

### Exercise sheet 8

Due before the lecture on Monday, 10 December 2018.

**Exercise 1.** (6 points) For a based space  $(Y, y_0)$ , let  $\Omega Y$  denote its based loop space equipped with the usual H-space structure given by concatenation of based loops. Define the *Moore loop space* of  $Y$  to be the space

$$\Omega'Y = (\{0\} \times \{*\}) \cup (\mathbb{R}_{>0} \times \Omega Y),$$

topologised as a subspace of  $\mathbb{R}_{\geq 0} \times \Omega Y$ , where  $*$  is the constant loop at the basepoint. It is a (strictly associative and strictly unital) topological monoid with multiplication  $\Omega'Y \times \Omega'Y \rightarrow \Omega'Y$  given by

$$(t_1, \gamma_1) \cdot (t_2, \gamma_2) = (t_1 + t_2, \gamma)$$

where  $\gamma$  is the based loop

$$\gamma(s) = \begin{cases} \gamma_1\left(\frac{s(t_1+t_2)}{t_1}\right) & 0 \leq s \leq \frac{t_1}{t_1+t_2} \\ \gamma_2\left(\frac{s(t_1+t_2)-t_1}{t_2}\right) & \frac{t_1}{t_1+t_2} \leq s \leq 1. \end{cases}$$

(These statements may be assumed without proof.)

(a) Show that the map

$$\lambda: \Omega Y \longrightarrow \Omega'Y, \quad \gamma \longmapsto (1, \gamma)$$

is a homotopy equivalence. Is it a map of H-spaces?

Now let  $(X, e)$  be a based space. For  $n \geq 2$ , the *n-fold reduced product* is the quotient space

$$J_n(X) = X^n / \sim$$

by the equivalence relation generated by

$$(x_1, \dots, x_i, e, \dots, x_n) \sim (x_1, \dots, e, x_i, \dots, x_n)$$

for all  $1 \leq i \leq n-1$ . We set  $J_1(X) = X$ . For each  $n \geq 1$ , there is an inclusion  $J_n(X) \hookrightarrow J_{n+1}(X)$  given by inserting a basepoint in one of the coordinates. We call

$$J(X) = \operatorname{colim}_n J_n(X)$$

the *James construction* on  $X$ .

(b) Assume that  $X$  is a CW-complex and that the basepoint  $e$  is a 0-cell. Show that each  $J_n(X)$  and hence  $J(X)$  naturally inherits the structure of a CW-complex.

- (c) Show that the inclusion  $\iota: X = J_1(X) \hookrightarrow J(X)$  satisfies the following universal property: for any map  $f: X \rightarrow M$  to a topological monoid, there exists a unique map of topological monoids  $\hat{f}: J(X) \rightarrow M$  such that  $\hat{f} \circ \iota = f$ .
- (d) Conclude that the adjunction unit  $X \rightarrow \Omega\Sigma X$  factors as the map  $\iota$  followed by a map  $g: J(X) \rightarrow \Omega\Sigma X$ .

*Remark:* It is a theorem of I. M. James that the map  $g: J(X) \rightarrow \Omega\Sigma X$  is a weak equivalence for every connected CW-complex  $X$ .

**Exercise 2.** (5 points)

- (a) Show that for simply-connected  $X$ , the suspension homomorphism

$$\Sigma: \pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X)$$

can be identified with the map  $\pi_i(\iota)$ , where  $\iota: X \rightarrow J(X)$  is as in Exercise 1. You may assume without proof the statement of the theorem given at the end of Exercise 1.

- (b) Show that, for  $X = S^2$  and  $i = 3$ , the kernel of  $\Sigma$  is generated by the Whitehead product  $[\text{id}_{S^2}, \text{id}_{S^2}]$ . (*Hint:* Consider the long exact sequence of the pair  $(J(S^2), S^2)$ .)
- (c) Conclude that  $\pi_4(S^3)$  is a group with two elements.

**Exercise 3.** (4 points) Show that every cohomology theory is represented by an  $\Omega$ -spectrum.

**Exercise 4.** (5 points) Let  $(X, A)$  be a CW pair with both  $X$  and  $A$  connected, such that the homotopy fibre of the inclusion  $A \hookrightarrow X$  is a  $K(\pi, n)$ , for  $n \geq 1$ . Show that  $A \hookrightarrow X$  is principal if the action of  $\pi_1(A)$  on  $\pi_{n+1}(X, A)$  is trivial. (*Hint:* Consider the homomorphism  $\pi_*(X, A) \rightarrow \pi_*(X/A)$ .)