sets is *half-exact*, i.e. it is homotopy invariant and satisfies the wedge and Mayer-Vietoris axioms. This is therefore a necessary condition for Brown's representability theorem.